Quenching of Flames by Fluid Advection

PETER CONSTANTIN

ALEXANDER KISELEV

AND

LEONID RYZHIK
University of Chicago

Abstract

We consider a simple scalar reaction-advection-diffusion equation with ignition-type nonlinearity and discuss the following question: What kinds of velocity profiles are capable of quenching any given flame, provided the velocity's amplitude is adequately large? Even for shear flows, the answer turns out to be surprisingly subtle.

If the velocity profile changes in space so that it is nowhere identically constant (or if it is identically constant only in a region of small measure), then the flow can quench any initial data. But if the velocity profile is identically constant in a sizable region, then the ensuing flow is incapable of quenching large enough flames, no matter how much larger the amplitude of this velocity is. The constancy region must be wider across than a couple of laminar propagating front widths.

The proof uses a linear PDE associated to the nonlinear problem, and quenching follows when the PDE is hypoelliptic. The techniques used allow the derivation of new, nearly optimal bounds on the speed of traveling-wave solutions. © 2001 John Wiley & Sons, Inc.

1 Introduction

We consider a mixture of reactants interacting in a region that may have a rather complicated spatial structure but is thin across. A mathematical model that describes a chemical reaction in a fluid is a system of two equations for concentration n and temperature T of the form

$$(1.1) \quad T_t + u \cdot \nabla T = \kappa \Delta T + \frac{v_0^2}{\kappa} g(T) n, \quad n_t + u \cdot \nabla n = \frac{\kappa}{\text{Le}} \Delta n - \frac{v_0^2}{\kappa} g(T) n.$$

The equations in (1.1) are coupled to the reactive Euler equations for the advection velocity $u(x, y, t)$. Two assumptions are usually made to simplify the problem: The first is a constant-density approximation [6] that allows us to decouple the Euler equations from system (1.1) and to consider $u(x, y, t)$ as a prescribed quantity that does not depend on T or n. The second assumption is that $\text{Le} = 1$ (equal thermal and material diffusivities). These two assumptions reduce the above system

to a single scalar equation for the temperature T. We assume in addition that the advecting flow is unidirectional. Then system (1.1) becomes

$$(1.2) \qquad T_t + Au(y)T_x = \kappa \Delta T + \frac{v_0^2}{\kappa} f(T), \quad T(0, x, y) = T_0(x, y),$$

with $f(T) = g(T)(1 - T)$. We are interested in strong advection and have accordingly written the velocity as a product of the amplitude A and the profile $u(y)$. In this paper we consider nonlinearity of the ignition type

$\quad$ (i) $\quad f(T)$ is Lipschitz-continuous on $0 \leq T \leq 1$,

$\quad$ (ii) $\quad f(1) = 0$, $\exists \theta_0 \in (0, 1)$ such that

$$(1.3) \qquad \begin{cases} f(T) = 0 \text{ for } 0 \leq T \leq \theta_0 \\ f(T) > 0 \text{ for } T > \theta_0, \end{cases}$$

$\quad$ (iii) $\quad f(T) \leq T$.

The last condition in (1.3) is just a normalization. We consider the reaction-diffusion equation (1.2) in a strip $D = \{x \in \mathbb{R}, y \in [0, H]\}$. Equation (1.2) may be considered as a simple model of flame propagation in a fluid [2] that is advected by a shear (unidirectional) flow. The physical literature on the subject is vast, and we refer to the recent review [25] for an extensive bibliography. The main physical effect of advection for frontlike solutions is the speedup of the flame propagation due to the large-scale distortion of the front. The role of the advection term in (1.2) for the frontlike initial data was also a subject of intensive mathematical scrutiny recently. Existence of unique frontlike traveling waves has been established in [2, 5, 22, 23], and their stability has been studied in [3, 17, 18, 22, 23, 24].

A traveling front is a solution of (1.2) of the form

$$(1.4) \qquad\qquad T(t, x, y) = U(x - c_A t, y)$$

such that

$$(1.5) \qquad \lim_{s \to -\infty} U(s, y) = 1, \quad \lim_{s \to +\infty} U(s, y) = 0, \quad U_s(s, y) < 0.$$

The monotonicity property is not required for traveling-wave solutions, but it is always present in the situation we consider. The speedup of the fronts by advection mentioned above may be quantified as the dependence of the traveling front speed c_A on the amplitude A. Variational formulas for c_A were derived in [9, 10] based on the methods of [20, 21, 22, 23] where related results were proven in a slightly different context. The paper [10] also contains results on the asymptotic behavior of c_A when A is small for some classes of shear flows as well as upper bounds on c_A linear in A. An alternative approach to quantifying advection effects was introduced by the present authors in [7, 14]. It is based on the notion of the bulk burning rate,

$$V(t) = \int_D T_t \frac{dx\,dy}{H} = \frac{v_0^2}{\kappa} \int_D f(T) \frac{dx\,dy}{H},$$

which extends the notion of front speed. We derived lower bounds for long-time averages of $V(t)$ that behave like CA for large A, with constant C depending on the geometry of the flow. These bounds are valid for a class of flows that we call percolating. They are characterized by infinite tubes of streamlines connecting $\pm\infty$ and include shear flows as a particular case. Our bounds imply the estimate $c_A \geq CA$ for traveling waves. Audoly, Berestycki, and Pomeau gave a formal argument [1] suggesting that in the case when the shear flow varies on the scale much larger than the laminar-flame width, one should have $c_A \sim A$. One of the by-products of this paper is a rigorous proof of this conjecture.

Our main goal in the present paper is to consider advection effects for a different physically interesting situation, where initial data are compactly supported. In this case, two generic scenarios are possible. If the support of the initial data is large enough, then two fronts form and propagate in opposite directions. Fluid advection speeds up the propagation, accelerating the burning. However, if the support of the initial data is small, then the advection exposes the initial hot region to diffusion that cools it below the ignition temperature, ultimately extinguishing the flame.

We consider for simplicity periodic boundary conditions

$$(1.6) \qquad T(t, x, y) = T(t, x, y + H)$$

in y and decay in x:

$$(1.7) \qquad T(t, x, y) \to 0 \quad \text{as } |x| \to \infty.$$

We take $u(y)$ to be periodic with period H and with mean equal to zero:

$$(1.8) \qquad \int_0^H u(y)dy = 0.$$

A constant nonzero mean can be easily taken into account by translation. We consider the case when the width of the domain is larger than the length scale for the laminar-front width: $H \gg l = \kappa/v_0$. We will always assume that initial data $T_0(x, y)$ is such that $0 \leq T_0(x, y) \leq 1$. Then we have $0 \leq T \leq 1$ for all $t > 0$ and $(x, y) \in D$. Moreover, we assume that for some L and $\eta > 0$ we have

$$(1.9) \qquad \begin{aligned} T_0(x, y) &> \theta_0 + \eta \quad \text{for } |x| \leq L/2, \\ T_0(x, y) &= 0 \qquad\quad \text{for } |x| \geq L. \end{aligned}$$

The main purpose of this paper is to study the possibility of quenching flames by strong fluid advection in a model (1.2). The phenomena associated with flame quenching are of great interest for physical, astrophysical, and engineering applications. The problem of extinction and flame propagation in the mathematical model (1.2) under conditions (1.3), (1.7), and (1.9) was first studied by Kanel' [13] in one dimension and with no advection. He showed that, in the absence of fluid motion, there exist two length scales $L_0 < L_1$ such that the flame becomes extinct for $L < L_0$ and propagates for $L > L_1$. More precisely, he has shown that there exist

L_0 and L_1 such that

(1.10)
$$T(t, x, y) \to 0 \quad \text{as } t \to \infty \text{ uniformly in } D \text{ if } L < L_0\,,$$
$$T(t, x, y) \to 1 \quad \text{as } t \to \infty \text{ for all } (x, y) \in D \text{ if } L > L_1\,.$$

In the absence of advection, the flame extinction is achieved by diffusion alone, given that the support of initial data is small compared to the scale of the laminar-flame width $l = \kappa/v_0$. However, in many applications the quenching is the result of strong wind and intense fluid motion and operates on larger scales. There are few results available for such situations in the framework of the reaction-diffusion model. Kanel''s result was extended to nonzero advection by shear flows by Roque-joffre [19], who has shown that (1.10) holds also for $u \neq 0$ with L_0 and L_1 depending, in particular, on A and $u(y)$. The second (propagation) part in (1.10) was also proved in [24] for general periodic flows.

However, the interesting question about more explicit quantitative dependence of L_0 and L_1 on A and $u(y)$ remained open. Is it possible to quench the initial data that previously led to an expanding solution by increasing A but not changing the profile? How does this possibility depend on the geometry of the profile $u(y)$? Anyone who has tried to light a match in the wind has some intuition about this phenomenon. Yet, the mathematical answer turns out to be surprisingly subtle.

In this paper, we also limit ourselves to shear flows. We are interested in understanding the behavior of L_0 and L_1 for large A. The answer depends strongly on the geometry of the flow. In some cases the maximal extinction size grows $L_0 \sim A$, and in others even the propagation size L_1 remains finite as A goes to infinity. In the first case, we will say that $u(y)$ is quenching.

DEFINITION 1.1 We say that the profile $u(y)$ is *quenching* if for any L and any initial data $T_0(x, y)$ supported inside the interval $[-L, L] \times [0, H]$ there exists A_0 such that the solution of (1.2) becomes extinct:

(1.11)
$$T(t, x, y) \to 0 \quad \text{as } t \to \infty \text{ uniformly in } D$$

for all $A \geq A_0$. We call the profile $u(y)$ strongly quenching if the critical amplitude of advection A_0 satisfies $A_0 \leq CL$ for some constant $C(u, \kappa, v_0, H)$ (which has the dimension of inverse time).

Note that (1.11) implies that the burning rate $V(t) = 0$ after some finite time T_0 since reaction is of the ignition type (1.3).

The key feature that distinguishes quenching from nonquenching velocities is the absence or presence of large enough flat parts in the profile $u(y)$.

DEFINITION 1.2 We say that the profile $u(y) \in C^\infty[0, H]$ satisfies the H-condition if

(1.12)   there is no point $y \in [0, H]$ where all derivatives of $u(y)$ vanish.

The *H*-condition guarantees that the operator

$$(1.13) \qquad \frac{\partial}{\partial t} + u(y)\frac{\partial}{\partial x} - \frac{\partial^2}{\partial y^2}$$

is hypoelliptic [11]. The study of existence of smooth fundamental solutions for such operators was initiated by Kolmogorov [15]. Kolmogorov's work with $u(y) = y$ served in part as a motivation for the fundamental result on characterization of hypoelliptic operators by Hörmander [11]. The hypoellipticity of the operator (1.13) plays a key role in some of our considerations. Our first result is that the *H*-condition implies strong quenching.

THEOREM 1.3 *Let $f(T)$ be an ignition-type nonlinearity, and let the advection $u \in C^\infty[0, H]$ satisfy the H-condition. Then $u(y)$ is strongly quenching. That means that there exists a constant $C(u, \kappa, v_0, H) > 0$ that may depend on H, κ, v_0, and $u(y)$ but is independent of A such that*

$$(1.14) \qquad T(t, x, y) \to 0 \quad \text{as } t \to \infty \text{ uniformly in } D$$

whenever the initial temperature $T_0(x, y)$ is supported in a set $[-L, L] \times [0, H]$ with $L < C(u, \kappa, v_0, H)A$.

The next result shows that a plateau on the order of the laminar-front width

$$(1.15) \qquad l = \frac{\kappa}{v_0}$$

in the profile $u(y)$ prohibits quenching (and therefore the conditions in Theorem 1.3 are natural).

THEOREM 1.4 *There exist universal constants $C_0, C_1 > 0$ such that, if $u(y) = \bar{u} = \text{const}$ for $y \in [a - h, a + h]$ for some $a \in [0, H]$ and $h \geq h_0 = C_0 l$, then*

$$(1.16) \qquad T(t, x, y) \to 1 \quad \text{as } t \to \infty$$

uniformly on compact sets for all $A \in \mathbb{R}$ whenever the initial temperature $T_0(x, y)$ satisfying (1.9) is supported in an interval $[-L, L] \times [0, H]$ with $L \geq C_1 h_0$.

An interesting by-product of the proof of Theorem 1.4 is an estimate for the speed of traveling front solutions of (1.2) when the shear flow varies slowly on the scale of the laminar-flame width l. Let us define

$$|u_+|_{h_0} = \max_{0 \leq y_0 \leq H} \left\{ \min_{y \in [y_0 - h_0, y_0 + h_0]} u_+ \right\}$$

with $h_0 = Cl$ given by Theorem 1.4 and $u_+ = \max(u(y), 0)$.

THEOREM 1.5 *The speed of the traveling front c satisfies the upper and lower bounds*

$$A|u_+|_{h_0} \leq c \leq A\|u_+\|_\infty + v_0.$$

The upper bound of Theorem 1.5 is contained in [7] (it is shown there for KPP-type reaction, but this immediately implies the corresponding bound for ignition nonlinearity by a simple application of the maximum principle). The left-hand side is close to $\|u_+\|_\infty$ if $u(y)$ is slowly varying on the scale h_0. This agrees with the formal prediction of Audoly, Berestycki, and Pomeau [1], and also (up to the addition of v_0) with the results of Majda and Souganidis [16] in the homogenization regime $\kappa \to 0$. The scale h_0 in the lower bound in Theorem 1.5 is natural. It is known that rapid oscillations in $u(y)$ reduce the enhancement of the traveling-front speed [7, 10, 14], so $\|u\|_\infty$ itself is not the right measure for a lower bound for c. The lower bound in Theorem 1.5 shows that the speedup of the traveling front is governed by a quantity that is neither local nor global: It is behavior on the scale h_0 that is important.

Unlike hypoellipticity, the quenching property is stable to small L^∞ perturbations: A small enough plateau (on the scale of the laminar-front width l) does not stop quenching.

THEOREM 1.6 *For every $\theta_0 > 0$ in (1.3) there exists a constant $B > 0$ such that, if a profile $u(y)$ satisfies the H-condition outside an interval $y \in [a-h, a+h]$ with $h \leq h_1 = Bl$, then it is strongly quenching.*

Moreover, the strongly quenching profiles are generic in the following sense:

THEOREM 1.7 *The set of all strongly quenching shear flows $u(y)$ contains a dense open set in $C[0, H]$ (here $C[0, H]$ is the space of continuous functions on $[0, H]$).*

In Section 3 we show that all results on quenching, namely, Theorems 1.3, 1.6, and 1.7, extend to the case of the full system (1.1) with Le $\neq 1$.

Finally, in the last section we prove that initial data of sufficiently small size (of the same order as in the case $u(y) = 0$) will be quenched by any shear flow $Au(y)$.

2 Quenching by a Shear Flow

We prove Theorem 1.3 in this section.

PROOF OF THEOREM 1.3: It suffices to show that there exists some time t_0 such that

$$(2.1) \qquad T(t_0, x, y) \leq \theta_0 \quad \text{for all } (x, y) \in D .$$

Then it follows from the maximum principle that $T(t, x, y) \leq \theta_0$ for all $t \geq t_0$, and hence T satisfies the linear advection-diffusion equation

$$T_t + Au(y)T_x = \kappa \Delta T$$

for $t \geq t_0$, which implies (1.14). We actually show that (2.1) holds at $t_0 = \kappa/v_0^2$ for sufficiently large A. Recall that $f(T) \leq T$, and hence $T(t, x, y)$ can be bounded from above using the maximum principle as follows:

$$(2.2) \qquad T(t, x, y) \leq \Phi(t, x, y)e^{v_0^2 t/\kappa} .$$

Here the function $\Phi(t, x, y)$ satisfies the linear problem

(2.3)
$$\Phi_t + Au(y)\Phi_x = \kappa \Delta \Phi, \quad \Phi(0, x, y) = T_0(x, y),$$
$$\Phi(t, x, y) = \Phi(t, x, y + H).$$

Furthermore, we have

$$\Phi(t, x, y) = \int_{-\infty}^{\infty} dz G(t, x - z)\Psi(t, z, y)$$

with the function $\Psi(t, x, y)$ satisfying the degenerate parabolic equation

(2.4)
$$\Psi_t + Au(y)\Psi_x = \kappa \Psi_{yy}, \quad \Psi(0, x, y) = T_0(x, y),$$
$$\Psi(t, x, y) = \Psi(t, x, y + H),$$

and

$$G(t, x) = \frac{1}{\sqrt{4\pi \kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right).$$

We note that if $u(y)$ satisfies the H-condition (1.12), then the diffusion process defined by (2.4) has a unique smooth transition probability density. Indeed, the Lie algebra generated by the operators ∂_y and $\partial_t + u(y)\partial_x$ consists of vector fields of the form

$$\frac{\partial}{\partial y}, \frac{\partial}{\partial t} + u(y)\frac{\partial}{\partial x}, u'(y)\frac{\partial}{\partial x}, u''(y)\frac{\partial}{\partial x}, \ldots, u^{(n)}(y)\frac{\partial}{\partial x}, \ldots,$$

which span $\mathbb{R}^2$ if $u(y)$ satisfies (1.12). Then the theory of Hörmander [11] and the results of Ichihara and Kunita [12] imply that there exists a smooth transition probability density $p_A(t, x, y, y')$ such that

$$\Psi(t, x, y) = \int_{\mathbb{R}} dx' \int_0^H dy' p_A(t, x - x', y, y')T_0(x', y').$$

In particular, the function $p_A(t)$ is uniformly bounded from above for any $t > 0$ [12]. Then we have

$$T(t, x, y) \leq e^{v_0^2 t/\kappa}\|\Phi(t)\|_{L_{x,y}^\infty} \leq e^{v_0^2 t/\kappa}\|\Psi(t)\|_{L_{x,y}^\infty} \leq e^{v_0^2 t/\kappa}\|p_A(t)\|_{L_{x,y}^\infty}\|T_0\|_{L_{x,y}^1}.$$

It is straightforward to observe that

$$p_A(t, x, y, y') = \frac{v_0}{A} p_0\left(t, \frac{v_0}{A}x, y, y'\right)$$

with p_0 being the transition probability density for (2.4) with $A = v_0$. That is, p_0 satisfies

$$\frac{\partial p_0}{\partial t} + v_0 u(y)\frac{\partial p_0}{\partial x} = \kappa \frac{\partial^2 p_0}{\partial y^2},$$
$$p_0(0, x, y, y') = \delta(x)\delta(y - y'), \quad p_0(t, x, y) = p_0(t, x, y + H).$$

Therefore we obtain

$$T(t, x, y) \leq e^{v_0^2 t/\kappa}\frac{v_0}{A}\|p_0(t)\|_{L_{x,y}^\infty}\|T_0\|_{L_{x,y}^1} \leq 2e^{v_0^2 t/\kappa}\frac{v_0}{A}\|p_0(t)\|_{L_{x,y}^\infty}LH,$$

and, in particular, at time $t_0 = \kappa/v_0^2$ we have

$$(2.5) \qquad T(t_0, x, y) \leq C \frac{v_0}{A} \|p_0(t_0)\|_{L^\infty_{x,y}} LH \leq \theta_0$$

as long as

$$\frac{A}{v_0} \geq C \frac{\|p_0(t_0)\|_{L^\infty_{x,y}} LH}{\theta_0} \,.$$

Theorem 1.3 follows from (2.5) as explained in the beginning of this section. $\square$

We now prove Theorem 1.6, which shows that a sufficiently small plateau in the profile $u(y)$ is not an obstruction to quenching.

PROOF OF THEOREM 1.6: Let us define the set

$$D_r = \mathbb{R} \times [a - r, a + r] \,.$$

As before, it suffices to show that the solution of (2.3) satisfies

$$(2.6) \qquad \Phi(t_0, x, y) \leq \frac{\theta_0}{e} \,, \qquad t_0 = \frac{\kappa}{v_0^2} \,,$$

and that is what we will do. First, we split the initial data for (2.4) into two parts: one supported on a strip D_{h_1} containing the flat part of $u(y)$, and another supported outside it. We will choose $h_1 = C_1 l > h$ such that any solution of (2.4) that is independent of x and with initial data supported inside D_{h_1} will be small at time $t_0 = \kappa/v_0^2$. The second part is supported away from the strip D_h, where $u(y)$ is flat. Therefore for a sufficiently small time it behaves like a solution of (2.4) with advection satisfying the H-condition.

We choose h_1 as follows. Let $\phi(t, y)$ be a periodic solution of

$$\phi_t = \phi_{yy} \,, \qquad \phi(0, y) = \phi_0(y) \,, \qquad 0 \leq \phi_0(y) \leq 1 \,,$$

given by

$$\phi(t, y) = \sum_{j \in Z} \phi_j(0) e^{2ij\pi y/H - 4\kappa t \pi^2 j^2/H^2} \,.$$

Then we have

$$\left| \phi(t, y) - \frac{\|\phi(0, y)\|_{L^1_y}}{H} \right| \leq 2 \frac{\|\phi(0, y)\|_{L^1_y}}{H} \sum_{j \geq 1} e^{-4\kappa t \pi^2 j^2/H^2} \,.$$

A simple estimate shows that

$$\sum_{j \geq 1} e^{-\frac{4\kappa^2 \pi^2 j^2}{v_0^2 H^2}} \leq C \frac{H v_0}{\kappa} \,,$$

and hence

$$(2.7) \qquad \left| \phi(t_0, y) - \frac{\|\phi(0, y)\|_{L^1_y}}{H} \right| \leq C \frac{\|\phi(0, y)\|_{L^1_y}}{l} \,, \qquad t_0 = \frac{\kappa}{v_0^2} \,.$$

Therefore we have

$$\phi(t_0, x, y) \leq \frac{\theta_0}{10}$$

as long as

(2.8) $\|\phi(0, y)\|_{L_y^1} \leq C_1 l \,,$

where $C_1 = \theta_0 H l / (10(l + CH))$. Let us choose h_1 so that (2.8) is automatically verified for initial data supported on D_{2h_1},

$$h_1 \leq C_1 \frac{l}{4}$$

with C_1 as in (2.8). Let us assume that the width of the interval $[a - h, a + h]$ on which $u(y)$ is constant satisfies

$$h \leq \frac{h_1}{4} \,.$$

This condition determines the constant B in the statement of Theorem 1.6. We may now split the initial data for (2.4) as follows:

$$T_0(x, y) \leq \chi_0(y) + \psi_0(x, y) \,.$$

Here the smooth function $\chi_0(y)$ is supported in the interval $[a - 2h_1, a + 2h_1]$, while the function $\psi_0(x, y)$ is supported outside the set $[a - h_1, a + h_1]$. Both of these functions satisfy in addition $0 \leq \chi_0(y), \psi_0(x, y) \leq 1$. Then the function $\Phi(t, x, y)$ satisfies the inequality

$$\Phi(t, x, y) \leq \chi(t, y) + \psi(t, x, y)$$

with the functions χ and ψ satisfying (2.4) with the initial data χ_0 and ψ_0, respectively. It follows from our choice of h_1 that

$$\chi(t_0, y) \leq \frac{\theta_0}{10} \,,$$

so it remains only to estimate $\psi(t_0, x, y)$. We will do it separately for (x, y) inside and outside of the strip $D_{h_1/2}$. For the points $(x, y) \in D_{h_1/2}$ we have

$$\psi(t, x, y) = \int_{\mathbb{R}} dx' \int_0^H dy' \, p_A(t, x - x', y, y') \psi_0(x', y')$$

$$\leq \int_{\mathbb{R}} dx' \int_{|y'-a| \geq h_1} dy' \, p_A(t, x - x', y, y') \leq \mathcal{P}\left\{|W(t)| \geq \frac{h_1}{2}\right\} \leq \frac{\theta_0}{10}$$

for sufficiently small t. Here $W(t)$ is the one-dimensional Brownian motion with diffusivity κ, and $\mathcal{P}$ denotes probability with respect to it. Thus (2.6) holds inside $D_{h_1/2}$.

In order to estimate the function $\psi(t, x, y)$ outside $D_{h_1/2}$, we introduce a profile $\tilde{u}(y)$ that coincides with $u(y)$ outside of the interval $[a - 3h/2, a + 3h/2]$

and satisfies the H-condition on the whole interval $[0, H]$. We define the process $(X(t), Y(t))$ by

$$(2.9) \quad dX(t) = u(Y(t))dt, \quad dY(t) = \sqrt{2\kappa}\, dW(t), \quad X(0) = x, \; Y(0) = y.$$

Consider the stopping time τ, which is the first time when $Y(t)$ enters the interval $[a - 3h/2, a + 3h/2]$. Then we have

$$
\begin{aligned}
\psi(t, x, y) &\leq P_{x,y}\{(X(t), Y(t)) \in \operatorname{supp} \psi_0\} \\
&= P_{x,y}\{(X(t), Y(t)) \in \operatorname{supp} \psi_0 : \tau > t\} P_y(\tau > t) \\
&\quad + P_{x,y}\{(X(t), Y(t)) \in \operatorname{supp} \psi_0 : \tau < t\} P_y(\tau < t) \\
&\leq P_{x,y}\{(\tilde{X}(t), \tilde{Y}(t)) \in \operatorname{supp} \psi_0 : \tau > t\} P_y(\tau > t) + P_y(\tau < t).
\end{aligned}
$$

Here $P_{x,y}$ denotes probability with respect to the process $(X(t), Y(t))$ starting at (x, y), while P_y denotes probability with respect to $Y(t)$ starting at y (recall that $Y(t)$ is independent of x). The process $X(t)$ for $t < \tau$ is identical to the process $(\tilde{X}(t), \tilde{Y}(t))$ defined by (2.9) with $u(Y)$ replaced by $\tilde{u}(Y)$. Therefore we have

$$
\begin{aligned}
\psi(t, x, y) &\leq P_{x,y}\{(\tilde{X}(t), \tilde{Y}(t)) \in \operatorname{supp} \psi_0 : \tau > t\} P_y(\tau > t) + P_y(\tau < t) \\
&\leq P_{x,y}\{(\tilde{X}(t), \tilde{Y}(t)) \in \operatorname{supp} \psi_0\} + P_y(\tau < t).
\end{aligned}
$$

Recall that $(x, y) \notin D_{h_1/2}$ and $h \leq h_1/4$. Therefore the point y is a fixed distance away from the interval $[a - 3h/2, a + 3h/2]$. Hence we may choose $t_1 < t_0$ sufficiently small so that

$$P_y(\tau < t_1) \leq \frac{\theta_0}{10}.$$

Furthermore, the function $\tilde{\psi}(t, x, y) = P_{x,y}\{(\tilde{X}(t), \tilde{Y}(t)) \in \operatorname{supp} \psi_0\}$ satisfies (2.4) with the initial data

$$
\tilde{\phi}(0, x, y) = \begin{cases} 1, & (x, y) \in \operatorname{supp} \psi_0, \\ 0, & (x, y) \notin \operatorname{supp} \psi_0. \end{cases}
$$

However, $\tilde{u}(y)$ is quenching, and thus we may choose A so large that

$$\tilde{\psi}(t_1, x, y) \leq \frac{\theta_0}{10}.$$

Therefore we have at $t = t_1$,

$$\psi(t_1, x, y) \leq \frac{\theta_0}{5},$$

and hence the same upper bound holds at $t = t_0 > t_1$. Therefore (2.6) holds also outside $D_{h_1/2}$, and Theorem 1.6 follows. The fact that $u(y)$ is strongly quenching follows from this property of $\tilde{u}(y)$. $\qquad \square$

Theorem 1.7 is a simple corollary of Theorem 1.3.

PROOF OF THEOREM 1.7: The set of all profiles $u(y)$ satisfying the H-condition is dense in $C[0, H]$, so by Theorem 1.3 the set of strongly quenching profiles is dense. To complete the proof, we will show that if $\tilde{u}(y)$ satisfies the H-condition, then there exists $\delta(\tilde{u})$ such that if

$$\|u(y) - \tilde{u}(y)\|_{C[0,H]} < \delta(\tilde{u}),$$

then $u(y)$ is strongly quenching. From the proof of Theorem 1.3, we know that there exists a constant $C(\tilde{u})$ such that the solution $\tilde{\Psi}(x, y, t)$ of the equation (2.4) with advection $A\tilde{u}(y)$ satisfies

$$\tilde{\Psi}(x, y, t_0) \leq \frac{\theta_0}{e}$$

if the initial data $\tilde{\Psi}(x, y, 0)$ is supported on the interval $[-L, L]$ with $L < C(\tilde{u})A$ (recall that such inequality implies quenching of T with the same initial data). Let $\Psi(x, y, t)$ be a solution of (2.4) with advection $Au(y)$ and initial data supported in $[-L', L']$. Then by the Feynman-Kac formula (see, e.g., [8]),

$$\Psi(x, y, t) \leq P_{x,y}\left(x + A\int_0^t u(y + W(s))ds \in [-L', L']\right)$$

$$\leq P_{x,y}\left(x + A\int_0^t \tilde{u}(y + W(s))ds \in [-L' - A\delta t, L' + A\delta t]\right)$$

$$\leq \tilde{\Psi}(x, y, t),$$

where $\tilde{\Psi}(x, y, t)$ is the solution of (2.4) with advection $Au(y)$ and initial data equal to the characteristic function of the interval $[-L' - A\delta t_0, L' + A\delta t_0]$ for $t \leq t_0$. Now choose $\delta < C(\tilde{u})/t_0$. Then

$$\Psi(x, y, t_0) \leq \tilde{\Psi}(x, y, t_0) \leq \frac{\theta_0}{e}$$

provided that $L' \leq (C(\tilde{u}) - \delta t_0)A$, and hence $u(y)$ is strongly quenching. $\square$

3 Quenching for a System

All results on quenching for equation (1.2) proved in the previous section extend directly to the case of system (1.1). In this respect, the situation is similar to the case $u(y) = 0$, where all results on quenching proved by Kanel' [13] for a single equation extend to the system case. We make an assumption $g(T) \leq T$, which is just a normalization and corresponds to the condition $f(T) \leq T$ in the case of a single equation. We take compactly supported initial data for the temperature, while for the concentration we assume $0 \leq n(x, y, 0) \leq 1$. Notice that by the maximum principle, $n(x, y, t) \leq 1$ for every t. Therefore, T satisfies

$$T_t + u(y)T_x - \kappa\Delta T \leq T,$$

and from this point the analysis proceeds in the same way as for a scalar equation (starting from (2.2)). We summarize the results as follows:

THEOREM 3.1 *Assume that $g(T) = 0$ for $T \leq \theta_0$ and $g(T) \leq T$. Suppose that the velocity profile $u(y) \in C_0^\infty[0, H]$ satisfies the H-condition; then it is strongly quenching. The same conclusion holds if the profile $u(y)$ has a plateau of the size $h \leq h_1 = Bl$ (where B is a universal constant). Moreover, the set of all quenching shear flows contains a dense open set in $C[0, H]$.*

4 Flame Propagation

We prove Theorem 1.4 in this section. Let $T(t, x, y)$ be a solution of (1.2) with the initial data as in (1.7). We will use the following result of Xin [24], which holds for more general types of advection (its version for the shear flow was also proved by Roquejoffre in [19]).

Consider

$$(4.1) \quad T_t + u(x, y) \cdot \nabla T = \kappa \Delta T + \frac{v_0^2}{\kappa} f(T), \quad T(0, x, y) = T_0(x, y) \in L^2(D),$$

with $u(x, y)$ being periodic in both variables and $0 \leq T_0(x, y) \leq 1$.

PROPOSITION 4.1 (Xin) *Assume that the initial data in (4.1) are such that*

$$\lim_{|x| \to \infty} T_0(x, y) = 0 \quad \textit{uniformly in } D \quad \textit{and} \quad T_0(x, y) > \theta_0 + \eta \textit{ for } |x| \leq L.$$

Then there exists $L_1(\eta, u)$ depending on η and $u(x, y)$ such that if $L \geq L_1$, then

$$\lim_{t \to \infty} T(x - st, y, t) = \begin{cases} 1 & \text{if } c_l < s < c_r \\ 0 & \text{if } c_l > s \text{ or } s > c_r. \end{cases}$$

Here c_l and c_r are the speeds of left and right traveling waves, respectively ($c_l < 0$ and $c_r > 0$).

The right traveling wave is described by (1.4) and (1.5), and the left traveling wave satisfies (1.4) and

$$\lim_{s \to -\infty} U(s, y) = 0, \quad \lim_{s \to +\infty} U(s, y) = 1, \quad U_s(s, y) > 0.$$

Notice that Theorems 1.3 and 1.6 imply an estimate $L_1 \geq CA$ if u has no flat parts larger than a certain critical size. On the other hand, Theorem 1.4 shows that $L_1 = h_0$ is independent of A if u has a sufficiently large flat part.

PROOF OF THEOREM 1.4: The proof of Theorem 1.4 proceeds in several steps. We consider the initial data satisfying (1.9). First we find h_0 such that there exists a C^2-function $\phi(x, y)$ such that $0 \leq \phi < \theta_0 + \eta$,

$$(4.2) \qquad\qquad \kappa \Delta \phi + \frac{v_0^2}{\kappa} f(\phi) \geq 0,$$

and ϕ vanishes on the boundary of the disc of radius h_0 centered at the point $(0, a)$,

$$\phi|_{\partial\Omega_0} = 0, \quad \Omega_0 = B((0, a); h_0).$$

Then in the system of coordinates that moves with the speed $\bar{u}$, the function $\phi(x, y)$ is a subsolution of (1.2) in Ω_0. Therefore, initial data that start above ϕ will not decay to zero. Next we consider the special solution $\Phi(t, x, y)$ of (1.2) with the initial data given by

$$(4.3) \qquad \Phi_0(x, y) \begin{cases} \phi(x, y) & (x, y) \in \Omega_0 \\ 0 & (x, y) \notin \Omega_0. \end{cases}$$

We show that $\Phi(t, x, y)$ satisfies (1.16), which implies that (1.16) holds for arbitrary initial data $T_0 \geq \Phi_0$, in particular, for such as described in Theorem 1.4.

Step 1: Construction of a Stationary Subsolution

Choose θ_1, θ_2 so that $\theta_0 + \eta > \theta_2 > \theta_1 > \theta_0$, and define $f_1(T)$ by

$$f_1(T) = \begin{cases} 0, & T \leq \theta_1, \\ \frac{f(\theta_2)(T-\theta_1)}{\theta_2 - \theta_1}, & \theta_1 \leq T \leq \theta_2, \\ f(T), & \theta_2 \leq T \leq 1. \end{cases}$$

The function $f(T)$ is Lipschitz-continuous, and hence we may choose θ_1 and θ_2 so that $f_1(T) \leq f(T)$. Therefore, if ϕ satisfies

$$(4.4) \qquad \kappa\Delta\phi + \frac{v_0^2}{\kappa} f_1(\phi) = 0,$$

then ϕ satisfies (4.2). We are going to construct an explicit radial solution $\phi(r)$ of (4.4) with the "initial" conditions

$$\phi(0) = \theta_2, \quad \frac{\partial\phi}{\partial r}(0) = 0.$$

Indeed, $\phi(r)$ is given explicitly by
(4.5)

$$\phi(r) = \theta_1 + (\theta_2 - \theta_1) J_0\left(\frac{r v_0 \sqrt{\alpha}}{\kappa}\right), \quad \alpha = \frac{f(\theta_2)}{\theta_2 - \theta_1}, \quad \text{for } r \leq R_1 = \frac{\kappa \xi_1}{v_0 \sqrt{\alpha}}.$$

Here $J_0(\xi)$ is the Bessel function of order zero, and ξ_1 is its first zero. Furthermore, we have

$$(4.6) \qquad \phi(r) = B \ln\frac{r}{R} \quad \text{for } R_1 \leq r$$

with B and R determined by matching (4.5) and (4.6) at $r = R_1$. Then we get

$$R = l\xi_1 \sqrt{\frac{\theta_2 - \theta_1}{f(\theta_2)}} \exp\left[\frac{\theta_1}{(\theta_2 - \theta_1)\xi_1 |J_0'(\xi_1)|}\right] = Cl, \quad l = \frac{\kappa}{v_0}.$$

Then $\phi(r)$ satisfies

$$\kappa \Delta \phi + \frac{v_0^2}{\kappa} f(\phi) \geq 0, \quad \phi(r) > 0, \; 0 \leq r < R, \; \phi(R) = 0.$$

Thus we will take the critical size of the plateau in the velocity profile to be $2R$ so that the disc of radius R will fit in.

Step 2: A Subsolution

Let us now assume that $h \geq h_0 = R$. We make a coordinate change

$$\xi = x - \bar{u}t.$$

In the new coordinates we have a function $T(t, \xi, y)$ that solves

$$(4.7) \qquad T_t + A(-\bar{u} + u(y))T_\xi = \kappa \Delta_{\xi,y}T + \frac{v_0^2}{\kappa} f(T)$$

with the initial data given by (4.3),

$$T(0, \xi, y) = \begin{cases} \phi(\xi, y), & (\xi, y) \in \Omega_0 = B((0, a); h_0) \\ 0, & (\xi, y) \notin \Omega_0. \end{cases}$$

Observe that $\phi(\xi, y)$ satisfies (4.2) inside Ω_0 since $u(y) = \bar{u}$ in Ω_0. Moreover, $T(t, \xi, y) \geq \phi(\xi, y)$ on $\partial \Omega_0$, where ϕ vanishes. Therefore the maximum principle implies that inside Ω_0 we have

$$(4.8) \qquad T(t, \xi, y) \geq \phi(\xi, y) \quad \text{for all } t > 0 \text{ and } (\xi, y) \in \Omega_0.$$

Note that $T_h(t, \xi, y) = T(t + h, \xi, y)$ solves (4.7) with the initial data

$$(4.9) \qquad T_h(0, \xi, y) = T(h, \xi, y) \geq T(0, \xi, y).$$

The inequality in (4.9) follows from (4.8) inside Ω_0 and the fact that $T(t, x, y) \geq 0$ outside Ω_0. Therefore we have

$$T(t + h, \xi, y) - T(t, \xi, y) \geq 0 \quad \text{for all } h > 0, t > 0, (\xi, y) \in D,$$

and thus the limit

$$\bar{T}(\xi, y) = \lim_{t \to \infty} T(t, \xi, y)$$

exists since $T \leq 1$. Moreover, the standard parabolic regularity implies that $T(t, \xi, y)$ converges to $\bar{T}(\xi, y)$ uniformly on compact sets together with its derivatives up to the second order. Therefore $\bar{T}$ satisfies the stationary problem

$$(4.10) \qquad A(-\bar{u} + u(y))\bar{T}_\xi = \kappa \Delta_{\xi,y}\bar{T} + \frac{v_0^2}{\kappa} f(\bar{T}).$$

We also have

$$\bar{T}(\xi, y) > \phi(\xi, y) \quad \text{for } (\xi, y) \in \Omega_0.$$

It is easy to show using the sliding method of Berestycki and Nirenberg [4] that for any $(\xi, y) \in D_h$

$$(4.11) \qquad \bar{T}(\xi, y) > \phi(\xi - r, y)$$

where the right side is any translation of ϕ along the ξ-axis.

Indeed, assume that there exists the smallest (positive) r such that $\bar{T}(\xi_0, y_0) = \phi(\xi_0 - r, y_0)$ at some point $(\xi_0, y_0) \in D_h$. Then the strong maximum principle implies that $\bar{T}(\xi, y) = \phi(\xi - r, y)$ for all (ξ, y) inside the translate $\Omega_r = \Omega_0 - re_1$ of the disc Ω_0. But that contradicts the fact that $\bar{T}(\xi, y) > \phi(\xi - r, y) = 0$ on the boundary $\partial\Omega_r$. Then (4.11) implies that

$$(4.12) \qquad \bar{T}(\xi, a) > \theta_2 = \sup_{(\xi, y)\in\Omega_0} \phi(\xi, y) \quad \text{for all } \xi \in \mathbb{R}.$$

The next two lemmas show that (4.10) and (4.12) imply that $\bar{T}(\xi, y) \equiv 1$.

LEMMA 4.2 *Let $\bar{T}$ be a solution of (4.10) such that $0 \leq \bar{T} \leq 1$. Then we have*

$$(4.13) \qquad \int_D d\xi\, dy\, f(\bar{T}(x, y)) < \infty, \quad \int_D d\xi\, dy\, |\nabla\bar{T}|^2 < \infty.$$

PROOF: In order to show that the integral of $f(\bar{T})$ is finite, we integrate (4.10) over the set $(-L + \zeta, L + \zeta) \times [0, H]$ with L large and $\zeta \in [0, l]$. We get

$$A \int_0^H dy[-\bar{u} + u(y)][\bar{T}(L + \zeta, y) - \bar{T}(-L + \zeta, y)] =$$
$$\kappa \int_0^H dy[\bar{T}_\xi(L + \zeta, y) - \bar{T}_\xi(-L + \zeta, y)] + \frac{v_0^2}{\kappa} \int_0^H dy \int_{-L+\zeta}^{L+\zeta} d\xi f(\bar{T}(\xi, y))$$

and average this equation in $\zeta \in [0, l]$:

$$\frac{A}{l} \int_0^l d\zeta \int_0^H dy[-\bar{u} + u(y)][\bar{T}(L + \zeta, y) - \bar{T}(-L + \zeta, y)]$$
$$= \frac{\kappa}{l} \int_0^H dy[\bar{T}(L + l, y) - \bar{T}(L, y) - \bar{T}(-L + l, y) + \bar{T}(-L, y)]$$
$$+ \frac{v_0^2}{l\kappa} \int_0^l d\zeta \int_0^H dy \int_{-L+\zeta}^{L+\zeta} d\xi f(\bar{T}(\xi, y)).$$

Therefore we obtain

$$\frac{v_0^2}{\kappa} \int_0^H dy \int_{-L+l}^L d\xi f(\bar{T}(\xi, y)) \leq \frac{4\kappa H}{l} + 2HA[\bar{u} + \|u\|_\infty]$$

for all L, and hence the first inequality in (4.13) holds. In order to obtain the second inequality we multiply (4.10) by $\bar{T}$ and perform the same integration and averaging

as above. This leads to

$$\frac{A}{2l} \int_0^l d\zeta \int_0^H dy [-\bar{u} + u(y)] [\bar{T}^2(L + \zeta, y) - \bar{T}^2(-L + \zeta, y)] =$$

(4.14)
$$\frac{\kappa}{2l} \int_0^H dy [\bar{T}^2(L + l, y) - \bar{T}^2(L, y) - \bar{T}^2(-L + l, y) + \bar{T}^2(-L, y)]$$

$$+ \frac{1}{l} \int_0^l d\zeta \int_0^H dy \int_{-L+\zeta}^{L+\zeta} d\xi \left[\frac{v_0^2}{\kappa} \bar{T} f(\bar{T}) - \kappa |\nabla \bar{T}|^2 \right],$$

and then the second inequality in (4.13) follows from (4.14) and the first inequality in (4.13). □

LEMMA 4.3 *The limit function* $\bar{T}(\xi, y) \equiv 1$.

PROOF: Notice that $\bar{T}$ cannot achieve local minima in D as follows from the maximum principle. Therefore if we define

$$\mu_{\alpha, \beta} = \min_{\alpha \leq \xi \leq \beta, 0 \leq y \leq H} \bar{T}(\xi, y) \quad \text{and} \quad \mu(\alpha) = \min_{0 \leq y \leq H} \bar{T}(\alpha, y),$$

then $\mu_{\alpha, \beta} = \mu(\alpha)$ or $\mu_{\alpha, \beta} = \mu(\beta)$. Furthermore, if $\bar{T}(\xi, y) = 1$ at some point, then $\bar{T}(\xi, y) \equiv 1$ everywhere by the strong maximum principle. In particular, if $\mu(0) = 1$, then $\bar{T}(\xi, y) \equiv 1$. Therefore we consider only the case that $\mu(0) < 1$ and argue by contradiction. We have either $\mu(\xi) < \mu(0)$ for any $\xi > 0$ or for any $\xi < 0$. Otherwise the minimum of $\bar{T}$ over the set $[-\xi, \xi] \times [0, H]$ would be achieved inside. Let us assume without loss of generality that $\mu(\xi) < \mu(0)$ for any $\xi > 0$. Consider

$$\delta = \min \left(\frac{1 - \mu(0)}{2}, \frac{\theta_2 - \theta_0}{2} \right).$$

For any $\xi > 0$, we have three options:

(1) $\bar{T}(\xi, y) \in [\mu(\xi), \mu(\xi) + \delta]$ for any $y \in [0, H]$, and $\mu(\xi) \geq \theta_0 + \delta$. In this case, by the definition of δ, properties (1.3) of f, and the fact that $\mu(\xi) \leq \mu(0) < 1$, we have

$$\int_0^H f(\bar{T}(\xi, y)) dy \geq CH.$$

(2) $\mu(\xi) < \theta_0 + \delta$. Inequality (4.12) implies that there exist y_1 and y_2 such that $\bar{T}(\xi, y_1) = \theta_2 - \delta$ and $\bar{T}(\xi, y_2) = \theta_2$. Then

$$\left| \int_{y_1}^{y_2} f(\bar{T}(\xi, y)) dy \right| \geq C|y_2 - y_1|,$$

$$\int_{y_1}^{y_2} |\bar{T}_y|^2 dy \geq |y_2 - y_1|^{-1} \left(\int_{y_1}^{y_2} |\bar{T}_y| dy \right)^2 \geq \delta^2 |y_2 - y_1|^{-1}.$$

Therefore,

(4.15)
$$\int_H^0 f(\bar{T}(\xi, y))dy \int_{y_1}^{y_2} |\bar{T}_y|^2 dy \geq C\delta^2$$

in this case.

(3) $\mu(\xi) \geq \theta_0 + \delta$ and there exists $y \in [0, H]$ such that $\bar{T}(\xi, y) > \mu(\xi) + \delta$. In this case, an argument identical to the reasoning of option (2) leads to the same bound (4.15).

Overall, we see that for any $\xi > 0$,

$$\int_0^H \left(f(\bar{T}(\xi, y) + |\nabla \bar{T}|^2(\xi, y)) \right) dy \geq C ,$$

where C depends only on H and δ. But this contradicts Lemma 4.2. □

Using Proposition 4.1, we can now complete the proof of Theorem 1.4. Notice that even though we showed $\bar{T}(\xi, y) \equiv 1$, we still have to show that the limiting function is the same in the original coordinates. Lemma 4.3 implies that there exists a time t_0 so that for all $\xi \in [0, L_0(\theta_2 - \theta_0, Au(y))]$ and all $y \in [0, H]$, we have

$$T(t_0, \xi, y) \geq \theta_0 + \eta .$$

Here $L_0(\eta, v)$ is defined by Proposition 4.1. Then we may apply Proposition 4.1 with $s = 0$ in the original coordinates (x, y) and initial data $T(t_0, x, y)$ and get (1.16). The fact that $c_l < 0 < c_r$ follows from (1.8) and, for instance, results of [2]. □

The subsolution we constructed in Theorem 1.4 is also useful for a proof of Theorem 1.5. As mentioned in the introduction, we only need to show the lower bound; the upper bound is contained in [7]. We begin the proof with two auxiliary lemmas.

LEMMA 4.4 *Let $u(y)$ be a function that is constant on an interval,*

$$u(y) = \bar{u} , \quad y \in [a - h, a + h] ,$$

that may or may not satisfy the mean zero condition (1.8). *Assume that $h \geq h_0$ with h_0 given in Theorem* 1.4. *Then we have $c_l(u) \leq \bar{u} \leq c_r(u)$ (recall that $c_l(u)$ and $c_r(u)$ are the velocities of unique left and right traveling fronts).*

PROOF: Let T_0 be an initial data as in Theorem 1.4 such that

$$T_0(x, y) \geq \begin{cases} \phi(x, y) , & (x, y) \in \Omega_0 = B((0, a); h_0) , \\ 0 , & (\xi, y) \notin \Omega_0 , \end{cases}$$

with the function $\phi(x, y)$ constructed in the proof of Theorem 1.4 and given explicitly by (4.5) and (4.6). Then we have

$$T(t, x - \bar{u}t, y) \geq \phi(x, y) ,$$

and therefore Proposition 4.1 implies that $c_l \leq \bar{u} \leq c_r$ because $T(t, x - \bar{u}t, y)$ may not go to zero as $t \to \infty$. $\qquad\square$

Let y_0 be a point at which $|u_+|_{h_0}$ is achieved: $u(y) \geq |u_+|_{h_0}$ for $y \in [y_0 - h_0, y_0 + h_0]$. We now define a new velocity field

(4.16)
$$v(y) = \begin{cases} u(y), & y \notin [y_0 - h_0, y_0 + h_0], \\ |u_+|_{h_0}, & y \in [y_0 - h_0, y_0 + h_0], \end{cases}$$

so that $u(y) \geq v(y)$.

LEMMA 4.5 *Let $u(y) \geq v(y)$ be two velocity fields. Then the corresponding right traveling front speeds satisfy $c_r(u) \geq c_r(v)$. Similarly, if $u(y) \leq v(y)$, then $c_l(u) \leq c_l(v)$.*

PROOF: Let $T^{u,v}(t, x, y)$ be solutions of

$$T_t^u + Au(y)T_x^u = \kappa \Delta T^u + \frac{v_0^2}{\kappa} f(T^u)$$

and

$$T_t^v + Av(y)T_x^v = \kappa \Delta T^v + \frac{v_0^2}{\kappa} f(T^v)$$

with the same initial data

$$T^u(0, x, y) = T^v(0, x, y) = T_0(x, y).$$

The function $T_0(x, y)$ is assumed to be frontlike and monotonic. That is,

$$T_0(x, y) = 1 \text{ for } x \leq -L, \quad T_0(x, y) = 0 \text{ for } x \geq L, \quad \text{and} \quad \frac{\partial T_0}{\partial x}(x, y) \leq 0.$$

Then by the maximum principle applied to the equation for $\partial T^{u,v}/\partial x$ we have

$$\frac{\partial T^{u,v}}{\partial x}(t, x, y) \leq 0 \quad \text{for all } t > 0$$

and hence

$$T_t^v + Au(y)T_x^v - \kappa \Delta T^v - \frac{v_0^2}{\kappa} f(T^v) = A(u(y) - v(y))T_x^v \leq 0.$$

Therefore by the maximum principle we have

(4.17)
$$T^u(t, x, y) \geq T^v(t, x, y).$$

However, by results of [24, 19] similar to Proposition 4.1 but for the frontlike initial data, we have

$$\lim_{t \to \infty} T^v(x - st, y, t) = \begin{cases} 1 & \text{if } s < c_r(v) \\ 0 & \text{if } s > c_r(v), \end{cases}$$

and similarly for T^u. Then $c_r(v) > c_r(u)$ would be incompatible with (4.17), and hence $c_r(v) \leq c_r(u)$. Naturally, an analogous result is true for left traveling fronts. $\qquad\square$

PROOF OF THEOREM 1.5: In order to finish the proof of Theorem 1.5, we apply Lemma 4.5 to $v(y)$ given by (4.16) and observe that $c_r(v)$ satisfies the estimate in Theorem 1.5 according to Lemma 4.4. $\qquad\square$

Finally, we remark that in the situation of Theorem 1.4, strong advection not only does not quench the flame but has the directly opposite effect, according to Theorem 1.5 and Proposition 4.1.

COROLLARY 4.6 *Let $u(y) = \bar{u} = \text{const}$ for $y \in [a-h, a+h]$ for some $a \in [0, H]$, and h be larger than or equal to the critical size h_0. Then, provided that the size of initial data $L \geq h_0$, we have*

$$\lim_{t\to\infty} T(x - st, y, t) = \begin{cases} 1 & \text{if } c_l < s < c_r \\ 0 & \text{if } c_l > s \text{ or } s > c_r. \end{cases}$$

Moreover, $c_r \geq |u_+|_{h_0}$ and $c_l \leq -|u_-|_{h_0}$.

PROOF: The first statement is a direct corollary of Proposition 4.1 and Theorem 1.4. The second statement is the content of Theorem 1.5. $\qquad\square$

5 A Lower Bound for the Quenching Size

Recall that the burning rate is defined by

$$V(t) = \int_D T_t(x, y) \frac{dx\,dy}{H}.$$

We also say that nonlinearity $f(T)$ is of concave KPP class if

$$f(0) = f(1) = 0, \quad f(T) > 0, \quad \text{for } 0 < T < 1, \ f''(T) < 0.$$

We have previously shown [7] that for such nonlinearities and for frontlike initial conditions, the burning rate in the presence of any advection is bounded from below by Cv_0; more precisely,

$$(5.1) \qquad\qquad V(t) \geq Cv_0\left(1 - e^{-Cv_0^2 t/\kappa}\right).$$

The physical meaning of (5.1) is that no advection may slow up the burning significantly. We now show that similarly there is a fixed size of initial data that is quenched by any shear flow. That means that no shear flow may help prevent quenching of initial data with a fixed small support. Let $T(t, x, y)$ be a solution of

$$(5.2) \qquad\qquad T_t + u(y)T_x = \kappa\Delta T + \frac{v_0^2}{\kappa}f(T)$$

with the initial data $T(0, x, y) = T_0(x, y)$ and periodic boundary conditions (1.6).

PROPOSITION 5.1 *There exists a constant $C > 0$ that depends only on the nonlinearity $f(T)$ such that given any initial data T_0 with*

$$\int_D T_0(x, y)dx\, dy \le Cl^2, \quad l = \frac{\kappa}{v_0},$$

we have

(5.3) $\qquad\qquad T(t, x, y) \to 0 \quad$ *uniformly in D as $t \to +\infty$*

for all $u(y) \in C[0, H]$.

PROOF: We will show that solution of (5.2) with $f(T)$ replaced by a KPP nonlinearity $\tilde{f}(T) = MT(1 - T)$ and the same initial data drops below θ_0 before time $t_0 = \kappa/v_0^2$. That will imply (5.3). Choose M such that $f(T) \le \tilde{f}(T) = MT(1 - T)$ (such M exists since f is Lipschitz-continuous) and let $\tilde{T}$ be the solution of

$$\tilde{T}_t + u(y)\tilde{T}_x = \kappa \Delta \tilde{T} + \frac{v_0^2}{\kappa} \tilde{f}(\tilde{T})$$

with initial data T_0 and periodic boundary conditions (1.6). Let us also define

$$\tilde{V}(t) = \int_D \tilde{T}_t(x, y)\frac{dx\, dy}{H} = \frac{Mv_0^2}{\kappa} \int_D \tilde{T}(1 - \tilde{T})\frac{dx\, dy}{H}.$$

Note that $\tilde{T}(t, x, y) \ge T(t, x, y)$ and hence Proposition 5.1 follows from the following lemma:

LEMMA 5.2 *There exists a constant $C > 0$ that depends only on the nonlinearity $f(T)$ such that given any initial data T_0 with*

$$\int_D T_0(x, y)dx\, dy \le Cl^2, \quad l = \frac{\kappa}{v_0},$$

and any $u(x, y) \in C^1(D)$, there exists a time $t_1 < t_0$ such that $\tilde{T}(t, x, y) \le \theta_0$ for all $(x, y) \in D$.

PROOF: The following inequality holds for $\tilde{V}(t)$:

$$\tilde{V}(t) + \frac{\kappa}{Mv_0^2} \frac{d\tilde{V}(t)}{dt} \ge 2\kappa \int_D |\nabla \tilde{T}|^2 \frac{dx\, dy}{H}.$$

We define the set

$$\mathcal{S}(t) = \left\{ y \in [0, H] : \exists x \in \mathbb{R} \text{ such that } T(t, x, y) \ge \theta_0/e^{2M} \right\} \subset [0, H].$$

Observe that there exists a constant $\delta > 0$ that depends only on M and θ_0 such that if the Lebesgue measure $|\mathcal{S}(\tau)| \le \delta l$ at some time $\tau \in [0, t_0/2]$, then $\|\tilde{T}(t_0)\|_{L^\infty} \le \theta_0$. Indeed, we then have

$$\tilde{T}(\tau, x, y) \le \chi(y) + \psi(x, y), \quad \|\chi(y)\|_{L^1} \le \delta l, \quad \psi(x, y) \le \theta_0/e^{2M}.$$

Therefore we have with C as in (2.7)

$$T(t_0, x, y) \le e^M \left[(C + 1)\delta + \frac{\theta_0}{e^{2M}} \right] \le \theta_0$$

for a sufficiently small δ. Hence it suffices to consider the case when $|\mathcal{S}(t)| \ge \delta l$ for all $t \in [0, t_0/2]$. We claim that then at any time $t \in [0, t_0/2]$ we have

$$(5.4) \qquad \int_D |\nabla \tilde{T}|^2 dx\, dy \int_D \tilde{f}(\tilde{T}) dx\, dy \ge Cl^2$$

with the constant C depending only on M and θ_0. This may be proved similarly to lemma 2 in [7]. Therefore we obtain

$$\tilde{V}(t) + \frac{\kappa}{Mv_0^2} \frac{d\tilde{V}(t)}{dt} \ge \frac{2Cv_0^2 l^2}{H^2 \tilde{V}(t)} .$$

Thus we have

$$\tilde{V}(t) \ge \frac{Cv_0 l}{H} \quad \text{for } \frac{t_0}{2} \le t \le t_0 .$$

Therefore

$$\int_D \tilde{T}(t_0, x, y) \ge \int_D T_0(x, y) dx\, dy + C\kappa t_0 .$$

However, we have an a priori bound

$$\int_D \tilde{T}(t_0, x, y) dx\, dy \le e^M \int_D T_0(x, y) dx\, dy ,$$

which contradicts the previous inequality if

$$(5.5) \qquad \int_D T_0(x, y) dx\, dy \le \frac{C\kappa t_0}{e^M - 1} = C_M l^2 .$$

Therefore $T(t, x, y)$ has to drop below θ_0 if the initial data satisfies (5.5), and Lemma 5.2 is proved. $\square$

Proposition 5.1 then follows from Lemma 5.2. $\square$

Remark. The uniform quenching size in Proposition 5.1 is optimal. Indeed, the subsolution we have constructed in Section 4 for shear flows with a flat part has L^1-norm Cl^2, and thus one cannot expect that initial data with L^1-norm larger than Cl^2 will be quenched by all shear flows.

Bibliography

[1] Audoly, B.; Berestycki, H.; Pomeau, Y. Rèaction diffusion en ècoulement stationnaire rapide. *C. R. Acad. Sci. Ser. IIB* **328** (2000), 255–262.

[2] Berestycki, H.; Larrouturou, B.; Lions, P.-L. Multi-dimensional traveling-wave solutions of a flame propagation model. *Arch. Rational Mech. Anal.* **111** (1990), no. 1, 33–49.

[3] Berestycki, H.; Larrouturou, B.; Roquejoffre, J.-M. Stability of traveling fronts in a model for flame propagation. I. Linear analysis. *Arch. Rational Mech. Anal.* **117** (1992), no. 2, 97–117.

[4] Berestycki, H.; Nirenberg, L. Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains. *Analysis, et cetera*, 115–164. Academic Press, Boston, 1990.

[5] Berestycki, H.; Nirenberg, L. Traveling fronts in cylinders. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **9** (1992), no. 5, 497–572.

[6] Clavin, P.; Williams, F. A. Theory of pre-mixed flame propagation in large-scale turbulence. *J. Fluid. Mech.* **90** (1979), 589–604.

[7] Constantin, P.; Kiselev, A.; Oberman, A.; Ryzhik, L. Bulk burning rate in passive-reactive diffusion. *Arch. Rational Mech. Anal.* **154** (2000), no. 1, 53–91.

[8] Freidlin, F. *Functional integration and partial differential equations.* Annals of Mathematics Studies, 109. Princeton University Press, Princeton, N.J., 1985.

[9] Hamel, F. Formules min-max pour les vitesses d'ondes progressives multidimensionnelles. *Ann. Fac. Sci. Toulouse Math. (6)* **8** (1999), no. 2, 259–280.

[10] Heinze, S.; Papanicolaou, G.; Stevens, A. Variational principles for propagation speeds in inhomogeneous media. *SIAM J. Appl. Math. A*, in press.

[11] Hörmander, L. Hypoelliptic second order differential equations. *Acta Math.* **119** (1967), 147–171.

[12] Ichihara, K.; Kunita, H. A classification of the second order degenerate elliptic operators and its probabilistic characterization. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **30** (1974), 235–254.

[13] Kanel′, Ja. I. Stabilization of solutions of the Cauchy problem for equations encountered in combustion theory. *Mat. Sb. (N.S.)* **59 (101)** (1962) suppl., 245–288.

[14] Kiselev, A.; Ryzhik, L. Enhancement of the traveling front speeds in reaction-diffusion equations with advection. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, in press.

[15] Kolmogorov, A. N. Zufällige Bewegungen. *Ann. of Math. (2)* **35** (1934), 116–117.

[16] Majda, A. J.; Souganidis, P. Large-scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales. *Nonlinearity* **7** (1994), no. 1, 1–30.

[17] Mallordy, J.-F.; Roquejoffre, J.-M. A parabolic equation of the KPP type in higher dimensions. *SIAM J. Math. Anal.* **26** (1995), no. 1, 1–20.

[18] Roquejoffre, J.-M. Stability of traveling fronts in a model for flame propagation. II. Nonlinear stability. *Arch. Rational Mech. Anal.* **117** (1992), no. 2, 119–153.

[19] Roquejoffre, J.-M. Eventual monotonicity and convergence to traveling fronts for the solutions of parabolic equations in cylinders. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), no. 4, 499–552.

[20] Volpert, A. I.; Volpert, V. A.; Volpert, V. A. *Traveling wave solutions of parabolic systems.* Translations of Mathematical Monographs, 140. American Mathematical Society, Providence, R.I., 1994.

[21] Volpert, V. A.; Volpert, A. I. Location of spectrum and stability of solutions for monotone parabolic system. *Adv. Differential Equations* **2** (1997), no. 5, 811–830.

[22] Volpert, V. A.; Volpert, A. I. Existence and stability of multidimensional traveling waves in the monostable case. *Israel J. Math.* **110** (1999), 269–292.

[23] Volpert, V. A.; Volpert, A. I. Spectrum of elliptic operators and stability of traveling waves. *Asymptot. Anal.* **23** (2000), no. 2, 111–134.

[24] Xin, J. Existence and nonexistence of traveling waves and reaction-diffusion front propagation in periodic media. *J. Statist. Phys.* **73** (1993), no. 5-6, 893–926.

[25] Xin, J. Front propagation in heterogeneous media. *SIAM Rev.* **42** (2000), no. 2, 161–230.

PETER CONSTANTIN
University of Chicago
Department of Mathematics
5734 S. University Avenue
Chicago, IL 60637
E-mail: `const@`
    `math.uchicago.edu`

ALEXANDER KISELEV
University of Chicago
Department of Mathematics
5734 S. University Avenue
Chicago, IL 60637
E-mail: `kiselev@`
    `math.uchicago.edu`

LEONID RYZHIK
University of Chicago
Department of Mathematics
5734 S. University Avenue
Chicago, IL 60637
E-mail: `ryzhik@`
    `math.uchicago.edu`