RELAXATION ENHANCEMENT BY TIME-PERIODIC FLOWS

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Abstract. We study enhancement of diffusive mixing by fast incompressible time-periodic flows. The class of *relaxation-enhancing* flows that are especially efficient in speeding up mixing has been introduced in [2]. The relaxation-enhancing property of a flow has been shown to be intimately related to the properties of the dynamical system it generates. In particular, time-independent flows u such that the operator $u \cdot \nabla$ has sufficiently smooth eigenfunctions are not relaxation-enhancing. Here we extend results of [2] to time-periodic flows $u(x,t)$ and in particular show that there exist flows such that for each fixed time the flow is Hamiltonian, but the resulting time-dependent flow is relaxation-enhancing. Thus we confirm the physical intuition that time dependence of a flow may aid mixing. We also provide an extension of our results to the case of a nonlinear diffusion model. The proofs are based on a general criterion for the decay of a semigroup generated by an operator of the form $\Gamma + iAL(t)$ with a negative unbounded self-adjoint operator Γ, a time-periodic self-adjoint operator-valued function $L(t)$, and a parameter $A \gg 1$.

1. Introduction

In the present paper we study enhancement of diffusive mixing by fast incompressible time-periodic flows. We let u be a time-periodic incompressible (i.e., $\nabla \cdot u = 0$) Lipschitz vector field (flow) on a smooth compact Riemannian manifold M, or on a bounded domain $M \subset \mathbb{R}^n$ with $\partial M \in C^2$. In the latter case we also require $u(x,t) \cdot \hat{n} = 0$ for $(x,t) \in \partial M \times \mathbb{R}$. We consider the PDE

$$\frac{d}{dt}\phi^A(x,t) + Au(x,At) \cdot \nabla \phi^A(x,t) = \Delta \phi^A(x,t), \qquad \phi^A(x,0) = \phi_0(x) \tag{1.1}$$

on M, with Neumann boundary conditions on ∂M if M is a bounded domain in $\mathbb{R}^n$. Here Δ is the Laplace-Beltrami operator on M and ∇ is the covariant derivative. We are interested in the case of fast flows with $A \gg 1$. Note that the choice of the term $Au(x, At)$ is natural here because all these flows have the same streamlines — solutions of $\frac{d}{dt}X(x,t) = Au(X(x,t), At)$, $X(x,0) = x$, have the same trajectories for different A but traverse them at different speeds (proportional to A).

It is well known that as time tends to infinity, the solution $\phi^A(x,t)$ tends to its average

$$\bar{\phi} \equiv \frac{1}{|M|}\int\limits_M \phi^A(x,t)\, d\mu = \frac{1}{|M|}\int\limits_M \phi_0(x)\, d\mu = \bar{\phi}_0,$$

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with $|M|$ the volume of M and μ the volume measure. We would like to understand how the speed of relaxation to the average depends on the properties of the flow and determine which flows are efficient in enhancing this process.

The question of the influence of advection on diffusion is very natural and physically relevant, and the subject has a long history. We refer to the recent paper [2] for a more detailed overview of the relevant literature. In [2], a class of *relaxation-enhancing* time-independent flows has been introduced, and a sharp characterization of such flows has been obtained. Our main goal here is to generalize the results of [2] to allow periodic time dependence, and also to provide some interesting examples. Let us recall the definition of these flows from [2], adjusted to our setting.

Definition 1.1. We say that the incompressible time-periodic flow $u \in \mathrm{Lip}(M \times \mathbb{R})$ is *relaxation-enhancing* if for any $\tau, \delta > 0$ there is $A_0 > 0$ such that for any $A > A_0$ and any initial datum $\phi_0 \in L^2(M)$ with $\|\phi_0\|_{L^2(M)} = 1$, the solution $\phi^A(x,t)$ satisfies

$$\|\phi^A(\cdot, \tau) - \bar{\phi}_0\|_{L^2(M)} < \delta. \tag{1.2}$$

Remark. We note that just as in [2], $\|\phi_0\|_{L^2(M)} = 1$ can be replaced by $\|\phi_0\|_{L^p(M)} = 1$ and the $L^2(M)$-norm in (1.2) by the $L^q(M)$-norm (with any $p, q \in [1, \infty]$) without a change to the class of relaxation-enhancing flows.

The flow u defines a unitary evolution $\{U(t)\}_{t\in\mathbb{R}}$ on $L^2(M)$ such that for any $\psi \in L^2(M)$,

$$(U(t)\psi)(X(x,t)) \equiv \psi(x) \tag{1.3}$$

with $X(x,t)$ the unique solution to the ODE

$$\frac{d}{dt}X(x,t) = u(X(x,t),t), \qquad X(x,0) = x. \tag{1.4}$$

That is,

$$\frac{d}{dt}(U(t)\psi) + u \cdot \nabla(U(t)\psi) = 0. \tag{1.5}$$

We also let $U(t,s) \equiv U(t)U(s)^*$ so that $(U(t,s)\psi)(X(x,t)) \equiv \psi(X(x,s))$. Unitarity of the group $\{U(t,s)\}_{s,t\in\mathbb{R}}$ is implied by incompressibility of u which guarantees that $X(\cdot,t)$ is area-preserving. We note that if $u(x,t) = u(x)$ is time independent, then $U(t,s) = e^{(-u\cdot\nabla)(t-s)}$.

The main result of this paper is

Theorem 1.2. *Let M be a smooth compact Riemannian manifold. A time p-periodic incompressible flow $u \in \mathrm{Lip}(M \times \mathbb{R})$ is relaxation enhancing if and only if the period operator $U(p)$ has no eigenfunctions in $H^1(M)$ other than the constant function.*

Remark. 1. When u is time-independent, then this is the main result of [2] (and $U(p)$ can be replaced by $u \cdot \nabla$ in the statement of the theorem).

2. In the case of time-independent u and M a bounded domain with Dirichlet boundary conditions, a necessary and sufficient condition for u to be relaxation-enhancing has been derived earlier in [1] by methods different from [2] and this paper. In particular, [1] provides estimates on the principal eigenvalue of the operator $-\Delta + Au \cdot \nabla$ and ties the behavior of

this eigenvalue with short-time evolution corresponding to (1.1). Such a link is currently not available in the case of compact manifolds or Neumann boundary conditions.

We will now discuss an example showing how important time dependence of the flow can be for relaxation enhancement. It is an example of a relaxation-enhancing time-periodic flow that, frozen at each instance of time, has closed streamlines and is not relaxation-enhancing as a stationary flow. This shows that relaxation enhancement can be achieved by flows of relatively simple structure if time dependence is allowed. This contrasts with the time independent case, where relaxation-enhancing flows must be quite complex (which is necessary to ensure purely continuous spectrum or only rough eigenfunctions of $u \cdot \nabla$).

We call a time-independent flow u on $M = \mathbb{T}^{2n}$ *Hamiltonian* if there is a C^1-function $H : M \to \alpha\mathbb{T}$ (for some $\alpha > 0$ and $\alpha\mathbb{T} \equiv [0, \alpha]$ with ends identified) or $H : M \to \mathbb{R}$ such that $u(x) = (-H_{x_{n+1}}(x), \dots, -H_{x_{2n}}(x), H_{x_1}(x), \dots, H_{x_n}(x))$. For instance, the flow $u(x) \equiv (0, 2)$ on $\mathbb{T}^2$ corresponds to the $2\mathbb{T}$-valued Hamiltonian $H(x) = 2x_1$. It is easy to see from Theorem 1.2 that no stationary Hamiltonian flow can be relaxation-enhancing. Indeed, any $\psi(x) \equiv \omega(H(x))$ with ω a smooth α-periodic function is an $H^1(M)$ eigenfunction of $u \cdot \nabla$. A part of our motivation was the question of existence of *time-periodic* Hamiltonian relaxation-enhancing flows which we now answer in the affirmative by providing the following example on the two-dimensional torus. We note that a stationary incompressible flow on $\mathbb{T}^2$ is Hamiltonian (and has closed streamlines) if and only if its mean $(\bar{u}_1, \bar{u}_2) \equiv \int_{\mathbb{T}^2} u(x)dx$ has rationally dependent coordinates. That is, $\bar{u}_1$ and $\bar{u}_2$ are integer multiples of the same number $\alpha > 0$, in which case the function $H : \mathbb{T}^2 \to \alpha\mathbb{T}$,

$$H(x_1, x_2) \equiv \int_0^{x_1} u_2(y, 0)\, dy - \int_0^{x_2} u_1(x_1, y)\, dy,$$

is a Hamiltonian for u. Notice that $H \in C^1(\mathbb{T}^2; \alpha\mathbb{T})$ because $\int_0^1 u_1(x_1, y)dy = \bar{u}_1$ and $\int_0^1 u_2(y, x_2)dy = \bar{u}_2$ for any x_1, x_2 due to incompressibility of u, and that a real-valued Hamiltonian exists for u only if $(\bar{u}_1, \bar{u}_2) = (0, 0)$.

Example 1.3. Let $v \in \mathrm{Lip}(\mathbb{T}^2)$ be any stationary smooth incompressible relaxation-enhancing flow, for instance, a flow with a purely continuous spectrum (see, e.g., [6, 7]). If $(\bar{v}_1, \bar{v}_2) \equiv \int_{\mathbb{T}^2} v(x)dx$ is its mean, then $\bar{v}_1, \bar{v}_2 \neq 0$ because v cannot be Hamiltonian. Let $b \equiv (\bar{v}_1, 0)$ and consider the time-$\bar{v}_1^{-1}$-periodic flow $u(x, t) \equiv v(x + bt) - b$. For any fixed time t the flow $u(x, t)$ has mean $(0, \bar{v}_2)$ and hence is Hamiltonian.

If now $X'(t) = u(X(t), t)$ and $Y'(t) = v(Y(t))$ with any $X(0) = Y(0) = x \in \mathbb{T}^2$, then $Y(t) = X(t) + bt$. This means that $X(\bar{v}_1^{-1}) = Y(\bar{v}_1^{-1})$, and so the period operator $U_u(\bar{v}_1^{-1})$ for u equals $U_v(\bar{v}_1^{-1}) \equiv e^{(-v\cdot\nabla)\bar{v}_1^{-1}}$. Since $U_v(\bar{v}_1^{-1})$ has no eigenfunctions in $H^1(\mathbb{T}^2)$ because v is relaxation-enhancing, Theorem 1.2 shows that the flow u is also relaxation-enhancing.

Thus, we have

Theorem 1.4. *There exists a time-periodic smooth incompressible flow u on $\mathbb{T}^2$ which is relaxation-enhancing but for each $t \in \mathbb{R}$, the flow $u(\cdot, t)$ is Hamiltonian.*

Just as in [2], our main result can be formulated and proved in an abstract Hilbert space setting. Let Γ be a self-adjoint, positive, unbounded operator with a discrete spectrum on a separable Hilbert space H. Let $0 < \lambda_1 \le \lambda_2 \le \dots$ be the eigenvalues of Γ, and e_j the corresponding eigenvectors forming an orthonormal basis in H. The (homogeneous) Sobolev space $H^m(\Gamma)$ associated with Γ is formed by all vectors $\psi = \sum_j c_j e_j$ such that

$$\|\psi\|_{H^m(\Gamma)} \equiv \sum_j \lambda_j^m |c_j|^2 < \infty.$$

We use $\langle \cdot, \cdot \rangle$ for the inner product in H and $\|\cdot\|$ and $\|\cdot\|_1$ for the norms in H and in $H^1(\Gamma)$, respectively. Note that $H^2(\Gamma)$ is the domain $D(\Gamma)$ of Γ.

Next, we assume that $L(t)$ is a periodic family of self-adjoint operators on H (without loss of generality assume that the period is 1) which satisfies

Condition 1. There is $C_0 < \infty$ such that for any $t \in \mathbb{R}$ and any $\psi \in H^1(\Gamma)$ we have

$$\|L(t)\psi\| \le C_0 \|\psi\|_1. \tag{1.6}$$

Let us also assume that the family $L(t)$ generates a strongly continuous unitary group $U(t)$ on H. That is, for each $\psi_0 \in H$, $\psi(t) \equiv U(t)\psi_0$ is a weak solution of

$$\frac{d}{dt}\psi(t) = iL(t)\psi(t), \qquad \psi(0) = \psi_0. \tag{1.7}$$

We let $V \equiv U(1)$ be the (unitary) period operator and $U(t,s) \equiv U(t)U(s)^*$. Note that due to periodicity of $L(t)$ we have

$$U(t,s) = U(t - \lfloor s \rfloor, s - \lfloor s \rfloor) \tag{1.8}$$

for any $s, t \in \mathbb{R}$. We will also assume

Condition 2. There is a function $B \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ such that for any $t, s \in \mathbb{R}$ and any $\psi \in H^1(\Gamma)$ we have

$$\|U(t,s)\psi\|_1 \le B(t-s)\|\psi\|_1 \tag{1.9}$$

Notice that (1.6) and (1.9) together imply that if $\psi_0 \in H^1(\Gamma)$, then $\psi(t) = U(t)\psi_0$ is a classical solution of (1.7) and belongs to $H^1(\Gamma)$.

We are now interested in the behavior of the solutions to the Bochner differential equation

$$\frac{d}{dt}\phi^A(t) = iAL(At)\phi^A(t) - \Gamma\phi^A(t), \qquad \phi^A(0) = \phi_0 \tag{1.10}$$

with $A \in \mathbb{R}$. When $H \equiv L^2(M) \ominus 1$ is the space of mean-zero functions from $L^2(M)$, $\Gamma \equiv -\Delta$ and $L(t) \equiv iu(t) \cdot \nabla$ on H, then this is exactly (1.1).

Definition 1.5. We say that the family $L(t)$ is *relaxation-enhancing* (with respect to Γ) if for any $\tau, \delta > 0$ there is $A_0 > 0$ such that for any $A > A_0$ and any $\phi_0 \in H$ with $\|\phi_0\| = 1$, the solution $\phi^A(t)$ satisfies

$$\|\phi^A(\tau)\| < \delta. \tag{1.11}$$

We now have the following abstract version of Theorem 1.2.

Theorem 1.6. *Assume Conditions 1 and 2. Then the periodic family $L(t)$ is relaxation-enhancing if and only if the unitary operator V has no eigenfunctions in $H^1(\Gamma)$.*

Notice that Theorem 1.2 now follows directly from this result.

Proof of Theorem 1.2. As mentioned above, we let $H \equiv L^2(M) \ominus 1$, and $\Gamma \equiv -\Delta$ and $L(t) \equiv iu(t) \cdot \nabla$ restricted to H. Conditions 1 and 2 are now implied by Lipschitzness of u with $C_0 \equiv \|u\|_\infty$ and $B(t) \equiv e^{|t| \|\nabla u\|_\infty}$ (see [2]), and so Theorem 1.6 gives Theorem 1.2 for all ϕ_0 with $\bar{\phi}_0 = 0$. Since $\bar{\phi}_0$ is conserved by (1.1), the result follows. □

The final extension we discuss in this paper is to the case of *porous medium equations*, where $\Delta\phi^A$ is replaced by $\Delta(\phi^A)^q$, $q > 1$. We discuss the setting and the result in Section 3.

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2. Proof of Theorem 1.6

In this section we prove Theorem 1.6. As in [2], we reformulate (1.10) as the small diffusion–long time problem

$$\frac{d}{dt}\phi^\epsilon(t) = iL(t)\phi^\epsilon(t) - \epsilon\Gamma\phi^\epsilon(t), \qquad \phi^\epsilon(0) = \phi_0 \tag{2.1}$$

by setting $\varepsilon \equiv A^{-1}$ and rescaling time by a factor of $1/\epsilon$. Notice that (1.11) now becomes

$$\|\phi^\epsilon(\tau/\epsilon)\| < \delta. \tag{2.2}$$

We first note the following existence and uniqueness result from [2].

Lemma 2.1. *Assume that Condition 1 is fulfilled. Then for any $\epsilon > 0$ and $T > 0$, there exists a unique solution $\phi^\epsilon(t)$ of the equation (2.1) on $[0, T]$ with initial data $\phi_0 \in H^1(\Gamma)$. This solution satisfies*

$$\phi^\epsilon(t) \in L^2([0,T], H^2(\Gamma)) \cap C([0,T], H^1(\Gamma)), \qquad \frac{d}{dt}\phi^\epsilon(t) \in L^2([0,T], H).$$

Remarks. 1. The proof of Lemma 2.1 is standard and proceeds by constructing a weak solution using Galerkin approximations and then establishing uniqueness and regularity. We refer, for example, to Evans [5] where the construction is carried out for parabolic PDEs. Given Condition 1, this can be applied verbatim to the general case.

2. The result is also valid for initial data $\phi_0 \in H$, but the solution has rougher properties on intervals containing $t = 0$, namely

$$\phi^\epsilon(t) \in L^2([0,T], H^1(\Gamma)) \cap C([0,T], H^{-1}(\Gamma)), \qquad \frac{d}{dt}\phi^\epsilon(t) \in L^2([0,T], H^{-1}(\Gamma)).$$

Existence of a rougher solution can also be derived from general semigroup theory, by checking that $iL - \epsilon\Gamma$ satisfies the conditions of the Hille-Yosida theorem and thus generates a strongly continuous contraction semigroup in H (see, e.g. [4]).

Proof of Theorem 1.6. Let us first assume that $V\psi = e^{iE}\psi$ for some $\psi \in H^1(\Gamma)$, $\|\psi\| = 1$. We will then show that the family $L(t)$ is not relaxation-enhancing. For $\varepsilon \geq 0$ let $\phi^\epsilon(t)$ be the solution of (2.1) with $\phi^\epsilon(0) = \psi$. Then we have

$$\left|\frac{d}{dt}\langle \phi^\epsilon(t), \phi^0(t)\rangle\right| = \epsilon|\langle \Gamma\phi^\epsilon(t), \phi^0(t)\rangle| \leq \frac{\epsilon}{2}(\|\phi^\epsilon(t)\|_1^2 + \|\phi^0(t)\|_1^2). \tag{2.3}$$

By multiplying equation (2.1) by $\phi^\epsilon(t)$ and integrating in time we obtain

$$2\epsilon\int_0^\infty \|\phi^\epsilon(t)\|_1^2 dt \leq \|\phi^\epsilon(0)\|^2 = 1. \tag{2.4}$$

Now $V^n\psi = e^{inE}\psi$ and periodicity of $L(t)$ imply $\phi^0(n+t) = e^{inE}\phi(t)$ for $n \in \mathbb{Z}$ and so due to Condition 2,

$$\int_0^{\tau/\epsilon} \|\phi^0(t)\|_1^2 dt = \sum_{n=0}^{\lfloor \tau/\epsilon\rfloor - 1} \int_0^1 \|\phi^0(t)\|_1^2 dt + \int_0^{\{\tau/\epsilon\}} \|\phi^0(t)\|_1^2 dt \leq \frac{\tau}{\epsilon} B_1^2 \|\psi\|_1^2 \tag{2.5}$$

with

$$B_1 \equiv \sup_{t\in[0,1]} B(t), \tag{2.6}$$

where $\lfloor x\rfloor$ and $\{x\}$ are the integer and fractional parts of x. Substituting (2.4) and (2.5) into (2.3) we obtain after integration

$$|\langle \phi^\epsilon(\tau/\epsilon), \phi^0(\tau/\epsilon)\rangle| \geq \langle \phi^\epsilon(0), \phi^0(0)\rangle - \frac{1}{4} - \frac{\tau}{2}B_1^2\|\psi\|_1^2 = \frac{3}{4} - \frac{\tau}{2}B_1^2\|\psi\|_1^2.$$

Thus for $\tau \leq B_1^{-2}\|\psi\|_1^{-2}$ we have $\|\phi^\epsilon(\tau/\epsilon)\| \geq 1/4$ for any ϵ, and hence the family $L(t)$ is not relaxation-enhancing.

Let us now assume that none of the eigenfunctions of V belong to $H^1(\Gamma)$. We will then show that the family $L(t)$ is relaxation-enhancing. We start with some auxiliary lemmas.

Lemma 2.2. *Suppose that for all $t \in (a,b)$ we have $\|\phi^\epsilon(t)\|_1^2 \geq N\|\phi^\epsilon(t)\|^2$. Then*

$$\|\phi^\epsilon(b)\|^2 \leq e^{-2\epsilon N(b-a)}\|\phi^\epsilon(a)\|^2.$$

Proof. This follows immediately from

$$\frac{d}{dt}\|\phi^\epsilon\|^2 = 2\Re\langle \phi^\epsilon, \phi^\epsilon_t\rangle = -2\langle \phi^\epsilon, \epsilon\Gamma\phi^\epsilon\rangle = -2\epsilon\|\phi^\epsilon\|_1^2 \leq -2\epsilon N\|\phi^\epsilon(t)\|^2 \tag{2.7}$$

and integration in time. □

This lemma shows that as long as the $H^1(\Gamma)$-norm of ϕ^ϵ stays large, its H-norm will decay rapidly relative to $e^{-\epsilon t}$ (which is what we need to establish (2.2)). We next need to consider the case when $\|\phi^\epsilon(\tau_0)\|_1^2 \leq N\|\phi^\epsilon(\tau_0)\|^2$ for some τ_0. First we show that in this case the evolution (2.1) will stay for some time relatively close (with respect to ϵ) to the "free" evolution $U(t,\tau_0)\phi(\tau_0)$.

Lemma 2.3. *Let $\phi^\epsilon(t)$ and $\phi^0(t)$ be solutions of the equation* (2.1) *with $\phi^\epsilon(\tau_0) = \phi^0(\tau_0) = \phi_0 \in H^1(\Gamma)$. Then for any $\tau \geq 0$ we have*

$$\|\phi^\epsilon(\tau_0+\tau) - \phi^0(\tau_0+\tau)\|^2 \leq \frac{\epsilon}{2}\|\phi_0\|_1^2 \int\limits_0^\tau B(t)^2 dt.$$

Proof. Regularity guaranteed by Conditions 1 and 2 and Lemma 2.1 allows us to multiply the equation

$$(\phi^\epsilon - \phi^0)' = iL(t)(\phi^\epsilon - \phi^0) - \epsilon\Gamma\phi^\epsilon$$

by $\phi^\epsilon - \phi^0$. We obtain

$$\frac{d}{dt}\|\phi^\epsilon(t) - \phi^0(t)\|^2 \leq 2\epsilon(\|\phi^\epsilon(t)\|_1\|\phi^0(t)\|_1 - \|\phi^\epsilon(t)\|_1^2) \leq \frac{\epsilon}{2}\|\phi^0(t)\|_1^2 \leq \frac{\epsilon}{2}B(t-\tau_0)^2\|\phi_0\|_1^2,$$

with the last inequality using Condition 2. Integration in time now gives the result. □

We now need to obtain suitable estimates on the free evolution. We denote by P_c the orthogonal projection in H on the continuous spectral subspace of the unitary operator V and by $P_p = I - P_c$ the orthogonal projection on the pure point spectral subspace of V. We also denote by P_N the orthogonal projection onto the subspace of H generated by eigenfunctions of Γ belonging to eigenvalues $\lambda_1, \dots, \lambda_N$. Note that P_N is a compact operator because Γ has a discrete spectrum.

Lemma 2.4. *Let C be any compact operator. Then the operator norm*

$$\left\|\frac{1}{T}\int_0^T U(t)^*CU(t)P_c dt\right\| \to 0 \quad \text{as } T \to \infty.$$

Proof. Denote $D = \lfloor T \rfloor$. We have

$$\left\|\frac{1}{T}\int_0^T U^*(t)CU(t)P_c dt\right\| = \left\|\int_0^1 \frac{1}{D}\sum_{n=1}^D (V^*)^{n-1}U(t)^*CU(t)V^{n-1}P_c dt\right\| + O(D^{-1}).$$

By the dominated convergence theorem it is sufficient to prove that for any $t \in [0,1]$

$$\left\|\frac{1}{D}\sum_{n=1}^D (V^*)^{n-1}U(t)^*CU(t)V^{n-1}P_c\right\| \to 0 \quad \text{as } D \to \infty.$$

The operator $\tilde{C} = U(t)^*CU(t)$ is compact, so we can reduce the problem to the case of $\tilde{C}$ being rank 1. The proof in this case is identical to that of Theorem 5.8 in [3] with integrals replaced by sums. □

Compactness of P_N and $\|P_N U(t)P_c\phi\|^2 = \langle P_c\phi, U(t)^*P_N U(t)P_c\phi\rangle$ now gives

Corollary 2.5. *For any N and $\sigma > 0$ there exists $T_c(N,\sigma)$ such that for any $T \geq T_c(N,\sigma)$ and any $\phi \in H$ with $\|\phi\| \leq 1$ we have*

$$\frac{1}{T}\int_0^T \|P_N U(t)P_c\phi\|^2 dt \leq \sigma.$$

Next we consider the free evolution of $P_p\phi$.

Lemma 2.6. *Let $K \subset S \equiv \{\phi \in H : \|\phi\| = 1\}$ be a compact set. Consider the set $K_1 \equiv \{\phi \in K : \|P_p\phi\| \geq 1/2\}$. Then for any $\Omega > 0$ we can find $N_p(\Omega, K)$ and $T_p(\Omega, K)$ such that for any $N \geq N_p(\Omega, K)$, any $T \geq T_p(\Omega, K)$, and any $\phi \in K_1$ we have*

$$\frac{1}{T}\int_0^T \|P_N U(t) P_p\phi\|_1^2 dt \geq \Omega.$$

Proof. Notice that with $D = \lfloor T \rfloor$,

$$\frac{1}{T}\int_0^T \|P_N U(t) P_p\phi\|_1^2 dt \geq \int_0^1 \frac{1}{D+1}\sum_{n=1}^{D} \|P_N U(t) V^{n-1} P_p\phi\|_1^2 \, dt. \tag{2.8}$$

The proof will now follow from

Lemma 2.7. *For any fixed $t \in [0,1]$ there are $N_p(t, \Omega, K)$ and $D_p(t, \Omega, K)$ such that for any $N \geq N_p(t, \Omega, K)$, any $D \geq D_p(t, \Omega, K)$, and any $\phi \in K_1$ we have*

$$\frac{1}{D+1}\sum_{n=1}^{D} \|P_N U(t) V^{n-1} P_p\phi\|_1^2 \geq 2\Omega. \tag{2.9}$$

Proof. Denote by e^{iE_j} the eigenvalues of V (distinct, without repetitions) and by Q_j the orthogonal projection on the space spanned by the eigenfunctions corresponding to e^{iE_j}. Then (2.9) can be rewritten as

$$\sum_{j,l} \frac{e^{i(E_j - E_l)D} - 1}{(e^{i(E_j - E_l)} - 1)(D+1)} \langle \Gamma P_N U(t) Q_j\phi, P_N U(t) Q_l \phi\rangle \geq 2\Omega$$

with the fraction equal to $D/(D+1)$ when $j = l$. The rest of the proof is identical to that of Lemma 3.3 from [2] with Q_j replaced by $U(t)Q_j$ and integrals replaced by sums, provided we can show that $U(t)Q_j\phi \notin H^1(\Gamma)$ whenever $Q_j\phi \neq 0$. But this is true because if $U(t)Q_j\phi \in H^1(\Gamma)$, then $VQ_j\phi = U(1,t)U(t)Q_j\phi \in H^1(\Gamma)$ by Condition 2, which is a contradiction with the assumption that V has no eigenfunctions in $H^1(\Gamma)$ (unless $Q_j\phi = 0$). $\square$

Using (2.8) and (2.9), it is now easy to finish the proof of Lemma 2.6. Indeed, one only needs to choose $N_p(\Omega, K)$ and $T_p(\Omega, K)$ to be larger than $N_p(t, \Omega, K)$ and $D_p(t, \Omega, K)$ for all $t \in E$ with $E \subset [0,1]$ some set of measure $1/2$. This is possible because $N_p(t, \Omega, K)$ and $D_p(t, \Omega, K)$ are finite for each t. $\square$

We can now proceed with the proof of Theorem 1.6. Recall that we assume that V has no eigenfunctions in $H^1(\Gamma)$. Given $\tau, \delta > 0$, we choose M large enough, so that $e^{-\lambda_M \tau/160} < \delta$. Define the sets $K \equiv \{\phi \in S : \|\phi\|_1^2 \leq B_1^2 \lambda_M\} \subset S$ and as before, $K_1 \equiv \{\phi \in K : \|P_p\phi\| \geq 1/2\}$ (recall that B_1 is from (2.6)). Choose N so that $N \geq M$ and $N \geq N_p(5\lambda_M, K)$ from Lemma 2.6. Define

$$\tau_1 \equiv \max\{T_p(5\lambda_M, K), T_c(N, \frac{\lambda_M}{20\lambda_N}), 1\},$$

where T_p is from Lemma 2.6 and T_c from Corollary 2.5. Finally, choose $\epsilon_0 > 0$ so that $\tau_1 + 1 < \tau/2\epsilon_0$, and

$$\epsilon_0 \int_0^{\tau_1+1} B(t)^2 dt \le \frac{1}{20\lambda_N},$$

where $B(t)$ is from Condition 2.

Take any $\epsilon < \epsilon_0$. If we have $\|\phi^\epsilon(s)\|_1^2 \ge \lambda_M \|\phi^\epsilon(s)\|^2$ for all $s \in [0, \tau/2\epsilon]$ then Lemma 2.2 implies that $\|\phi^\epsilon(\tau/2\epsilon)\|^2 \le e^{-\lambda_M \tau} \le \delta$ by the choice of M and we are done. Otherwise, let τ_0 be the first time in the interval $[0, \tau/2\epsilon]$ such that $\|\phi^\epsilon(\tau_0)\|_1^2 \le \lambda_M \|\phi^\epsilon(\tau_0)\|^2$. We now let $\phi^0(t) \equiv U(t, \tau_0)\phi^\epsilon(\tau_0)$ solve (2.1) with initial condition $\phi^0(\tau_0) = \phi^\epsilon(\tau_0)$. Lemma 2.3 then gives

$$\|\phi^\epsilon(t) - \phi^0(t)\|^2 \le \frac{\lambda_M}{40\lambda_N} \|\phi^\epsilon(\tau_0)\|^2 \tag{2.10}$$

for all $t \in [\tau_0, \lceil \tau_0 \rceil + \tau_1]$. We also have

$$\|\phi^0(\lceil \tau_0 \rceil)\|_1^2 \le B_1^2 \lambda_M \|\phi^\epsilon(\tau_0)\|^2 = B_1^2 \lambda_M \|\phi^0(\lceil \tau_0 \rceil)\|^2$$

by Condition 2 and so $\phi^0(\lceil \tau_0 \rceil)/\|\phi^0(\lceil \tau_0 \rceil)\| \in K$. We now claim that the following estimate holds:

$$\|\phi^\epsilon(\lceil \tau_0 \rceil + \tau_1)\|^2 \le e^{-\lambda_M \epsilon \tau_1/20} \|\phi^\epsilon(\tau_0)\|^2. \tag{2.11}$$

Indeed, given our choice of τ_1, Corollary 2.5, Lemma 2.6, (2.10), and $U(t + \lceil \tau_0 \rceil, \lceil \tau_0 \rceil) = U(t)$, the proof is the same as that of the almost identical estimate (3.8) in [2] (which has τ_0 in place of $\lceil \tau_0 \rceil$). Then we have

$$\|\phi^\epsilon(\tau_0 + \tau_1 + 1)\|^2 \le \|\phi^\epsilon(\lceil \tau_0 \rceil + \tau_1)\|^2 \le e^{-\lambda_M \epsilon \tau_1/20} \|\phi^\epsilon(\tau_0)\|^2 \le e^{-\lambda_M \epsilon (\tau_1+1)/40} \|\phi^\epsilon(\tau_0)\|^2, \tag{2.12}$$

where we used (2.7) in the first inequality. The same method can be repeated with τ_0 replaced by the first time after $\tau_0 + \tau_1 + 1$ at which $\|\phi^\epsilon(t)\|_1^2 \le \lambda_M \|\phi^\epsilon(t)\|^2$, etc. On the other hand, for any interval $I = [a, b]$ such that $\|\phi^\epsilon(t)\|_1^2 \ge \lambda_M \|\phi^\epsilon(t)\|^2$ on I, we have by Lemma 2.2 that

$$\|\phi^\epsilon(b)\|^2 \le e^{-2\lambda_M \epsilon (b-a)} \|\phi^\epsilon(a)\|^2. \tag{2.13}$$

Combining all the decay factors gained from (2.12) and (2.13), and using $\tau_1 + 1 < \tau/2\epsilon$, we find that there exists $\tau_2 \in [\tau/2\epsilon, \tau/\epsilon]$ such that

$$\|\phi^\epsilon(\tau_2)\|^2 \le e^{-\lambda_M \epsilon \tau_2/40} \le e^{-\lambda_M \tau/80} < \delta^2$$

by our choice of M. Then (2.7) gives $\|\phi^\epsilon(\tau/\epsilon)\| \le \|\phi^\epsilon(\tau_2)\| < \delta$, thus finishing the proof of Theorem 1.6. □

3. Relaxation for the porous medium equation

In this section, we indicate how to generalize our results on relaxation enhancement to some nonlinear equations. The arguments of the previous section and [2] are sufficiently robust to remain applicable in this more general setting. Here we focus on the case of the *porous medium equation with advection*

$$\frac{d}{dt}\phi^A(x,t) + Au(x, At) \cdot \nabla \phi^A(x,t) = \Delta(\phi^A(x,t))^q, \qquad \phi^A(x,0) = \phi_0(x), \tag{3.1}$$

with $q > 1$ and on a smooth compact Riemannian manifold M without boundary. We restrict our considerations to initial data ϕ_0 which are positive and bounded: $0 < h \leq \phi_0(x) \leq h^{-1}$. This is the physically relevant case, and such choice of data also ensures uniform parabolicity. We refer to [8] (which mainly concentrates on (3.1) without the advection term) for the overview of history, basic properties, and applications of the equation (3.1). In particular, a unique classical solution to (3.1) exists under our assumptions on the initial data provided $u \in C^\infty(M \times \mathbb{R})$ (see [8, Section 3.1] and references therein).

We again define relaxation-enhancing flows via Definition 1.1 but this time with the initial data also satisfying $h \leq \phi_0 \leq h^{-1}$ for some $h > 0$, and A_0 can additionally depend on h. Notice that the mean $\bar{\phi} = \bar{\phi}_0$ of the solution is again preserved by the evolution (3.1). We now have

Theorem 3.1. *Let M be a smooth compact Riemannian manifold. Consider equation* (3.1) *with real-valued positive initial data bounded away from 0 and ∞. A time p-periodic incompressible flow $u \in C^\infty(M \times \mathbb{R})$ is relaxation enhancing for* (3.1) *if and only if the period operator $U(p)$ has no eigenfunctions in $H^1(M)$ other than the constant function.*

Remarks. 1. The same result also holds in the case of the *generalized porous medium equation with advection*

$$\frac{d}{dt}\phi^A(x,t) + Au(x,At) \cdot \nabla\phi^A(x,t) = \Delta\Psi(\phi^A(x,t)), \qquad \phi^A(x,0) = \phi_0(x), \tag{3.2}$$

where Ψ is any smooth increasing function with $\Psi(0) = 0$ and Ψ' bounded away from zero on each interval $[h, h^{-1}]$, $h > 0$.

2. Similarly to Theorem 1.2, this theorem can be stated in a more abstract form. We do not pursue this more general formulation here since it requires a number of technical assumptions. The role of the $H^1(\Gamma)$-norm is then typically played by an expression derived from the nonlinear term. In our case this expression is $\int \psi^{q-1}|\nabla\psi|^2\,dx$ which is equivalent to the $H^1(\Gamma)$-norm for all $h \leq \psi \leq h^{-1}$ (and $h \leq \phi^\varepsilon(t) \leq h^{-1}$ is guaranteed by $h \leq \phi_0 \leq h^{-1}$ and the maximum principle).

Proof. Most of the proof is parallel to that of Theorem 1.2 and Theorem 1.6, so we just indicate the necessary changes. Let us switch to the equivalent small-diffusion formulation

$$\frac{d}{dt}\phi^\epsilon(x,t) + u(x,t) \cdot \nabla\phi^\epsilon(x,t) = \epsilon\Delta(\phi^\epsilon(x,t))^q, \qquad \phi^\epsilon(x,0) = \phi_0(x). \tag{3.3}$$

If ψ is a nonconstant H^1 eigenfunction of $U(p)$, assume that ψ is bounded by $M < \infty$ (otherwise consider $\arg(\psi)\min\{|\psi|, M\}$ instead, which is an H^1 eigenfunction of $U(p)$ with the same eigenvalue). Without loss of generality assume $\Re\psi \neq 0$ and define $\phi_0 \equiv m(\Re\psi + 2M)$ where $m > 0$ is such that $\|\phi_0 - \bar{\phi}_0\| = 1$. Now $h \leq \phi_0 \leq h^{-1}$ for some $h > 0$, and let $\phi^0(t)$ and $\phi^\epsilon(t)$ solve (3.3). It is easy to see that $\phi^0(t) = m(\Re\psi^0(t) + 2M) \geq h$, where $\psi^0(t)$ solves (3.3) with $\varepsilon = 0$ and initial condition ψ. As a result we have

$$\|\nabla\phi^0(t)\| \leq m\|\nabla\psi^0(t)\| \leq mB_1\|\nabla\psi^0(\lfloor t \rfloor)\| = mB_1\|\nabla\psi\|.$$

Instead of (2.3) in the proof of Theorem 1.6 we now obtain

$$\left|\frac{d}{dt}\langle \phi^\epsilon(t)-\bar\phi_0, \phi^0(t)-\bar\phi_0\rangle\right| \le \frac{\epsilon q}{2}\left(\int_M (\phi^\epsilon)^{q-1}|\nabla\phi^\epsilon|^2\,dx + \int_M (\phi^\epsilon)^{q-1}|\nabla\phi^0|^2\,dx\right).$$

Similarly to (2.4), we have

$$2\epsilon q\int_0^\infty\int_M (\phi^\epsilon)^{q-1}|\nabla\phi^\epsilon|^2\,dxdt \le 1.$$

Since $|\phi^\epsilon|\le h^{-1}$, (2.5) carries over without changes and we obtain

$$|\langle \phi^\epsilon(\tau/\epsilon)-\bar\phi_0, \phi^0(\tau/\epsilon)-\bar\phi_0\rangle| \ge 1-\frac14-\frac{\tau q}{2}h^{1-q}m^2B_1^2\|\nabla\psi\|^2,$$

from which lack of relaxation enhancement follows.

The only argument in the proof of the opposite implication that requires a slight adjustment is Lemma 2.3, where we now have

$$\begin{aligned}\frac{d}{dt}\|\phi^\epsilon(t)-\phi^0(t)\|^2 &\le 2\epsilon\int_M \Delta(\phi^\epsilon)^q(\phi^\epsilon-\phi^0)\,dx\\
&\le 2\epsilon q\left(\int_M(\phi^\epsilon)^{q-1}|\nabla\phi^\epsilon|^2dx\right)^{1/2}\left(\int_M(\phi^\epsilon)^{q-1}|\nabla\phi^0|^2dx\right)^{1/2} - 2\epsilon q\int_M(\phi^\epsilon)^{q-1}|\nabla\phi^\epsilon|^2dx\\
&\le \frac{\epsilon q}{2}\int_M(\phi^\epsilon)^{q-1}|\nabla\phi^0|^2dx\\
&\le \frac{\epsilon q h^{1-q}}{2}B(t-\tau_0)^2\|\nabla\phi_0\|^2.\end{aligned}$$

The rest of the proof involves only estimates on the linear dynamics with $\varepsilon=0$. Thus all the bounds on the $H^1(\Gamma)$-norm from the proof of Theorem 1.6 translate immediately into estimates on the decay rate for $\|\phi^\varepsilon-\bar\phi_0\|$, with possibly h-dependent constants. $\square$

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