Differential Equations and Mathematical Physics
Proceedings of an International Conference held at the
University of Alabama at Birmingham
March 16 – 20, 1999
Pages: 61 – 72

Approximate Eigenvectors and Spectral Theory

M. Christ, A. Kiselev, and Y. Last

Abstract. We develop relations between spectral and eigenfunction properties of self-adjoint operators and properties of approximate eigenvectors of those operators. In particular, we establish a new general criterion for studying continuity properties of spectral measures. It can be viewed as a generalization of the Weyl criterion, and provides information on pointwise continuity properties of spectral measures (particularly, on positivity of upper α-derivatives). The criterion is formulated as a necessary and sufficient condition involving certain sequences of approximate eigenvectors. We also show that appropriately chosen sequences of approximate eigenvectors converge to generalized eigenfunctions in an appropriate weak sense. We apply these results to study spectral properties of some concrete Schrödinger operators.

1. Introduction and main results

Let A be a self-adjoint operator acting on a separable Hilbert space $\mathcal{H}$. Our main goal in this paper is to develop relations between spectral properties of A and sequences of corresponding approximate eigenvectors. It is well known (see Theorem VII.12 of [**24**]) that a real number E is in the spectrum of A if and only if there exists a sequence $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{H}$, with $\|\psi_n\| = 1 \quad \forall n$, such that $\|(A-E)\psi_n\| \to 0$ as $n \to \infty$. Moreover, E is in the essential spectrum of A if and only if the above ψ_n's can be chosen so that they converge weakly to 0. (This is often called the Weyl criterion.) We are interested here in extending the framework of such relations, and in particular, in being able to determine continuity properties of spectral measures from the nature of such sequences of approximate eigenvectors. One of our primary motivations for developing such relations is that they yield a new method for spectral analysis of self-adjoint operators. Our aim in this paper is mainly to describe our results. While we provide full proofs for some of them (including our core results), the complete development of the full-blown theory (as well as some of its applications) will be given in a future publication.

1991 *Mathematics Subject Classification.* Primary: 35J10; 81Q10 Secondary: 35P05.
The first author was supported in part by NSF grant DMS-9970660.
The second author was supported in part by NSF grant DMS-9801530.
The third author was supported in part by NSF grant DMS-9801474 and by an Allon fellowship.

Much of the recent research in spectral theory has focused on systems with complicated spectral properties and low degree of symmetry, such as Schrödinger operators with random, almost periodic, quasiperiodic, and other long-range potentials. When we are interested in the fine structure of the essential spectrum of Schrödinger operators for which the methods of scattering theory are not applicable, there are very limited tools, especially in higher dimensions, which may be effectively used for spectral analysis. On the other hand, for one-dimensional Schrödinger operators the subordinacy theory created by Gilbert and Pearson [**10, 11**] and further extended recently by Jitomirskaya and Last [**15, 16, 17**] provides a new powerful method for spectral analysis. The main results of the above mentioned papers give a necessary and sufficient link between the behavior of solutions of the time-independent Schrödinger equation and the singularity of the spectral measures. Subordinacy theory played an important role in many recent results in one-dimensional spectral theory [**5, 6, 14, 15, 16, 17, 18, 19, 21, 22, 25, 30**]. In this paper, we establish a very general criterion for studying spectral continuity properties in terms of sequences of approximate eigenvectors—something that we can study, in principle, for any Schrödinger operator. We hope that this criterion will be useful in spectral analysis, and applicable in higher-dimensional problems as well as in one dimension.

Fix a vector $\phi \in \mathcal{H}$, and let $\langle \cdot , \cdot \rangle$ denote the scalar product in $\mathcal{H}$. By elementary principles, there exists a unique real regular Borel measure μ^ϕ on $\mathbb{R}$, such that $\langle f(A)\phi, \phi\rangle = \int f(E) d\mu^\phi(E)$ for any bounded complex valued Borel function f. μ^ϕ is called the spectral measure for A and ϕ. Spectral measures are of great importance both in the general theory of self-adjoint operators and in its applications. For example, one can learn a great deal about dynamical properties of quantum systems by looking at properties of spectral measures of the corresponding Hamiltonians (see, e.g., [**2, 12, 13, 23**]). One relevant piece of information is that of α-continuity (or singularity) of the spectrum, which is directly related to positivity of the upper α-derivative, D^α_μ, on the support of the spectral measure (see [**23**] for more details). Given $E \in \mathbb{R}$ and $\epsilon > 0$, we denote by $I_\epsilon(E)$ the open interval of size 2ϵ around E, namely, $I_\epsilon(E) \equiv (E-\epsilon, E+\epsilon)$. Recall that the upper α-derivative of a Borel measure μ is defined by

$$D^\alpha_\mu(E) = \limsup_{\epsilon \to 0} \frac{\mu(I_\epsilon(E))}{(2\epsilon)^\alpha} \tag{1.1}$$

(where we are interested only in $\alpha \in [0,1]$). Its characteristics can also be deduced from the local scaling behavior of the Borel transform of μ, which is defined by $\mathcal{M}_\mu(z) = \int (x-z)^{-1} \, d\mu(x)$. As shown in [**8**], we have $D^\alpha_\mu(E) = \infty$ if and only if $\limsup_{\epsilon\to 0} \epsilon^{1-\alpha} \mathrm{Im}\, \mathcal{M}_\mu(E+i\epsilon) = \infty$, and similarly $D^\alpha_\mu(E) = 0$ if and only if $\limsup_{\epsilon\to 0} \epsilon^{1-\alpha} \mathrm{Im}\, \mathcal{M}_\mu(E+i\epsilon) = 0$. Borel transforms of spectral measures arise very often in spectral theory, since they coincide with resolvent matrix elements. Let $\mathcal{M}_\phi$ denote $\mathcal{M}_\mu$ for $\mu = \mu^\phi$, then by the spectral theorem [**24**], we have $\mathcal{M}_\phi(z) = \langle (A-z)^{-1}\phi, \phi\rangle$.

Our first (and, in a sense, main) result in this paper shows that for energies E in the essential support of the spectral measure μ^ϕ, $\mathrm{Im}\, \mathcal{M}_\phi(E+i\epsilon)$ can be essentially determined by supremizing the scalar product $\langle \psi, \phi\rangle$ using vectors ψ that are approximate eigenvectors "up to an error ϵ."

THEOREM 1.1. *Let E be such that $\int (x-E)^{-4}\, d\mu^{\phi}(x) = \infty$, then for any $0 < \epsilon < \|(A-E)\phi\|/\|\phi\|$,*

$$\begin{aligned}(1.2)\quad \epsilon \operatorname{Im} \mathcal{M}_{\phi}(E+i\epsilon) \;&<\; \sup\{|\langle\psi,\phi\rangle|^2 \mid \|\psi\|=1, \|(A-E)\psi\|=\epsilon\} \\ &\le \sup\{|\langle\psi,\phi\rangle|^2 \mid \|\psi\|=1, \|(A-E)\psi\|\le\epsilon\} \;\le\; 2\epsilon \operatorname{Im} \mathcal{M}_{\phi}(E+i\epsilon).\end{aligned}$$

Remarks. 1. The middle inequality is, of course, trivial.

2. The theorem is empty if ϕ happens to be an eigenvector of A with eigenvalue E (in which case $\|(A-E)\phi\| = 0$). Throughout the rest of this paper we often assume that this is not the case, and we do not bother saying so over and over again. (This degenerate case is easy to treat separately.)

DEFINITION 1.2. Let $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{H}$ with $\|\psi_n\| = 1\ \forall n$. We say that $\{\psi_n\}_{n=1}^{\infty}$ is a SOAEV (which stands for Sequence Of Approximate EigenVectors) for A and E if $\|(A-E)\psi_n\| \to 0$ as $n \to \infty$.

As pointed out above, the spectrum of A is well known to be given by

$$(1.3)\qquad \sigma(A) = \{E \mid \text{there exists a SOAEV for } A \text{ and } E\},$$

and moreover, the essential spectrum of A coincides with the set of energies E for which there exist SOAEV's which converge weakly to zero. SOAEV's that converge weakly to zero are often called Weyl sequences. In general, a SOAEV need not say anything about continuity properties of spectral measures of A or about the nature of its eigenfunctions. We are interested in isolating types of SOAEV's that would encapsulate this type of information. To this end, we define

DEFINITION 1.3. Let $\{\psi_n\}_{n=1}^{\infty}$ be a SOAEV for A and E. We say that $\{\psi_n\}_{n=1}^{\infty}$ is rooted at a vector $\phi \in \mathcal{H}$, if

$$(1.4)\qquad \liminf_{n\to\infty} \frac{|\langle\psi_n,\phi\rangle|^2}{\|(A-E)\psi_n\|} > 0.$$

We also define

DEFINITION 1.4. Let $\{\psi_n\}_{n=1}^{\infty}$ be a SOAEV for A and E, let $\phi \in \mathcal{H}$, and let μ be a finite positive Borel measure on $\mathbb{R}$. We say that $\{\psi_n\}_{n=1}^{\infty}$ is optimally rooted at ϕ with respect to μ, if

$$(1.5)\qquad \liminf_{n\to\infty} \frac{|\langle\psi_n,\phi\rangle|^2}{\|(A-E)\psi_n\|\,(1+\operatorname{Im}\mathcal{M}_{\mu}(E+i\|(A-E)\psi_n\|))} > 0.$$

Remark. Note that an optimally rooted SOAEV must also be rooted, and that any rooted SOAEV is also optimally rooted if $\limsup_{\epsilon\to 0} \operatorname{Im} \mathcal{M}_{\mu}(E+i\epsilon) < \infty$. The concept of optimal rooting is thus of interest only for E's where $\limsup_{\epsilon\to 0} \operatorname{Im} \mathcal{M}_{\mu}(E+i\epsilon) = \infty$, namely, on the essential support of the singular part of μ.

Our next theorem is a fairly immediate consequence of Theorem 1.1. It relates local scaling behavior of spectral measures to the existence of appropriate types of SOAEV's.

THEOREM 1.5. *Let $\phi \in \mathcal{H}$. Then*
(i) *The following three sets coincide*:

$$\left\{ E \;\middle|\; \limsup_{\epsilon\to 0} \frac{\mu^{\phi}(I_{\epsilon}(E))}{2\epsilon} > 0 \right\},$$

$$\{E \mid \textit{there exists a SOAEV for } A \textit{ and } E \textit{ which is rooted at } \phi\},$$

and

$$\{E \mid \textit{there exists a SOAEV for } A \textit{ and } E \textit{ which is optimally rooted at } \phi \textit{ w.r.t. } \mu^\phi\}.$$

(ii) *For* $\mu = \mu_\phi$ *and every* $E \in \mathbb{R}$ *and* $\alpha \in [0,1]$, $D_\mu^\alpha(E) = \infty$ (*respectively* > 0) *if and only if there exists a SOAEV,* $\{\psi_n\}_{n=1}^\infty$, *for* A *and* E, *such that*

$$\liminf_{n\to\infty} \frac{|\langle \psi_n, \phi\rangle|^2}{\|(A-E)\psi_n\|^\alpha} = \infty \ (\textit{respectively} \ > 0). \tag{1.6}$$

(iii) *Let* ν *be any finite positive Borel measure on* $\mathbb{R}$, *and let* $S \subset \mathbb{R}$ *be a Borel set. If for each* $E \in S$ *there exists a SOAEV for* A *and* E *that is optimally rooted at* ϕ *w.r.t.* ν, *then* $\nu \restriction S$ *is absolutely continuous w.r.t.* μ^ϕ.

For any finite positive Borel measure μ on $\mathbb{R}$, the set

$$S_\mu \equiv \{E \mid \limsup_{\epsilon\to 0}(\mu(I_\epsilon(E))/2\epsilon) > 0\}$$

supports μ (namely, $\mu(\mathbb{R} \setminus S_\mu) = 0$) and it is minimal in the sense that no set of strictly smaller Lebesgue measure supports μ. S_μ is thus often called the essential support of μ. The essence of Theorem 1.5 can be understood as saying that the essential support of μ^ϕ coincides with the set of energies E for which there exist SOAEV's for A and E that are optimally rooted at ϕ with respect to μ^ϕ. Our more elaborate formulation and the fact that we define rooted SOAEV's separately from optimally rooted ones are motivated by two main things. First, we want to have a formulation that is useful from the perspective of spectral analysis. That is, we want a theorem that would be helpful in situations where, a priori, we do not know anything about the spectral measure μ_ϕ, and where we may hope to characterize its properties by constructing appropriate SOAEV's. Second, we will see below that both rooted and optimally rooted SOAEV's are natural objects when one considers the issue of convergence to generalized eigenfunctions. Roughly speaking, requiring SOAEV's to be rooted is a sufficient condition to ensure that they converge in a certain very weak sense to generalized eigenfunctions of A. Requiring them to be optimally rooted is a sufficient condition to ensure that they converge to generalized eigenfunctions in a stronger, more natural, sense. We believe that these conditions are also necessary (at least from the practical perspective that one cannot find alternative "sensible" conditions) in order to ensure convergence in the precise senses that we consider below.

Fix $\phi \in \mathcal{H}$ and an orthonormal basis $\{\delta_n\}_{n=1}^\infty$ of $\mathcal{H}$. Let

$$\mathcal{H}_\phi \equiv \overline{\{f(A)\phi \mid f \in C_\infty(\mathbb{R})\}}$$

be the cyclic subspace spanned by A and ϕ, and let P_ϕ be the orthogonal projection on $\mathcal{H}_\phi$. By the spectral theorem, $A \restriction \mathcal{H}_\phi$ is unitarily equivalent to multiplication by E on $L^2(\mathbb{R}, d\mu^\phi(E))$, and so the vectors $\{P_\phi\delta_n\}_{n=1}^\infty$ have representations as functions in $L^2(\mathbb{R}, d\mu^\phi(E))$. For each n, we let $u_E(n)$ be the $L^2(\mathbb{R}, d\mu^\phi(E))$ function that corresponds to $P_\phi\delta_n$. Since $u_E(n)$ is defined a.e. w.r.t. μ^ϕ, we also have that the sequence $\{u_E(n)\}_{n=1}^\infty$ is defined a.e. w.r.t. μ^ϕ. We call $\{u_E(n)\}_{n=1}^\infty$ the generalized eigenfunction of A that corresponds to ϕ and E in the representation $\{\delta_n\}_{n=1}^\infty$. Our next theorem relates rooted and optimally rooted SOAEV's to generalized eigenfunctions.

THEOREM 1.6. *Let $\phi \in \mathcal{H}$. Then*
(i) *There exists an orthonormal basis, $\{\delta_m\}_{m=1}^{\infty}$, of $\mathcal{H}$, such that for a.e. E w.r.t. μ^ϕ, every SOAEV, $\{\psi_n\}_{n=1}^{\infty}$, for A and E, that is rooted at ϕ, obeys for every m, $\langle\psi_n,\phi\rangle^{-1}\langle\psi_n,P_\phi\delta_m\rangle \to u_E(m)$ as $n\to\infty$ (where $\{u_E(m)\}_{m=1}^{\infty}$ is the generalized eigenfunction of A that corresponds to ϕ and E in the representation $\{\delta_m\}_{m=1}^{\infty}$).*
(ii) *For any orthonormal basis, $\{\delta_m\}_{m=1}^{\infty}$, of $\mathcal{H}$ and for any finite positive Borel measure ν on $\mathbb{R}$, we have that for a.e. E w.r.t. ν, every SOAEV, $\{\psi_n\}_{n=1}^{\infty}$, for A and E, that is optimally rooted at ϕ w.r.t. ν, obeys for every m, $\langle\psi_n,\phi\rangle^{-1}\langle\psi_n,P_\phi\delta_m\rangle \to u_E(m)$ as $n\to\infty$ (where $\{u_E(m)\}_{m=1}^{\infty}$ is the generalized eigenfunction of A that corresponds to ϕ and E in the representation $\{\delta_m\}_{m=1}^{\infty}$).*

Remarks. 1. The existence of an appropriate basis in (i) comes from choosing a basis for which the $u_E(m)$'s are continuous functions of E. The need for optimally rooted SOAEV's in (ii) comes from the $u_E(m)$'s being arbitrary L^2 functions rather than continuous.

2. It may appear strange that (ii) is formulated without any reference to μ^ϕ. Note, however, that by (iii) of Theorem 1.5, the restriction of ν to the set where the optimally rooted SOAEV's exist is absolutely continuous w.r.t. μ^ϕ. The eigenfunctions $\{u_E(m)\}_{m=1}^{\infty}$ are thus well defined a.e. w.r.t. ν on this set.

3. The generalized eigenfunctions that we consider here are connected with the cyclic subspace $\mathcal{H}_\phi$, and it is sometimes more natural to consider a representation in a basis of $\mathcal{H}_\phi$ rather than in a basis of $\mathcal{H}$. The theorem remains true for such representations. Of particular interest is the case where linear combinations of the monomials $\{A^n\phi\}_{n=1}^{\infty}$ happen to be dense in $\mathcal{H}_\phi$ (this is true for any ϕ if A happens to be bounded, and it is also true for many natural choices of a vector ϕ in many cases of unbounded A's). In such a case, $\mathcal{H}_\phi$ has a natural basis which is obtained from $\{A^n\phi\}_{n=1}^{\infty}$ by Gram-Schmidt orthonormalization (this basis coincides with the orthogonal polynomials of the spectral measure μ^ϕ), and $A\restriction\mathcal{H}_\phi$ is well known to have a tridiagonal matrix representation in this basis (see, e.g., [**29**]). In such a case, the basis in (i) can be chosen to be this natural basis.

Theorem 1.6 relates SOAEV's to general discrete representations of generalized eigenfunctions which are well defined for arbitrary self-adjoint operators. In many practical cases, however, one wants to identify generalized eigenfunctions with solutions of a generalized eigenvalue equation in a particular (natural) space representation. While there are cases, like discrete Schrödinger operators, where the natural representation is a special case of our more general discrete representations, there are other cases, like continuous Schrödinger operators (or other differential operators), where the natural representations of generalized eigenfunctions are continuous. Let

$$H_V^\Omega = -\Delta + V(x) \tag{1.7}$$

denote the Schrödinger operator defined on a domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary and satisfying Dirichlet boundary conditions on $\partial\Omega$. (Note that we are primarily interested in cases where the domain Ω is unbounded, and the case $\Omega = \mathbb{R}^d$ is also included.) Recall that for a wide class of Schrödinger operators of this type, one has a generalized eigenfunction expansion theorem with continuum eigenfunctions. (see [**1, 3, 4, 9, 22, 27, 28**]). That is, for each vector $\phi \in \mathcal{H} = L^2(\Omega, dx)$ (where dx denotes the d-dimensional Lebesgue measure on Ω), the unitary map U_ϕ, which maps $H_V^\Omega \restriction \mathcal{H}_\phi$ to multiplication by the variable on $L^2(\mathbb{R}, d\mu^\phi)$, is known to be an integral operator with kernel $\overline{u_\phi(x,E)}$, where $u_\phi(x,E)$, for fixed E, is in L^2_{Loc} (in x) and solves

$$(H_V^\Omega - E)u_\phi(x,E) = 0 \tag{1.8}$$

(in a distributional sense if the potential is not sufficiently well-behaved). We will say that the $u_\phi(x,E)$'s are the generalized eigenfunctions of H_V^Ω, corresponding to ϕ, if their complex conjugates constitute the kernel of the unitary map U_ϕ. These generalized eigenfunctions are generally only defined a.e. w.r.t. μ^ϕ. In this setting, (ii) of Theorem 1.6 has the following analog.

THEOREM 1.7. *Let $\phi \in \mathcal{H} = L^2(\Omega, dx)$ and let H_V^Ω be a self-adjoint operator on $L^2(\Omega, dx)$ for which the generalized eigenfunctions $u_\phi(x,E)$ are well defined (as functions in L^2_{Loc} in x for a.e. E w.r.t μ_ϕ, whose complex conjugates constitute the kernel of the unitary map U_ϕ). Let P_ϕ be the orthogonal projection on $\mathcal{H}_\phi$, and let ν be any finite positive Borel measure on $\mathbb{R}$. Then for any compactly supported $\eta \in L^2(\Omega, dx)$, for a.e. E w.r.t. ν, every SOAEV, $\{\psi_n\}_{n=1}^\infty$, for H_V^Ω and E, that is optimally rooted at ϕ w.r.t. ν, obeys*

$$\int \langle \psi_n, \phi\rangle^{-1}(P_\phi \psi_n)(x)\overline{\eta(x)}\, dx \to \int u_\phi(x,E)\overline{\eta(x)}\, dx. \tag{1.9}$$

Remarks. 1. Note that (1.9) ensures that $\langle \psi_n, \phi\rangle^{-1}(P_\phi \psi_n)(x)$ converges to $u_\phi(x,E)$ in L^2_{Loc}.

2. While we are primarily interested in Schrödinger operators here, note that the theorem only relies on properties of the generalized eigenfunctions $u_\phi(x,E)$, and it does not require any explicit assumptions on H_V^Ω itself (beyond it being a self-adjoint operator on the appropriate space).

Using sequences of approximate eigenvectors, we can also study the multiplicity of the spectrum. Namely, the following natural extension of (iii) of Theorem 1.5 holds:

THEOREM 1.8. *Let ν be any finite positive Borel measure on $\mathbb{R}$, and let $S \subset \mathbb{R}$ be a Borel set. Let $\{\phi_k\}_{k=1}^n$ be an orthonormal set of vectors. Assume that for each $E \in S$, there exist n SOAEV's, $\{\psi_j^k\}$, $k = 1, \ldots, n$, for A and E, such that $\{\psi_j^k\}$ is optimally rooted at ϕ_k w.r.t. ν. Then the multiplicity of the spectrum on the set S is at least n, and $\nu \restriction S$ is absolutely continuous with respect to the spectral measures in at least n orthogonal cyclic subspaces of A.*

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1, and in Section 3, we prove Theorem 1.5. In Section 4, we describe some general consequences of Theorems 1.1 and 1.5, and in Section 5, we describe the application of these results to study spectral properties of a concrete family of Schrödinger operators. Note that we do not provide here proofs of Theorems 1.6, 1.7, and 1.8, nor do we provide proofs for the results described in Sections 4 and 5. Those will be given in a future publication.

We would like to thank F. Gesztesy, S. Jitomirskaya, R. Killip, and B. Simon for useful discussions.

2. Proof of Theorem 1.1

It is clearly sufficient to consider the case where $\|\phi\| = 1$. Since the cyclic subspace $\mathcal{H}_\phi$ is invariant under A, we note that it is sufficient to consider $\psi \in \mathcal{H}_\phi$, and we can identify $\mathcal{H}_\phi$ with $L^2(\mathbb{R}, d\mu^\phi)$, where A acts as the operator of multiplication by the variable. Fix E with $\int (x-E)^{-4}\, d\mu^\phi(x) = \infty$, and for every $\delta > 0$, define

$$C_{E,\delta} \equiv \left[\int ((x-E)^2+\delta^2)^{-2}\, d\mu^\phi(x)\right]^{-1/2}, \tag{2.1}$$

and a function $\psi_{E,\delta} \in L^2(\mathbb{R}, d\mu^\phi)$ with $\|\psi_{E,\delta}\| = 1$, by

$$\psi_{E,\delta}(x) \equiv C_{E,\delta}((x-E)^2+\delta^2)^{-1}. \tag{2.2}$$

Define also a function $\epsilon = \epsilon(E,\delta)$, by requiring the equality

$$\epsilon = \|(A-E)\psi_{E,\delta}\| = C_{E,\delta}\left[\int \frac{(x-E)^2}{((x-E)^2+\delta^2)^2}\, d\mu^\phi(x)\right]^{1/2}. \tag{2.3}$$

One can easily verify that ϵ is a continuous function of δ, that $\epsilon \to 0$ as $\delta \to 0$ (because we assume $\int (x-E)^{-4}\, d\mu^\phi(x) = \infty$), and that $\epsilon \to \left[\int (x-E)^2\, d\mu^\phi(x)\right]^{1/2} = \|(A-E)\phi\|$ as $\delta \to \infty$. Thus, given $0 < \epsilon < \|(A-E)\phi\|$, we can always find a positive δ for which (2.3) is obeyed. Recall that $\int ((x-E)^2+\delta^2)^{-1}\, d\mu^\phi(x) = \operatorname{Im} \mathcal{M}_\phi(E+i\delta)/\delta$. We have

$$|\langle\psi_{E,\delta},\phi\rangle|^2 = \left[\int \psi_{E,\delta}(x)\, d\mu^\phi(x)\right]^2 = \left[C_{E,\delta}\frac{\operatorname{Im} \mathcal{M}_\phi(E+i\delta)}{\delta}\right]^2, \tag{2.4}$$

and from (2.3) we obtain

$$\epsilon^2 = (C_{E,\delta})^2 \int \frac{(x-E)^2+\delta^2-\delta^2}{((x-E)^2+\delta^2)^2}\, d\mu^\phi(x) = (C_{E,\delta})^2 \frac{\operatorname{Im} \mathcal{M}_\phi(E+i\delta)}{\delta} - \delta^2. \tag{2.5}$$

By combining (2.4) and (2.5), we obtain

$$|\langle\psi_{E,\delta},\phi\rangle|^2 = (\epsilon^2+\delta^2)\frac{\operatorname{Im} \mathcal{M}_\phi(E+i\delta)}{\delta}. \tag{2.6}$$

One can easily verify that $\operatorname{Im} \mathcal{M}_\phi(E+i\delta)/\delta$ is a monotonely decreasing function of δ, while $\delta \operatorname{Im} \mathcal{M}_\phi(E+i\delta)$ is a monotonely increasing function of δ. Thus, if $\epsilon > \delta$, we have $\operatorname{Im} \mathcal{M}_\phi(E+i\delta)/\delta > \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon)/\epsilon$, and if $\epsilon \le \delta$, we have $\delta \operatorname{Im} \mathcal{M}_\phi(E+i\delta) \ge \epsilon \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon)$. In either case, we see that (2.6) implies

$$|\langle\psi_{E,\delta},\phi\rangle|^2 > \epsilon \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon). \tag{2.7}$$

This proves the leftmost inequality in (1.2). Since the middle inequality is obvious, it remains to prove the rightmost inequality. Consider any $\psi \in L^2(\mathbb{R}, d\mu^\phi)$ with $\|\psi\| = 1$ and $\|(A-E)\psi\| \le \epsilon$. Then

$$\begin{aligned} |\langle\psi,\phi\rangle|^2 &= \left[\int \psi(x)\, d\mu^\phi(x)\right]^2 = \left[\int \frac{|x-E-i\epsilon|\psi(x)}{|x-E-i\epsilon|}\, d\mu^\phi(x)\right]^2 \\ &\le \left[\int ((x-E)^2+\epsilon^2)|\psi(x)|^2\, d\mu^\phi(x)\right]\left[\int ((x-E)^2+\epsilon^2)^{-1}\, d\mu^\phi(x)\right] \\ &= (\|(A-E)\psi\|^2+\epsilon^2)\frac{\operatorname{Im} \mathcal{M}_\phi(E+i\epsilon)}{\epsilon} \le 2\epsilon \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon), \end{aligned} \tag{2.8}$$

which completes the proof. □

Remarks. 1. Note that if we happen to get $\epsilon = \delta$ in (2.3), then both suprema in (1.2) are attained (with the function $\psi_{E,\delta}$ defined above) and are equal to the upper bound. It is not hard to see that there are measures and ϵ's for which this actually happens.

2. Note that the condition $\int (x-E)^{-4}\, d\mu^\phi(x) = \infty$ is used only to ensure that the leftmost inequality holds for arbitrarily small ϵ's. In case that $\int (x-E)^{-4}\, d\mu^\phi(x) < \infty$, this inequality would still hold for ϵ's larger than $[\int (x-E)^{-2}\, d\mu^\phi(x)/\int (x-E)^{-4}\, d\mu^\phi(x)]^{1/2}$. The condition $\epsilon < \|(A-E)\phi\|$ is required in order to have an inequality for $\sup\{|\langle\psi,\phi\rangle|^2 \mid \|\psi\| = 1, \|(A-E)\psi\| = \epsilon\}$. Without it, $\|(A-E)\psi\| = \epsilon$ might not even be attainable. The inequality

$$\epsilon \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon) < \sup\{|\langle\psi,\phi\rangle|^2 \mid \|\psi\| = 1, \|(A-E)\psi\| \le \epsilon\} \tag{2.9}$$

holds without any restriction from above on ϵ. The rightmost inequality is independent of those two conditions and it holds whenever the appropriate supremum is defined, namely, when the distance of E from $\sigma(A)$ is less then ϵ (alternatively, we can say that it always holds if we define the supremum of the empty set to be zero).

3. Proof of Theorem 1.5

Proof of (i). The first set is well known to coincide with the set of E's where $\limsup_{\epsilon\to 0} \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon) > 0$. Let us denote this set by S_μ. Let us also denote the second set by S_{r} and the third set by S_{or}. We will show that these three sets coincide by showing $S_\mu \subset S_{\mathrm{or}} \subset S_{\mathrm{r}} \subset S_\mu$. Let $E \in S_\mu$, then clearly $\int (x-E)^{-4}\, d\mu^\phi(x) = \infty$ and there exists a sequence $\{\epsilon_n\}_{n=1}^\infty$ with $\epsilon_n < \|(A-E)\phi\|$ $\forall n$, $\epsilon_n \to 0$ as $n\to\infty$, so that $\inf_n \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon_n) > C > 0$. Theorem 1.1 thus implies that there exists a SOAEV $\{\psi_n\}_{n=1}^\infty$ with $\|(A-E)\psi_n\| = \epsilon_n$ $\forall n$, such that $|\langle\psi_n,\phi\rangle|^2 \ge \epsilon_n \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon_n)$ $\forall n$. Since $\operatorname{Im} \mathcal{M}_\phi(E+i\epsilon_n) > C$, we see that there must be some constant $\tilde{C} > 0$, so that $|\langle\psi_n,\phi\rangle|^2 \ge \tilde{C}\epsilon_n(1+\operatorname{Im} \mathcal{M}_\phi(E+i\epsilon_n))$. Thus, $\{\psi_n\}_{n=1}^\infty$ is optimally rooted at ϕ w.r.t. μ^ϕ, and it follows that $E \in S_{\mathrm{or}}$. We have thus shown $S_\mu \subset S_{\mathrm{or}}$. Since the inclusion $S_{\mathrm{or}} \subset S_{\mathrm{r}}$ is obvious, it remains to show $S_{\mathrm{r}} \subset S_\mu$.

Let $E \in S_{\mathrm{r}}$, then there exists a SOAEV, $\{\psi_n\}_{n=1}^\infty$, with $\inf_n |\langle\psi_n,\phi\rangle|^2/\|(A-E)\psi_n\| > C > 0$. Let $\epsilon_n = \|(A-E)\psi_n\|$, then by Theorem 1.1 we have $2\epsilon_n \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon_n) \ge |\langle\psi_n,\phi\rangle|^2$. Thus, we obtain $\inf_n \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon_n) > C/2 > 0$, and so $\limsup_{\epsilon\to 0} \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon) > 0$. This implies $E \in S_\mu$, and we thus conclude that $S_{\mathrm{r}} \subset S_\mu$. □

Proof of (ii). Suppose that there exists a SOAEV, $\{\psi_n\}_{n=1}^\infty$, for A and E, such that (1.6) holds. Let $\epsilon_n = \|(A-E)\psi_n\|$, then by Theorem 1.1 we have $\epsilon_n^{1-\alpha} \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon_n) \ge |\langle\psi_n,\phi\rangle|^2/2\|(A-E)\psi_n\|^\alpha$, and so clearly

$$\liminf_{n\to\infty} \epsilon_n^{1-\alpha} \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon_n) = \infty \ (>0).$$

Similarly, if there exists a sequence, $\{\epsilon_n\}_{n=1}^\infty$, where $\epsilon_n \to 0$ as $n\to\infty$, and $\liminf_{n\to\infty} \epsilon_n^{1-\alpha} \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon_n) = \infty\ (>0)$, then we can choose an appropriate SOAEV, $\{\psi_n\}_{n=1}^\infty$, with $\|(A-E)\psi_n\| = \epsilon_n$, so that (1.6) holds. Since we know that $D_\mu^\alpha(E) = \infty\ (>0)$ if and only if $\limsup_{\epsilon\to 0} \epsilon^{1-\alpha} \operatorname{Im} \mathcal{M}_\phi(E+i\epsilon) = \infty\ (>0)$, this proves (ii). □

Proof of (iii). Let E be such that there exists a SOAEV, $\{\psi_n\}_{n=1}^\infty$, for A and E that is optimally rooted at ϕ w.r.t. ν. Let $\epsilon_n = \|(A-E)\psi_n\|$, then $\epsilon_n \to 0$ as

$n \to \infty$, and there must be a positive constant C such that for sufficiently small ϵ_n, $|\langle\psi_n, \phi\rangle|^2 > C\epsilon_n(1 + \mathrm{Im}\,\mathcal{M}_\nu(E + i\epsilon_n)) > C\epsilon_n \mathrm{Im}\,\mathcal{M}_\nu(E + i\epsilon_n)$. By Theorem 1.1, we also have $2\epsilon_n \mathrm{Im}\,\mathcal{M}_\phi(E + i\epsilon_n) \geq |\langle\psi_n, \phi\rangle|^2$, and thus we get $2\mathrm{Im}\,\mathcal{M}_\phi(E + i\epsilon_n) > C\mathrm{Im}\,\mathcal{M}_\nu(E + i\epsilon_n)$ for small ϵ_n. This implies that

$$\liminf_{\epsilon\to 0} \frac{\mathrm{Im}\,\mathcal{M}_\nu(E + i\epsilon)}{\mathrm{Im}\,\mathcal{M}_\phi(E + i\epsilon)} < \infty. \tag{3.1}$$

It is well known that the part of ν which is singular w.r.t. μ_ϕ is supported on the set of E's where $\lim_{\epsilon\to 0}(\mathrm{Im}\,\mathcal{M}_\nu(E+i\epsilon)/\mathrm{Im}\,\mathcal{M}_\phi(E+i\epsilon)) = \infty$, and thus we conclude that the restriction of this part to S must vanish. □

4. Some general consequences

Let H_V^Ω be the Schrödinger operator on $\Omega \subset \mathbb{R}^d$ defined by (1.7). Denote by B_R the ball of radius R centered at the origin, and denote by W_m^l the usual Sobolev spaces of functions f such that $D^l f$ exists in the distributional sense and $\int(|u|^m + |D^l u|^m)\,dx < \infty$. One general consequence of Theorem 1.5 is that we can recover (and provide a simpler proof of) Theorem 1 of [**20**]. That is

THEOREM 4.1. *Assume that the potential $V(x)$ belongs to L^∞_{loc} and is bounded from below, and Ω is a domain with piecewise smooth boundary. Suppose that there exists a $W^2_{2,\mathrm{Loc}}$ distributional solution $u(x, E)$ of the generalized eigenfunction equation*

$$(H_V^\Omega - E)u(x, E) = 0 \tag{4.1}$$

satisfying the (*Dirichlet*) *boundary conditions and such that for some α, $0 \leq \alpha \leq 1$, we have*

$$\liminf_{R\to\infty} R^{-\alpha} \int_{B_R\cap\Omega} |u(x, E)|^2\,dx < \infty\ (= 0). \tag{4.2}$$

Fix some compactly supported $\phi(x) \in L^2(\Omega)$, such that

$$\int_\Omega \phi(x)u(x, E)\,dx \neq 0.$$

Then we have

$$D^\alpha_{\mu^\phi}(E) > 0\ (= \infty).$$

Remark. Our new proof of this theorem is based on using appropriately smoothened cutoffs of the solution in order to construct an appropriately rooted SOAEV.

Theorem 4.1 is related to the well-known theorem of Sch'nol (see [**3, 26, 27**]) which says that the existence of a polynomially bounded (but not L^2) solution of (4.1) implies that the energy E belongs to the essential spectrum of H_V^Ω. However, Sch'nol's theorem gives no further information on the structure of the essential spectrum or on spectral multiplicity. A major advantage of our new approach is that it allows us to go beyond merely characterizing the behavior of spectral measures, and to also probe spectral multiplicity through the existence of appropriate solutions. Roughly speaking, the existence of a number, say m, of linearly independent solutions having appropriate decay properties on a sufficiently large set of energies can be used to deduce that the spectrum has multiplicity of at least m on this set. There are various precise formulations that one can give to this result. We will present a formulation with respect to Hausdorff measures.

Let h^α be the α-dimensional Hausdorff measure on $\mathbb{R}$. Recall that for a measurable set S,

$$h^\alpha(S) = \lim_{\delta \to 0} \inf_{\delta\text{-covers}} \sum_\gamma |I_\gamma|^\alpha, \tag{4.3}$$

where the infimum is taken over all coverings of S by countable unions of intervals I_γ with size not exceeding δ. When $\alpha = 1$, h^α coincides with the Lebesgue measure.

THEOREM 4.2. *Let H_V^Ω be the Schrödinger operator on $\Omega \subset \mathbb{R}^d$ defined by* (1.7), *with $V \in L^\infty_{\text{loc}}$. Assume that for every energy E in a Borel set S of positive h^α measure, we have n linearly independent $W^2_{2,\text{Loc}}$ solutions $u_i(x,E)$ of the generalized eigenfunction equation*

$$(H_V^\Omega - E)u_i(x,E) = 0 \tag{4.4}$$

satisfying the (*Dirichlet*) *boundary conditions and such that for some α, $0 \le \alpha \le 1$, we have*

$$\liminf_{R \to \infty} R^{-\alpha} \|u_i\|^2_{B_R} < \infty. \tag{4.5}$$

Then the spectrum has multiplicity at least n on the set S.

Remarks. 1. Potentially the most useful is the case $\alpha = 1$, where Theorem 1.5 implies the existence of absolutely continuous spectrum of multiplicity n on the set S.

2. Results parallel to Theorems 4.1 and 4.2 hold in the discrete setting, where the Schrödinger operator h_v^Ω is defined by

$$h_v^\Omega u(n) = \sum_{m \in \Omega,\, |m-n|=1} u(m) + v(n)u(n). \tag{4.6}$$

No restrictions on the potential are needed in this case.

5. A concrete application

In this section we sketch a concrete application of our results to spectral analysis of Schrödinger operators. Complete proofs will appear elsewhere. The result we present here can be viewed as a generalization of recent results concerning the absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials [**5, 7, 25**] to Schrödinger operators on strips.

THEOREM 5.1. *Let Ω be a cylinder in $\mathbb{R}^d$, $\Omega = D \times (-\infty, \infty)$, where D is a compact domain in $\mathbb{R}^{d-1}$ with a smooth boundary. Let $\{E_n\}_{n=1}^\infty$ be the eigenvalues of the Dirichlet Laplacian $-\Delta^D$ on D in increasing order* (*counting multiplicity*). *If the potential $V(x)$ satisfies $|V(x)| \le C(1+|x|)^{-\frac{1}{2}-\epsilon}$ for some $\epsilon > 0$, then the absolutely continuous spectrum of the operator H_V^Ω has the same structure as the absolutely continuous spectrum of the unperturbed operator H_0^Ω. Namely, the absolutely continuous spectrum covers $[E_1, \infty)$ and has multiplicity $2n$ on $[E_n, E_{n+1}]$.*

Remarks. 1. We use methods of [**5**] to prove this result.

2. Our proof also finds the asymptotic behavior of solutions of the generalized eigenvalue equation for a.e. E.

The outline of the proof is the following: Given $E \neq E_n$ in the spectrum, let N be such that $E_N < E < E_{N+1}$. We call all modes (eigenfunctions) $\phi_1(y), ..., \phi_N(y)$ of $-\Delta^D$ corresponding to $E_1, ..., E_N$ open, while the rest of the modes $\phi_{N+1}(y), ...$ are closed. We consider the decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}_{\rm O} + \mathcal{H}_{\rm C}$, where $\mathcal{H}_{\rm O}$ is generated by functions of the type $f(x)\phi_i(y)$, $i = 1, ..., N$, $f \in L^2(dx)$, while $\mathcal{H}_{\rm C}$ is generated by functions of the type $f(x)\phi_i(y)$, $i = N+1, \dots$. Let $P_{\rm O}$, $P_{\rm C}$ be the orthogonal projections on these subspaces. One can use the methods of [**5**] to study the finite dimensional system

$$(-\Delta P_{\rm O} + P_{\rm O} V P_{\rm O}) u_{\rm o} = E u_{\rm o}, \tag{5.1}$$

and to show that for a.e. $E \in [E_n, E_{n+1}]$, it has $2n$ linearly independent bounded solutions. Using a certain compactness argument to control modes of $-\Delta^D$ that are closed for a given energy region, we also show that for a.e. $E \in [E_n, E_{n+1}]$, there exist $2n$ linearly independent bounded solutions of the generalized eigenvalue equation $(H_V^\Omega - E)u = 0$. By Theorem 4.2, it follows that the multiplicity of the absolutely continuous spectrum in this region is at least $2n$. We then use bounds on generalized eigenfunctions proved in [**22**] to show that the multiplicity of the absolutely continuous spectrum cannot be higher.

References

[1] N. Aronszajn, *On a problem of Weyl in the theory of Sturm-Liouville equations,* Am. J. Math. **79** (1957), 597–610

[2] J. Avron and B. Simon, *Transient and recurrent spectrum,* J. Funct. Anal. **43** (1981), 1–31

[3] J. Berezanskii, Expansions in Eigenfunctions of Selfadjoint Operators, Transl. Math. Mono., Vol. **17**, Amer. Math. Soc., Providence, R.I., 1968

[4] F.A. Berezin and M.A. Shubin, *The Schrödinger Equation,* Kluwer Academic Publishers, Dordrecht, Boston, 1991

[5] M. Christ and A. Kiselev, *Absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials: Some optimal results,* J. Amer. Math. Soc. **11** (1998), 771–797

[6] D. Damanik, *α-continuity properties of one-dimensional quasicrystals,* Commun. Math. Phys. **192** (1998), 169–182

[7] P. Deift and R. Killip, *On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials,* Commun. Math. Phys. **203** (1999), 341–347

[8] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon, *Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization,* J. Anal. Math. **69** (1996), 153–200

[9] W. Donoghue, *On perturbation of spectra,* Commun. Pure Appl. Math. **18** (1965), 559–579

[10] D. Gilbert and D. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators,* J. Math. Anal. Appl. **128** (1987), 30–56

[11] D. Gilbert, *On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints,* Proc. Roy. Soc. Edinburgh Sect. A **112** (1989), 213–229

[12] I. Guarneri *Spectral properties of quantum diffusion on discrete lattices*, Europhys. Lett. **10** (1989), 95–100

[13] I. Guarneri *On an estimate concerning quantum diffusion in the presence of a fractal spectrum*, Europhys. Lett. **21** (1993), 729–733

[14] J. Janas and S. Naboko, *Jacobi matrices with absolutely continuous spectrum,* Proc. Amer. Math. Soc. **127** (1999), 791–800

[15] S. Jitomirskaya and Y. Last, *Dimensional Hausdorff properties of singular continuous spectra,* Phys. Rev. Lett. **76** (1996), 1765–1769

[16] S. Jitomirskaya and Y. Last, *Power law subordinacy and singular spectra. I. Half-line operators,* Acta Math., to appear

[17] S. Jitomirskaya and Y. Last, *Power law subordinacy and singular spectra. II. Line operators,* preprint

[18] A. Kiselev, *Absolutely continuous spectrum of one-dimensional Schrödinger operators and Jacobi matrices with slowly decreasing potentials,* Commun. Math. Phys. **179** (1996), 377–400

[19] A. Kiselev, *Absolutely continuous spectrum of perturbed Stark operators,* to appear in Trans. Amer. Math. Soc.

[20] A. Kiselev and Y. Last, *Solutions, spectrum and dynamics of Schrödinger operators on infinite domains,* preprint

[21] A. Kiselev, Y. Last, and B. Simon, *Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators,* Commun. Math. Phys. **194** (1998), 1–45

[22] Y. Last and B. Simon, *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators,* Invent. math. **135** (1999), 329–367

[23] Y. Last, *Quantum dynamics and decompositions of singular continuous spectra,* J. Funct. Anal. **142** (1996), 406–445

[24] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I. Functional Analysis*, Academic Press, London, San Diego, 1980

[25] C. Remling, *The absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials,* Commun. Math. Phys. **193** (1998), 151–170

[26] I. Sch'nol, *On the behavior of the Schrödinger equation,* Math. Sb. **42** (1957), 273–286

[27] B. Simon, *Schrödinger semigroups,* Bull. Amer. Math. Soc. **7**(1982), 447–525

[28] B. Simon, *Spectral analysis of rank one perturbations and applications,* CRM Lecture Notes Vol. 8 (J. Feldman, R. Froese, L. Rosen, eds.), pp. 109–149, Amer. Math. Soc., Providence, RI, 1995

[29] B. Simon, *The classical moment problem as a self-adjoint finite difference operator,* Advances in Math. **137** (1998), 82–203

[30] G. Stolz, *Bounded solutions and absolute continuity of Sturm-Liouville operators,* J. Math. Anal. Appl. **169** (1992), 210–228

Department of Mathematics, University of California, Berkeley CA 94720
E-mail address: `mchrist@math.berkeley.edu`

Department of Mathematics, University of Chicago, 5734 S. University Ave, Chicago, IL 60637
E-mail address: `kiselev@math.uchicago.edu`

Insitute of Mathematics, The Hebrew University, 91904 Jerusalem, Israel and Department of Mathematics, California Institute of Technology, Pasadena, CA 91125
E-mail address: `ylast@its.caltech.edu`