The explosion problem in a flow

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March 19, 2008

Abstract

We consider the explosion problem in an incompressible flow introduced in [5]. We use a novel $L^p - L^\infty$ estimate for elliptic advection-diffusion problems to show that the explosion threshold obeys a positive lower bound which is uniform in the advecting flow. We also identify the flows for which the explosion threshold tends to infinity as their amplitude grows and obtain an effective description of the explosion threshold in the strong flow asymptotics in a two-dimensional one-cell flow.

1 Introduction

1.1 The explosion problem

The explosion problem concerns existence and regularity of positive solutions of nonlinear elliptic equations of the form

$$-\Delta\phi = \lambda g(\phi), \tag{1.1}$$

in a domain $\Omega \subset \mathbb{R}^n$ with the Dirichlet boundary conditions: $\phi = 0$ on the boundary $\partial\Omega$. The nonlinearity $g(\phi)$ is convex and increasing with $g(0) > 0$ and

$$\int_0^\infty \frac{ds}{g(s)} < +\infty. \tag{1.2}$$

Two typical examples to keep in mind are $g(s) = e^s$ and $g(s) = (1+s)^m$ with $m > 1$. The positive parameter $\lambda > 0$ measures the non-dimensional strength of the nonlinearity. It has been shown in the pioneering works of Keener and H. Keller [24], Joseph and Lundgren [20], and Crandall and Rabinowitz [12] that there exists a critical threshold $\lambda^* > 0$ so that (1.1) admits positive solutions for $0 < \lambda < \lambda^*$, while no positive solutions exist for $\lambda > \lambda^*$. The regularity of solutions at $\lambda = \lambda^*$ is a delicate issue: the linearized problem was studied by Brezis and Vazquez in [7] in great detail. In particular, when the domain is a ball, and for the exponential and power nonlinearities mentioned above, the solutions at the critical value λ^* are uniformly bounded in dimensions less or equal to $N = 9$ and $N = 10$, respectively, while in higher dimensions they are unbounded. For more general nonlinearities $g(s)$ and domains Ω, regularity of solutions at $\lambda = \lambda^*$ in dimensions $N = 2, 3$ has been established by Nedev [30], and more recently in dimension $N = 4$ by Cabré [8].

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In the present paper we consider the non-selfadjoint elliptic problem

$$-\Delta\phi + u\cdot\nabla\phi = \lambda g(\phi) \text{ in } \Omega, \qquad (1.3)$$
$$\phi = 0 \text{ on } \partial\Omega,$$

with a prescribed incompressible flow $u(x)$ so that $\nabla\cdot u = 0$. Extension of the aforementioned results to the case when a flow is present is a natural question in the context of the original motivation for the study of (1.1) as the explosion problem [15, 36, 38]. Of particular interest is to understand how the presence of an underlying flow and its features affect the explosion limit.

Existence of a critical explosion threshold $\lambda^*(u)$ can be established as a straightforward generalization of existing methods. We are mostly interested here in the qualitative dependence of $\lambda^*(u)$ on the flow u – whether a flow may raise or lower the explosion threshold, and in the asymptotic behavior of $\lambda^*(u)$ in the limit of a strong flow. Intuitively, a flow improves mixing and interaction with the boundary – hence one may expect that an incompressible flow would always raise the explosion threshold. Somewhat surprisingly, this was shown not necessarily to be the case in [5]. More precisely, Berestycki, Kagan, Joulin and Sivashinsky have considered in [5], the problem (1.3) for a two-dimensional cellular flow and observed numerically that while the explosion threshold increases for flows oscillating on a small scale, it may actually decrease if the flow has large scale variations. This is because such flows may promote creation of hot spots where the explosion would happen faster than without any flow. The authors of [5] have also presented a formal asymptotic analysis and found an effective problem in the limit of the large flow amplitude. The fully nonlinear problem when the flow itself satisfies a Navier-Stokes type equation coupled to the explosion problem for temperature has been studied in [2, 23, 29] using numerics and formal asymptotics. Recently, some rigorous results for the behavior of the solutions to the coupled system in the regime of a strong gravity have been obtained in [11]. Here we derive several qualitative properties of the explosion threshold $\lambda^*(u)$ in terms of the geometry and the amplitude of the flow u.

1.2 The main results

In the following we always assume that Ω is a smooth bounded domain in $\mathbb{R}^n$, $u(x)$ is a $C^1(\bar{\Omega})$ divergence-free flow ($\nabla\cdot u = 0$ in Ω). Our first proposition establishes the direct analog of the classical results for (1.1) and allows us to define the critical parameter $\lambda^*(u)$.

Proposition 1.1 *There exists $\lambda^*(u) \in (0,\infty)$ such that (i) for every $0 < \lambda < \lambda^*(u)$ the problem (1.3) has a unique positive classical solution $\phi_\lambda(x)$ such that the principal eigenvalue κ_1 of the linearized operator $M\psi = -\Delta\psi + u\cdot\nabla\psi - \lambda g'(\phi_\lambda)\psi$ is positive; (ii) if (1.3) admits another non-negative solution $v(x)$ then $v(x) \geq \phi_\lambda$; (iii) the function $\phi_\lambda(x)$ is increasing in λ; (iv) there exists no classical solution of (1.3) for $\lambda > \lambda^*(u)$.*

The proof of this result is very close to that in [12] – we present it below both for the convenience of the reader and since we will use some of the intermediate steps in what follows. Another reason to discuss the proofs for $u \not\equiv 0$ is that some of the basic results in the self-adjoint case $u = 0$ rely on the variational characterization of the principal eigenvalue of the linearized operator M which we do not have when the flow is present.

The next theorem shows that the possible creation of hot spots cannot drop the explosion threshold arbitrarily close to zero, no matter what the incompressible flow $u(x)$ is.

Theorem 1.2 *For any domain Ω and nonlinearity $g(\phi)$ there exists $\lambda_0 > 0$ so that the critical threshold $\lambda^*(u)$ for (1.3) satisfies $\lambda^* \geq \lambda_0 > 0$ for all incompressible flows $u(x)$ in Ω. The constant λ_0 depends on Ω and the function g.*

This result does not hold without the restriction that the flow $u(x)$ is incompressible – we describe in Section 2.3 examples of flows for which λ^* may be as small as one wishes. The proof of Theorem 1.2 involves the following uniform $L^p - L^\infty$ bound for solutions of the Dirichlet problem for elliptic diffusion-advection problems with the constant independent of the incompressible flow.

Lemma 1.3 *Let the flow $u(x)$ be divergence-free and let $q(x)$ be the solution of the elliptic problem*

$$\begin{aligned} &-\Delta q + u \cdot \nabla q = f(x) \text{ in } \Omega, \\ &q = 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.4}$$

with $f(x) \in L^p(\Omega)$, $p > n/2$. There exists a constant $C(\Omega, n, p) > 0$ which depends on p and the domain Ω but not on the flow $u(x)$, so that $\|q\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}$.

These results raise several interesting open questions. First, can one identify the optimal constant C in (1.4)? More importantly, we would like to pose as an open problem to know whether in all domains with smooth boundaries a flow realizing the best constant in Lemma 1.3 exists. The same question pertains to the smallest possible explosion threshold in Theorem 1.2: does the flow minimizing $\lambda^*(u)$ over all incompressible flows exist in any domain Ω? If so, what are its geometric characteristics? Numerical simulations in [5] indicate that a natural guess that $u = 0$ turns out not to be correct for all domains as an incompressible flow may create additional hot spots. More precisely, it has been numerically computed in [5] that in a very long rectangle the explosion threshold corresponding to a cellular flow with a certain finite positive amplitude is smaller than that corresponding to $u = 0$. However, it is not clear whether in the situation when $u = 0$ is not a minimizer of $\lambda^*(u)$, such a minimizer exists at all, or if minimizing flows do not exist. When it exists, how is it determined?

Let us now fix a flow profile $u(x)$ and consider the explosion problem (1.3) with a strong flow $Au(x)$, with a large flow amplitude $A \gg 1$:

$$\begin{aligned} &-\Delta\phi + Au \cdot \nabla\phi = \lambda g(\phi) \text{ in } \Omega \\ &\phi = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.5}$$

We are interested in the behavior of the explosion threshold $\lambda^*(A)$ for (1.5) in the limit $A \to +\infty$. Let us recall that a function $\psi \in H^1(\Omega)$ is a first integral of u if $u \cdot \nabla\psi = 0$ a.e. in Ω.

Theorem 1.4 *We have $\lambda^*(A) \to +\infty$ as $A \to +\infty$ if and only if u has no non-zero first integrals in $H^1_0(\Omega)$.*

This theorem provides a sharp characterization of the flows capable of preventing an explosion for an arbitrary $\lambda > 0$ provided that the flow amplitude A is sufficiently large. The proof uses the ideas from [4] and [10] together with some techniques of [6]. Not surprisingly, the explosion threshold tends to infinity as $A \to +\infty$ under the same assumptions as the principal Dirichlet eigenvalue of the operator $-\Delta + Au \cdot \nabla$ (see [4]), as both quantities measure the effectiveness of the enhancement of the boundary cooling due to the flow.

Finally, in Section 3.3, we consider the effective problem for (1.5) in the limit $A \to +\infty$ for the class of two-dimensional cellular flows. We show that in this limit, the various cells of the flow "do not talk to each other". The main result of that section is Theorem 3.4. In particular, when $A \to +\infty$, the explosion threshold $\lambda^*(A)$ is close to the explosion threshold on the "largest" cell in Ω. Moreover, the explosion threshold for each of the individual flow cells in the limit $A \to +\infty$ has an asymptotic description in terms of the Freidlin problem. We recall that the fast flow asymptotics for the parabolic reaction-diffusion equations in flows without cells has been treated by M. Freidlin in [16], and our results for the elliptic explosion problem in the special case of one cell flows are what one would expect formally from [16]. The most interesting and delicate new ingredient is the independent behavior of the solutions in various cells when $A \to +\infty$. We mention that unlike in [16], our proofs are not probabilistic in nature. Actually, as a by-product of the present paper, one can use our arguments to recover some of the results of [16] by analytic techniques.

Another natural variable coefficients extension of the classical results for (1.1) is to allow the nonlinearity to be spatially dependent – work in this direction has been recently done in [18, 19].

Acknowledgment. This work has been carried as the first author was visiting the University of Chicago. It has been supported by ASC Flash Center at the University of Chicago. AK was supported by NSF grant DMS-0314129, AN was supported by NSF grant DMS-0604600, LR was supported by NSF grant DMS-0604687.

2 Existence and basic properties of λ^*

2.1 The critical parameter

We begin with the proof of Proposition 1.1 which is well known for $u = 0$. We do not assume in this section that the flow $u(x)$ is incompressible. As we have mentioned, the proof is very close to that in [12], with some minor modifications. Let $\mu_1[u]$ and $\eta(x)$ be the principal eigenvalue and the normalized positive eigenfunction of the adjoint problem

$$\begin{aligned} &-\Delta\eta - \nabla \cdot (u\eta) = \mu_1[u]\eta \text{ in } \Omega \\ &\eta = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.1}$$

Note that $\mu_1[u] > 0$ as the operator $-\Delta + u \cdot \nabla$ has no zero order term.

Lemma 2.1 *The problem (1.3) admits no non-negative classical solutions for $\lambda > \mu_1[u]/g'(0)$.*

Proof. Since the function $g(s)$ is convex and $g(0) > 0$ we have $g(s) \geq g'(0)s$. Therefore, any classical solution $\phi_\lambda \geq 0$ of (1.3) satisfies

$$\begin{aligned} &-\Delta\phi_\lambda + u \cdot \nabla\phi_\lambda \geq \lambda g'(0)\phi_\lambda \text{ in } \Omega, \\ &\phi_\lambda = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.2}$$

Multiplying (2.2) by η and integrating by parts we conclude that

$$\lambda g'(0) \int \eta\phi_\lambda \leq \int \phi_\lambda[-\Delta\eta - u \cdot \nabla\eta] = \mu_1[u] \int \eta\phi_\lambda.$$

It follows that for a positive solution of (1.3) to exist we must have $\mu_1[u] \geq g'(0)\lambda$ and thus no non-negative solution of (1.3) exists if $\lambda > \mu_1[u]/g'(0)$. □

Next, we show that for a sufficiently small $\lambda > 0$ a positive solution of (1.3) exists. Let $\tau(x)$ be the expected value of the exit-time, solution of

$$\begin{aligned} &-\Delta\tau + u \cdot \nabla\tau = 1 \text{ in } \Omega, \\ &\tau = 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.3}$$

and let

$$\theta_u = \max_{x \in \bar{\Omega}} \tau(x). \tag{2.4}$$

Lemma 2.2 *There exists a constant $C > 0$ which depends only on the nonlinearity $g(s)$ but not on the flow $u(x)$ so that problem (1.3) admits a minimal non-negative solution ϕ_λ for all $\lambda \leq C/\theta_u$.*

The proof is by constructing a super-solution and using it to show that a positive solution of (1.3) exists. Let us recall the following fact.

Lemma 2.3 *Assume that there exists a smooth function $\bar{\phi}(x) \geq 0$ satisfying*

$$-\Delta\bar{\phi} + u \cdot \nabla\bar{\phi} \geq \lambda g(\bar{\phi}) \text{ in } \Omega, \qquad (2.5)$$
$$\bar{\phi} \geq 0 \text{ on } \partial\Omega.$$

Then there exists a classical solution ϕ_λ of (1.3) which is minimal in the sense that for any other non-negative solution ψ of (1.3) we have $\phi_\lambda(x) \leq \psi(x)$ for all $x \in \Omega$.

Proof. We construct an approximating sequence $\phi_n(x)$ by setting $\phi_0(x) = 0$ and letting ϕ_{n+1} be the smooth solution of

$$-\Delta\phi_{n+1} + u \cdot \nabla\phi_{n+1} = \lambda g(\phi_n) \text{ in } \Omega, \qquad (2.6)$$
$$\phi_{n+1} = 0 \text{ on } \partial\Omega.$$

The difference $w_1 := \phi_1 - \phi_0 (= \phi_1)$ satisfies

$$-\Delta w_1 + u \cdot \nabla w_1 = \lambda g(0) \geq 0 \text{ in } \Omega, \qquad (2.7)$$
$$w_1 = 0 \text{ on } \partial\Omega.$$

It follows from the maximum principle that $w_1 \geq 0$ and thus $\phi_1 \geq \phi_0$. Similarly, we have for the higher differences $w_n = \phi_n - \phi_{n-1}$:

$$-\Delta w_n + u \cdot \nabla w_n = \lambda[g(\phi_{n-1}) - g(\phi_{n-2})] \geq 0 \text{ in } \Omega, \qquad (2.8)$$
$$w_1 = 0 \text{ on } \partial\Omega.$$

Then by induction we conclude that $0 \leq \phi_n \leq \phi_{n+1}$ since the function $g(s)$ is increasing. The same induction argument shows that $\phi_n(x) \leq \bar{\phi}(x)$ for all $n \geq 1$. Therefore, the sequence ϕ_n converges to a limit ϕ_λ which has to be a solution of (1.3) and satisfy $0 \leq \phi_\lambda \leq \bar{\phi}(x)$. As the sequence ϕ_n does not depend on the choice of the super-solution $\bar{\phi}$, the limit ϕ_λ is a minimal solution of (1.3). □

Proof of Lemma 2.2. Observe that for $\lambda > 0$ sufficiently small the function $\bar{\tau}(x) = 2g(0)\lambda\tau(x)$ satisfies

$$-\Delta\bar{\tau} + u \cdot \nabla\bar{\tau} \geq \lambda g(\bar{\tau}) \text{ in } \Omega, \qquad (2.9)$$
$$\bar{\tau} = 0 \text{ on } \partial\Omega.$$

Here $\tau(x)$ is the solution of (2.3). This is true provided that $2g(0) \geq g(2g(0)\lambda\tau)$. As the function $g(s)$ is increasing, for this inequality to hold it suffices to require that $2g(0) \geq g(2g(0)\lambda\theta_u)$. This condition is clearly satisfied if $\lambda \leq C/\theta_u$ with a constant C which depends only on the function $g(s)$. Now, existence of a minimal solution to (1.3) follows from Lemma 2.3. □

Recall that a solution ϕ_λ of (1.3) is stable if the principal eigenvalue $\kappa_1(\lambda, \phi_\lambda)$ of the linearized operator

$$M_\lambda\psi = -\Delta\psi + u \cdot \nabla\psi - \lambda g'(\phi_\lambda)\psi$$

is positive.

Lemma 2.4 *Any minimal solution of (1.3) has $\kappa_1(\lambda, \phi_\lambda) \geq 0$.*

Proof. Let ϕ_λ be a minimal solution of (1.3) and assume that the principal eigenvalue $\kappa_1(\lambda, \phi_\lambda)$ of the problem

$$-\Delta\psi + u \cdot \nabla\psi - \lambda g'(\phi_\lambda)\psi = \kappa_1(\lambda, \phi_\lambda)\psi, \quad \psi = 0 \text{ on } \partial\Omega \tag{2.10}$$

is negative. Consider the function $\psi_\varepsilon = \phi_\lambda - \varepsilon\psi$, then we have

$$\begin{aligned} -\Delta\psi_\varepsilon + u \cdot \nabla\psi_\varepsilon - \lambda g(\psi_\varepsilon) &= \lambda g(\phi_\lambda) - \varepsilon\lambda g'(\phi_\lambda)\psi - \varepsilon\kappa_1(\phi_\lambda)\psi - \lambda g(\phi_\lambda - \varepsilon\psi) \\ &= -\varepsilon\kappa_1(\lambda, \phi_\lambda)\psi - \frac{\varepsilon^2 g''(\xi)}{2}\psi^2 \geq 0, \end{aligned}$$

provided that ε is sufficiently small and $\kappa_1(\lambda, \phi_\lambda) < 0$. This contradicts minimality of ϕ. Therefore, we have $\kappa_1(\lambda, \phi_\lambda) \geq 0$ if ϕ_λ is a minimal solution. □

Lemma 2.5 *Assume that ϕ_λ is a solution of (1.3) such that $\kappa_1(\lambda, \phi_\lambda) = 0$. Then no classical solutions of (1.3) with $\tilde{\lambda} > \lambda$ exists.*

Proof. Assume that $\tilde{\lambda} > \lambda$ and there exists a function $\tilde{\phi} \geq 0$ such that

$$-\Delta\tilde{\phi} + u \cdot \nabla\tilde{\phi} = \tilde{\lambda} g(\tilde{\phi}), \quad \tilde{\phi} = 0 \text{ on } \partial\Omega.$$

Let us also denote by ψ the positive eigenfunction of the adjoint problem

$$-\Delta\psi - \nabla \cdot (u\psi) - \lambda g'(\phi_\lambda)\psi = 0, \quad \psi = 0 \text{ on } \partial\Omega. \tag{2.11}$$

Set $\eta = \phi_\lambda + \tau(\tilde{\phi} - \phi_\lambda)$ with $\tau \in [0, 1]$. Then convexity of g implies that

$$\begin{aligned} -\Delta\eta + u \cdot \nabla\eta - \lambda g(\eta) &= -\Delta\eta + u \cdot \nabla\eta - \lambda g(\phi_\lambda + \tau(\tilde{\phi} - \phi_\lambda)) \\ &\geq -\Delta\eta + u \cdot \nabla\eta - \lambda(1 - \tau)g(\phi_\lambda) - \lambda\tau g(\tilde{\phi}) = (\tilde{\lambda} - \lambda)\tau g(\tilde{\phi}) \geq 0, \end{aligned} \tag{2.12}$$

for all $\tau \in [0, 1]$. Moreover, we have equality in (2.12) when $\tau = 0$. Differentiating (2.12) with respect to τ at $\tau = 0$ gives the following inequality for $\zeta = \tilde{\phi} - \phi_\lambda$:

$$-\Delta\zeta + u \cdot \nabla\zeta - \lambda g'(\phi_\lambda)\zeta \geq (\tilde{\lambda} - \lambda)g(\tilde{\phi}) > 0. \tag{2.13}$$

Multiplying (2.13) by the eigenfunction ψ of (2.11) and integrating we obtain

$$0 < \int \psi \left[-\Delta\zeta + u \cdot \nabla\zeta - \lambda g'(\phi_\lambda)\zeta\right] = \int \zeta \left[-\Delta\psi - \nabla \cdot (u\psi) - \lambda g'(\phi_\lambda)\psi\right] = 0.$$

This contradiction shows that no solution of (1.3) for $\tilde{\lambda} > \lambda$ may exist if $\kappa_1(\lambda, \phi_\lambda) = 0$. □

This also finishes the proof of Proposition 1.1. The critical threshold $\lambda^*(u)$ is the supremum of all λ for which a stable solution of (1.3) exists. We summarize the upper and lower bounds for $\lambda^*(u)$ in Lemmas 2.1 and 2.2 as

$$\frac{C}{\theta_u} \leq \lambda^*(u) \leq \frac{\mu_1(u)}{g'(0)} < +\infty. \tag{2.14}$$

We will use these bounds in the sequel.

2.2 A uniform bound away from λ^*

Uniform L^∞-bounds for the functions ϕ_{λ^*} at $\lambda = \lambda^*$ are difficult to obtain and will be investigated elsewhere [3]. However, we have the following universal estimate for $\lambda < \lambda^*$ which will prove useful later.

Proposition 2.6 *For any $\delta > 0$ there exists a constant $C(\delta) > 0$ which depends only on δ and nonlinearity $g(s)$ but not on the domain Ω or the incompressible flow $u(x)$ so that the minimal positive solution $\phi_\lambda(x)$ of (1.3) satisfies $0 \le \phi_\lambda(x) \le C(\delta)$ for all $\lambda \in (0, (1-\delta)\lambda^*)$.*

Proof. The proof is based on an idea from [6]. Fix $\delta \in (0,1)$, let $\lambda_0 < (1-\delta)\lambda^*$ and take any $\lambda_1 \in ((1-\delta/3)\lambda^*, \lambda^*)$. We denote by ϕ_0 and ϕ_1 the corresponding classical solutions of (1.3) with $\lambda = \lambda_0$ and $\lambda = \lambda_1$, respectively.

Following [6], set

$$h(s) = \int_0^s \frac{ds'}{g(s')}.$$

It follows from the positivity of the function $g(s)$ and (1.2) that $h(s)$ is an increasing positive function with $h(+\infty) < +\infty$. We now define the rescaled inverse function

$$\Phi(s) = h^{-1}\left(\frac{\lambda_0}{\lambda_1} h(s)\right). \tag{2.15}$$

Note that, since $\lambda_0 < (1-\delta)\lambda^*$ and $\lambda_1 > (1-\delta/3)\lambda^*$ we have

$$0 \le \frac{\lambda_0}{\lambda_1} h(s) < \frac{\lambda_0}{\lambda_1} h(+\infty) < \frac{1-\delta}{1-\delta/3} h(+\infty).$$

Therefore, the function $\Phi(s)$ is well-defined for all $s \ge 0$, and there exists a constant $K(\delta)$ which depends only on the parameter $\delta > 0$ and the nonlinearity $g(s)$ so that $0 \le \Phi(s) \le K(\delta)$ for all $s \ge 0$ and all $\lambda \in (0, (1-\delta)\lambda^*)$.

In addition, as $g(s) \ge g(0) = 1$, we have $\Phi(s) \le s$ and

$$\Phi'(s) = \left[h'\left(h^{-1}\left(\frac{\lambda_0}{\lambda_1} h(s)\right)\right)\right]^{-1} \frac{\lambda_0}{\lambda_1} h'(s) = \frac{\lambda_0 g(\Phi(s))}{\lambda_1 g(s)}. \tag{2.16}$$

Hence, as $g(s)$ is increasing and $\Phi(s) \le s$, the function $\Phi(s)$ is increasing, with

$$0 < \Phi'(s) \le \lambda_0/\lambda_1 < 1.$$

Moreover, Φ is concave:

$$\begin{aligned}\Phi''(s) &= \frac{\lambda_0}{\lambda_1} \frac{g'(\Phi(s))\Phi'(s) g(s) - g(\Phi(s)) g'(s)}{g^2(s)} = \frac{\lambda_0}{\lambda_1 g^2(s)} \left[\frac{\lambda_0 g(\Phi(s))}{\lambda_1 g(s)} g'(\Phi(s)) g(s) - g(\Phi(s)) g'(s)\right] \\ &= \frac{\lambda_0 g(\Phi(s)}{\lambda_1 g^2(s)} \left[\frac{\lambda}{\lambda_1} g'(\Phi(s)) - g'(s)\right] \le 0\end{aligned}$$

because $g(s)$ is convex, $\Phi(s) \le s$ and $0 < \lambda < \lambda_1$.

Recall that ϕ_1 is the minimal positive classical solution to (1.3) with $\lambda = \lambda_1$ and set $\bar\phi = \Phi(\phi_1)$. Using concavity of the function $\Phi(s)$ and expression (2.16) we observe that the function $\bar\phi$ satisfies the inequality

$$\begin{aligned}-\Delta\bar\phi + u \cdot \nabla\bar\phi &= -\Phi''(\phi_1)|\nabla\phi_1|^2 - \Phi'(\phi_1)\Delta\phi_1 + \Phi'(\phi_1)(u \cdot \nabla\phi_1) \\ &= -\Phi''(\phi_1)|\nabla\phi_1|^2 + \lambda_1 \Phi'(\phi_1) g(\phi_1) \ge \lambda_1 \Phi'(\phi_1) g(\phi_1) = \lambda_0 g(\bar\phi).\end{aligned}$$

Moreover, as $\Phi(0) = 0$ the function $\bar{\phi}$ obeys the Dirichlet boundary conditions $\bar{\phi} = 0$ on $\partial\Omega$. Therefore, the function $\bar{\phi}$ is a super-solution for (1.3) with $\lambda = \lambda_0$. Employing the same iterative procedure as in the proof of Proposition 2.2 we may then construct a non-negative solution ϕ_0 of (1.3) with $\lambda = \lambda_0$ which is smaller than $\bar{\phi}(x)$. However, by construction we have $0 \le \bar{\phi}(x) \le K(\delta)$ and the conclusion of Proposition 2.6 holds. □

2.3 A uniform lower bound for λ^*

We prove Theorem 1.2 in this section. Let $\phi(x)$ be the minimal positive solution of (1.3):

$$\begin{aligned} &-\Delta\phi + u \cdot \nabla\phi = \lambda g(\phi) \text{ in } \Omega, \\ &\phi = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.17}$$

According to (2.14), in order to obtain a uniform lower bound for the explosion threshold λ^*, it suffices to bound from above θ_u, the supremum of the exit time, defined by (2.3) and (2.4). That is, it suffices to prove that there exists a constant $M > 0$ so that

$$\theta_u \le M, \tag{2.18}$$

for all divergence free flows $u(x)$ in Ω. The constant M should not depend on the flow $u(x)$. This bound is an immediate consequence of Lemma 1.3.

Proof of Lemma 1.3

We write $q(x)$, the solution of (1.4), as

$$q(x) = \int_0^\infty \bar{\psi}(t,x)dt. \tag{2.19}$$

The function $\bar{\psi}(t,x)$ satisfies the parabolic initial value problem

$$\begin{aligned} &\bar{\psi}_t - \Delta\bar{\psi} + u \cdot \nabla\bar{\psi} = 0 \text{ in } \Omega, \\ &\bar{\psi}(t,x) = 0 \text{ on } \partial\Omega, \\ &\bar{\psi}(0,x) = f(x) \text{ in } \Omega. \end{aligned} \tag{2.20}$$

We will now show that there exists a pair of constants $C > 0$ and $\alpha > 0$ so that for any incompressible flow u and any solution of (2.20) with initial data $f(x)$ we have a uniform bound

$$|\psi(t,x)| \le \frac{Ce^{-\alpha t}}{t^r}\|f\|_{L^1(\Omega)}, \tag{2.21}$$

with any $r > d/2$. The proof is as in [10] with a slight modification, we present the details for the convenience of the reader. First, multiplying (2.20) by ψ and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_2^2 = -\|\nabla\psi\|_2^2. \tag{2.22}$$

Using the Poincaré inequality in Ω we conclude that there exists a constant $\alpha > 0$ so that

$$\|\psi(t_2)\|_2 \le e^{-\alpha(t_2-t_1)}\|\psi(t_1)\|_2 \tag{2.23}$$

for any pair of times $t_2 \ge t_1 \ge 0$. On the other hand, we have, using the Poincaré inequality again, for all $1 < p < 2d/(d-2)$:

$$\|\psi\|_p \le C\|\nabla\psi\|_2.$$

Next, using the Hölder inequality, with $1/p + 1/q = 1$ we obtain:

$$\|\psi\|_2^2 = \int |\psi|^2 \leq \left(\int |\psi|\right)^{1/p} \left(\int |\psi|^{(2-1/p)q}\right)^{1/q} \leq C\|\psi\|_1^{1/p}\|\nabla\psi\|_2^{2-1/p},$$

provided that

$$\left(2 - \frac{1}{p}\right) q = \left(2 - \frac{1}{p}\right) \frac{p}{p-1} = \frac{2p-1}{p-1} < \frac{2d}{d-2},$$

or, equivalently, that $p > (d+2)/4$. Therefore, we have the following Nash-type inequality in Ω:

$$\|\nabla\psi\|_2^2 \geq C\frac{\|\psi\|_2^{4p/(2p-1)}}{\|\psi\|_1^{2/(2p-1)}} = C\frac{\|\psi\|_2^{s+2}}{\|\psi\|_1^s},$$

with $s = 2/(2p-1)$. However, incompressibility of the flow, the Hopf lemma and the boundary conditions imply that $\|\psi(t)\|_1 \leq \|f\|_1$. It follows that

$$\|\nabla\psi\|_2^2 \geq C\frac{\|\psi\|_2^{s+2}}{\|f\|_1^s}.$$

Going back to (2.22) we conclude that

$$\frac{d}{dt}\|\psi\|_2^2 = -2\|\nabla\psi\|_2^2 \leq -C\frac{\|\psi\|_2^{s+2}}{\|f\|_1^s}. \tag{2.24}$$

Therefore we have a bound

$$\|\psi(t)\|_2 \leq \frac{C}{t^{1/s}}\|f\|_1.$$

Combining this inequality with (2.23) evaluated with $t_1 = t$, $t_2 = 2t_1$ we conclude that

$$\|\psi(t)\|_2 \leq \frac{Ce^{-\alpha t}}{t^{1/s}}\|f\|_1, \tag{2.25}$$

with $1/s > d/4$.

Consider now the solution operator $\mathcal{P}_t: \ \psi_0 \to \psi(t)$. We have shown that

$$\|\mathcal{P}_t\|_{L^1\to L^2} \leq \frac{Ce^{-\alpha t}}{t^{1/s}}.$$

The adjoint operator to $\mathcal{P}_t^*$ is simply the solution operator corresponding to the (also incompressible) flow $(-u)$. Therefore, we have the dual bound

$$\|\mathcal{P}_t^*\|_{L^1\to L^2} \leq \frac{Ce^{-\alpha t}}{t^{1/s}},$$

which in turn implies that

$$\|\mathcal{P}_t\|_{L^2\to L^\infty} \leq \frac{Ce^{-\alpha t}}{t^{1/s}}.$$

Putting these bounds together we obtain

$$\|\psi(t)\|_\infty = \|\mathcal{P}_t f\|_\infty = \|\mathcal{P}_{t/2}\mathcal{P}_{t/2}f\|_\infty \leq \|\mathcal{P}_{t/2}\|_{L^2\to L^\infty}\|\mathcal{P}_{t/2}\|_{L^1\to L^2}\|f\|_1 \leq \frac{Ce^{-\alpha t}}{t^{2/s}}\|f\|_1,$$

which is (2.21). The maximum principle also implies that we have a trivial bound $\|\psi\|_{L^\infty} \leq \|f\|_{L^\infty}$. Interpolating between these two bounds we get the estimate

$$|\psi(t,x)| \leq \frac{C_\varepsilon e^{-\alpha_p t}}{t^{n/(2p)+\varepsilon}}\|f\|_{L^p}, \tag{2.26}$$

for any $\varepsilon > 0$. Now, for any $p > n/2$ we may choose $\varepsilon > 0$ sufficiently small so that the kernel would be integrable at $t = 0$, and (2.19) would imply that $\|q\|_{L^\infty} \leq C\|f\|_{L^p}$ and the constant $C > 0$ is independent of the incompressible flow u. This finishes the proof of Lemma 1.3 and hence that of Theorem 1.2. □

Explosion threshold in compressible flows

As we have mentioned in the introduction, without the incompressibility constraint the explosion threshold may be arbitrarily small. Indeed, according to Proposition 2.1 we have an upper bound $\lambda^*(u) \leq \mu_1(u)/g'(0)$. Therefore, to see that no uniform in the flow lower bound for λ^* in compressible flows exists, it suffices to construct flows $u_n(x)$ such that the principal eigenvalue $\mu_1(u_n) \to 0$, as $n \to +\infty$. Such example is provided by the radial flows $u_n(x) = 4nx$, say, in two-dimensions:

$$\begin{aligned} &-\Delta\phi_n + 4nx \cdot \nabla\phi_n = \mu_n\phi_n, \ \phi_n > 0 \text{ in } B(0,1) \subset \mathbb{R}^2, \\ &\phi_n = 0 \text{ on } |x| = 1. \end{aligned} \tag{2.27}$$

Then $\mu_n \leq Ce^{-cn} \to 0$ as $n \to +\infty$ – this can be seen either from the general theory in [17, 25] or by an explicit computation. Indeed, setting $\phi_n = e^{n|x|^2}\psi_n$ we obtain a self-adjoint problem for ψ_n:

$$\begin{aligned} &-\Delta\psi_n + 4n^2|x|^2\psi_n = (\mu_n + 4n)\mu_n\psi_n, \ \psi_n > 0 \text{ in } B(0,1) \subset \mathbb{R}^2, \\ &\psi_n = 0 \text{ on } |x| = 1. \end{aligned} \tag{2.28}$$

Hence, μ_n satisfies the variational principle

$$\mu_n = -4n + \inf_{\psi \in H_0^1(B)} \frac{\int |\nabla\psi|^2 + 4n^2 \int |x|^2|\psi|^2}{\int |\psi|^2}.$$

In addition, we have $\mu_n > 0$, as follows from the maximum principle applied to (2.27) For a test function of the form $\psi(x) = e^{-n|x|^2}q(x)$, where $0 \leq q(x) \leq 1$, $q(x) = 1$ for $0 \leq |x| \leq 1/2$ and $q(x) = 0$ for $|x| \geq 3/4$ we obtain by a straightforward computation

$$\frac{\int |\nabla\psi|^2 + 4n^2 \int |x|^2|\psi|^2}{\int |\psi|^2} = 8n^2\frac{\int |x|^2|\psi|^2}{\int |\psi|^2} + O(e^{-cn}) = 8n^2\frac{\int_{\mathbb{R}^2} |x|^2 e^{-2n|x|^2}}{\int_{\mathbb{R}^2} e^{-2n|x|^2}} + O(e^{-cn}) = 4n + O(e^{-cn}),$$

with $c > 0$. Therefore, $\mu_n \to 0$ as $n \to +\infty$ and hence $\lambda_n^* \to 0$ as well.

3 The strong flow asymptotics

In this section we consider the elliptic problem (1.3) when the advecting flow is strong. Accordingly, we introduce a large parameter $A \gg 1$ and re-write (1.3) as

$$\begin{aligned} &-\Delta\phi_\lambda + Au \cdot \nabla\phi_\lambda = \lambda g(\phi_\lambda) \text{ in } \Omega, \\ &\phi_\lambda = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.1}$$

We are interested in the behavior of the solution $\phi(x)$ of (3.1) for large A, as well as in the dependence of the explosion threshold λ^* on the amplitude A. With a slight abuse of notation we will denote here by $\lambda^*(A)$ the explosion threshold of the problem (3.1).

3.1 Equidistribution on the flow streamlines

Our first result shows that, when the flow is strong, solution becomes nearly constant on the flow streamlines, at least in an average sense and for λ away from $\lambda^*(A)$. This is a common phenomenon in diffusion-advection problems: a strong flow induces stratification.

Proposition 3.1 *Assume in that $u \cdot n = 0$ on the boundary $\partial\Omega$. Then the solution ϕ_λ of (1.3) is nearly constant on the streamlines of u for sufficiently large A in the sense that for any $\delta > 0$ there exists $C(\delta)$ so that*

$$\int |u \cdot \nabla \phi_\lambda|^2 \leq \frac{C(\delta)\lambda}{A},$$

for all $\lambda \leq (1-\delta)\lambda^*(A)$.

Proof. First, we multiply (3.1) by $u \cdot \nabla \phi_\lambda$ and integrate over Ω. The emerging integrals over the boundary vanish since $\phi_\lambda = u \cdot n = 0$ on $\partial\Omega$, and we obtain the following estimate:

$$\int |u \cdot \nabla \phi_\lambda|^2 = \frac{1}{A}\int (u \cdot \nabla \phi_\lambda)[\Delta \phi_\lambda + \lambda g(\phi_\lambda)] = \frac{1}{A}\int (u \cdot \nabla \phi_\lambda)\Delta \phi_\lambda = -\frac{1}{A}\int_\Omega \frac{\partial \phi_\lambda}{\partial x_k}\frac{\partial}{\partial x_k}\left[u_j \frac{\partial \phi_\lambda}{\partial x_j}\right]$$

$$= -\frac{1}{A}\int_\Omega \frac{\partial \phi_\lambda}{\partial x_k}\frac{\partial u_j}{\partial x_k}\frac{\partial \phi_\lambda}{\partial x_j} - \frac{1}{A}\int_\Omega \frac{\partial \phi_\lambda}{\partial x_k} u_j \frac{\partial^2 \phi_\lambda}{\partial x_j \partial x_k} = -\frac{1}{A}\int_\Omega \frac{\partial \phi_\lambda}{\partial x_k}\frac{\partial u_j}{\partial x_k}\frac{\partial \phi_\lambda}{\partial x_j} \leq \frac{C}{A}\int |\nabla \phi_\lambda|^2.$$

This means that the variation along the streamlines is smaller than across. Now, we have to bound the L^2-norm of $\nabla \phi_\lambda$. However, multiplying (3.1) by ϕ_λ and integrating by parts we see that

$$\int |\nabla \phi_\lambda|^2 = \lambda \int g(\phi_\lambda)\phi_\lambda.$$

Moreover, Proposition 2.6 shows that ϕ_λ satisfies a uniform bound $0 \leq \phi_\lambda \leq C(\delta)$ as long as $\lambda \in (0, (1-\delta)\lambda^*)$. Therefore, for such λ we know that

$$\int |\nabla \phi|^2 \leq C\lambda, \tag{3.2}$$

for all $A > 0$. It follows that

$$\int |u \cdot \nabla \phi|^2 \leq \frac{C\lambda}{A}.$$

In that sense solution becomes uniform over the streamlines. □

As a consequence, for each fixed $\lambda < \limsup_{A\to\infty} \lambda^*(A)$ we know that

$$\int |u \cdot \nabla \phi_\lambda|^2 \to 0, \text{ as } A \to +\infty.$$

We will improve this statement for two-dimensional cellular flows in Section 3.3.

An interesting by-product of the estimate (3.2) is that there are no boundary or internal layers in this problem unlike in the problems with boundary forcing in cellular flows [9, 13, 22, 27, 31, 33, 35, 37]. The reason is that ϕ_λ is set to be constant on the boundary and the normal component of the flow vanishes on the boundary – hence, there is no "conflict" between the boundary data and uniformization along the streamlines of the flow.

3.2 The critical parameter in the limit of a strong flow

We prove here Theorem 1.4. We recall that the assumption that there is no $H_0^1(\Omega)$ first integral $H(x)$ such that $u \cdot \nabla H = 0$ almost everywhere is equivalent to the fact that the principal Dirichlet eigenvalue $\mu_1(A)$ of the operator $-\Delta + A \cdot \nabla$ on Ω tends to infinity as $A \to +\infty$ [4].

First, assume that $\mu_1(A)$ is bounded as $A \to +\infty$. Then the upper bound in (2.14) for $\lambda^*(A)$ implies that $\limsup_{A\to+\infty} \lambda^*(A)$ is finite as well.

Next, we show that if u has no first integral in $H_0^1(\Omega)$ then $\lambda^*(A) \to +\infty$ as $A \to +\infty$. The proof is based on the lower bound for $\lambda^*(A)$ in (2.14). The next lemma is contained in [25] – we provide a proof in the spirit of [10].

Lemma 3.2 *We have $\theta_A \to 0$ as $A \to +\infty$ if u has no first integrals in $H_0^1(\Omega)$.*

Proof. As in the proof of Theorem 1.2 we represent the function $\tau_A(x)$ using the Duhamel formula as

$$\tau_A(x) = \int_0^\infty \psi(t,x)dt,$$

with the function $\psi(t,x)$ that solves the parabolic problem

$$\begin{aligned} &\psi_t - \Delta\psi + Au \cdot \nabla\psi = 0 \text{ in } \Omega, \\ &\psi(0,x) = 1, \\ &\psi = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.3}$$

It follows from the Theorem 5.3 of [10] that if u has no first integrals in $H_0^1(\Omega)$ then for any $t_0 > 0$ we can find a flow amplitude $A_0(t_0) > 0$ so that $\|\psi(nt_0)\|_{L^\infty(\Omega)} \le 2^{-n}|\Omega|$ for all $A \ge A_0$. Therefore, we have an upper bound

$$|\tau_A(x)| \le t_0|\Omega| \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2t_0|\Omega| \text{ for } A \ge A_0(t_0).$$

We conclude that $\|\tau_A\|_{L^\infty(\Omega)} \to 0$ as $A \to +\infty$. □

This also finishes the proof of Theorem 1.4. □

3.3 The explosion problem in a two-dimensional cellular flow

We now consider the explosion problem

$$\begin{aligned} &-\Delta\phi_A + Au \cdot \nabla\phi_A = \lambda f(\phi_A) \text{ in } \Omega, \\ &\phi_A = 0 \text{ on } \partial\Omega, \end{aligned} \tag{3.4}$$

in a two-dimensional simply connected domain Ω. The flow has the form $u = (\Psi_y, -\Psi_x)$ with a stream-function $\Psi(x,y)$ which we assume to be sufficiently smooth. We assume that the boundary of the domain Ω is a level set $\{\Psi = 0\}$ which may contain finitely many saddle critical points of the function Ψ – thus, the boundary is a union of streamlines of the flow away from the critical points. We also assume that inside Ω the flow has a cellular structure: the saddles of Ψ are all non-degenerate and are connected by the flow separatrices which divide Ω into a finite number of invariant regions, called the flow cells, that we will denote by $\mathcal{C}_j$. The stream-function $\Psi(x,y)$ has only one critical point (x_0, y_0) inside each of $\mathcal{C}_j$, which is a non-degenerate maximum or minimum. A prototype example of such flow has the stream-function $\Psi(x,y) = \sin\pi x \sin\pi y$ – its cells are squares $[n, n+1] \times [m, m+1]$ with integer m and n, and the domain Ω is a finite union of such squares. A more general flow of such type is depicted in Figure 3.1.

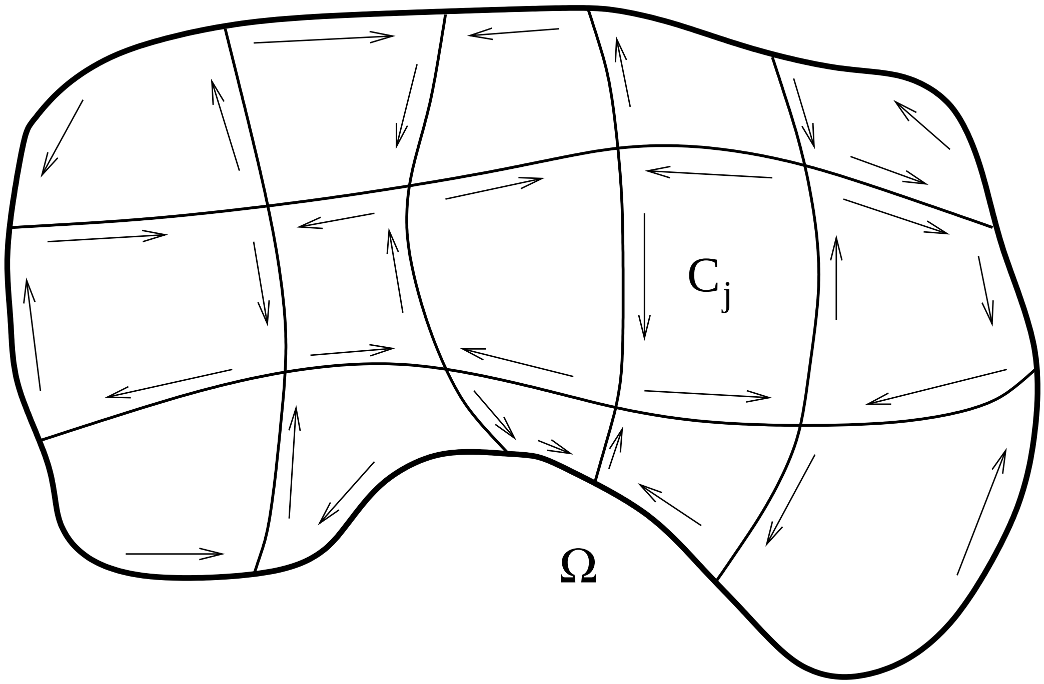

Figure 3.1: A schematic description of a cellular flow.

The Freidlin problem

The strong flow asymptotics for parabolic reaction-diffusion equations for two-dimensional flows with Morse class stream-functions has been considered in [16]. This class does not include the cellular flows under our consideration as we allow the stream-function to have several saddles on the level set $\{\Psi = 0\}$. Nevertheless, the limit problem of [16] is crucial in the explosion problem in a cellular flow. The limit problem in [16] was formulated as a system of reaction-diffusion equations on the Reeb graph of the function Ψ. We recall and re-derive these results below in the context of the explosion problem in the one-cell setting, as that is what we will need below. The single cell is also the situation addressed numerically in [5]. For a one-cell flow the Reeb graph is simply an interval $[0, H_0]$, where H_0 is the value of Ψ at the critical point inside the cell $\mathcal{C}$ that we assume to be a maximum, and $\{\Psi = 0\}$ is the boundary of the cell. We are interested in the behavior of solutions and of the explosion threshold in the limit of a large flow amplitude.

The effective Freidlin problem on the interval $0 \le h \le H_0$ is to find a function $\bar{\phi}(h)$ satisfying

$$-\frac{1}{T(h)}\frac{d}{dh}\left(p(h)\frac{d\bar{\phi}}{dh}\right) = \lambda g(\bar{\phi}), \qquad (3.5)$$
$$\bar{\phi}(0) = 0,\ \bar{\phi}'(h) \text{ is bounded for } 0 \le h \le H_0,$$

with the coefficients

$$T(h) = \oint_{\Psi(x,y)=h} \frac{dl}{|\nabla\Psi|}, \quad p(h) = \oint_{\Psi(x,y)=h} |\nabla\Psi| dl. \qquad (3.6)$$

Under our assumptions on the stream-function, the average turnover time $T(h)$ is bounded from above and below away from zero:

$$0 < T_0 \le T(h) \le T_1 |\ln h|. \qquad (3.7)$$

The uniform bound from below by T_0 in (3.7) comes from the fact that the maximum of $\Psi(x,y)$ is a non-degenerate critical point. The term $O(|\ln h|)$ for small $h > 0$ appears in (3.7) because the boundary may contain non-degenerate saddle points of Ψ so that the turnover time blows up as $h \downarrow 0$. The coefficient $p(h)$ is positive for $h > 0$ and behaves as $p(h) \sim C(H_0 - h)$ close to $h = H_0$. In particular we have $p(H_0) = 0$ (diffusivity vanishes at this point), while the drift satisfies

$$p'(h) = \oint_{\Psi(x,y)=h} \frac{\Delta\psi}{|\nabla\Psi|} dl \leq -\alpha_0, \quad \text{with } \alpha_0 > 0,$$

for h near H_0, and points away from $h = H_0$. Therefore, the end-point $h = H_0$ is inaccessible for the diffusion process corresponding to the left side of (3.5), and one does not need to prescribe the boundary condition at $h = H_0$ in order for (3.5) to be well-posed. The following proposition defines the explosion threshold for the effective problem.

Proposition 3.3 *There exists $\bar\lambda^* > 0$ so that a positive solution of the effective problem (3.5) exists for all $0 \leq \lambda < \bar\lambda^*$ and there is no positive solution of (3.5) for $\lambda > \bar\lambda^*$.*

Proof. The proof follows the same steps as in Section 2.1 – the only required modification is due to the degeneracies at $h = 0$ and $h = H_0$. This can be addressed using the general theory in [14] and [28] but in the present case the boundary value problem with a prescribed right side

$$-\frac{1}{T(h)}\frac{d}{dh}\left(p(h)\frac{d\psi}{dh}\right) = f(h) \tag{3.8}$$
$$\psi(0) = 0,\ \psi'(h) \text{ is bounded for } 0 \leq h \leq H_0,$$

has an explicit unique solution

$$\psi(h) = \int_0^h \frac{1}{p(s)}\left(\int_s^{H_0} f(\xi)T(\xi)d\xi\right) ds = \int_0^{H_0} f(\xi)T(\xi)P(\min(h,\xi))d\xi,$$

where

$$P(\xi) = \int_0^\xi \frac{ds}{p(s)}.$$

In particular, we have $|P(\xi)| \leq C|\ln(H_0 - \xi)|$, so that

$$|\psi(h)| \leq \int_0^{H_0} |f(\xi)|T(\xi)P(\min(h,\xi))d\xi \leq C\|f\|_\infty \int_0^{H_0} |\ln\xi||\ln(H_0-\xi)|d\xi \leq C\|f\|_\infty,$$

and we also have

$$|\psi'(h)| \leq \frac{1}{p(h)}\int_h^{H_0} |f(\xi)|T(\xi)d\xi \leq \frac{C\|f\|_\infty}{H_0 - h}\int_h^{H_0} |\ln\xi|d\xi \leq C\|f\|_\infty,$$

so that $\|\psi\|_{W^{1,\infty}} \leq C\|f\|_{L^\infty}$. Therefore, the mapping $f(h) \to \psi(h)$ is a compact map on $C[0,H_0]$ and the Krein-Rutman theory applies to the operator in the left side of (3.8). We may then repeat the proof of Proposition 1.1 essentially verbatim and conclude that the critical threshold $\bar\lambda$ for (3.5) exists. □

The explosion threshold for strong cellular flows: the main result

The main result of this section is the following theorem. We assume that the flow has a cellular structure and satisfies the assumptions outlined at the beginning of this section. Then for each cell $\mathcal{C}_j$ one may formulate the corresponding one-cell Freidlin problem (3.5) for a function $\bar{\phi}_j$, posed now on an interval $[0, H_j]$, where the outer boundary of $\mathcal{C}_j$ is the level set $\{\Psi = 0\}$ and H_j is the value of the function Ψ at the (unique) extremal point inside $\mathcal{C}_j$. For the Freidlin problem the Dirichlet boundary condition $\bar{\phi}(0) = 0$ is prescribed at $h = 0$, and the derivative $\bar{\phi}_j'(H_j)$ is imposed to be bounded. This defines the explosion threshold $\bar{\lambda}_j^*$ for each cell $\mathcal{C}_j$. The following theorem shows that in the limit of a large flow the explosion threshold for the whole domain Ω approaches the Freidlin explosion threshold for the "largest" cell $\mathcal{C}_j$.

Theorem 3.4 *Let $\lambda^*(A)$ be the explosion threshold for (3.4) and $\bar{\lambda}_j^*$ be the threshold for the aforementioned effective problem (3.5) posed in the cell $\mathcal{C}_j$ of the domain Ω. Then we have*

$$\lim_{A\to\infty} \lambda^*(A) = \min_j \bar{\lambda}_j^*.$$

A numerical illustration of the main result of Theorem 3.4 is depicted in Figure 3.2.

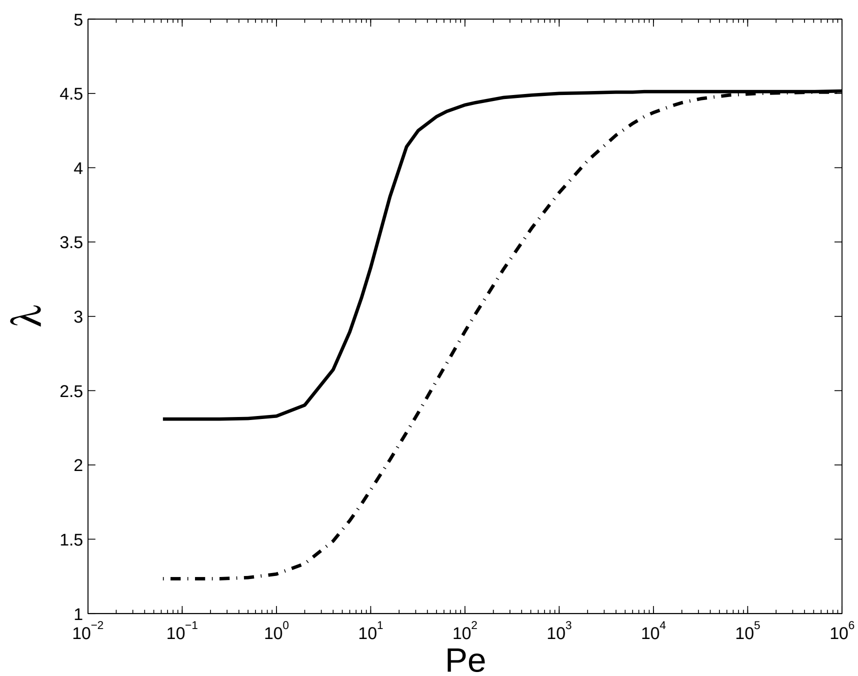

Figure 3.2: The value of λ^* for various values of the Péclet number $Pe = A$ for the cellular flow with the stream-function $\psi(x, y) = \sin(2\pi((2x/3 + 1)^3/8 + 1)) \sin(2\pi((y + 1)^2/4 + 1))$ on the the domain $[0, 2\pi] \times [0, 2\pi]$ with four cells. Dashed-dotted line: λ^* for the whole domain $[0, 2\pi] \times [0, 2\pi]$, solid line – the minimum of λ^* of the four individual cells.

The proof of Theorem 3.4 proceeds in several steps. First, we prove a stratification lemma for solutions of forced advection-diffusion problems in cellular flows. It shows that solution of the Dirichlet problem is small not only on the outer boundary but also on the whole skeleton of separatrices and cells "do not talk to each other". The second step is to establish the result of Theorem 3.4 for domains consisting of one cell where one just has to show that for one cell the explosion threshold converges in the strong flow limit to that of the Freidlin problem. The last step is to generalize this result to a domain consisting of finitely many cells.

3.4 Cellular flows: a stratification lemma

We begin the proof of Theorem 3.4 with the following lemma, of an independent interest. Let $\mathcal{D}_0$ be the union of all cell boundaries (separatrices of the flow) inside Ω including the outside boundary $\partial\Omega$. First, we show that solutions of a linear problem with the homogeneous Dirichlet data on the outer boundary are small on $\mathcal{D}_0$.

Lemma 3.5 *Let $\psi_A(x)$ be the exit time from Ω, solution of*

$$\begin{aligned} &-\Delta\psi_A + Au\cdot\nabla\psi_A = 1 \text{ in } \Omega, \\ &\psi_A = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.9}$$

For any $\delta > 0$ there exists $A_0 > 0$ so that for all $A > A_0$ we have $0 \le \psi_A(x) \le \delta$ for all $x \in \mathcal{D}_0$.

Intuitively, this lemma says that once a diffusive particle obeying an SDE

$$dX_t = Au(X_t)dt + \sqrt{2}dW_t,$$

comes close to the skeleton of separatrices, somewhere inside Ω, it exits the domain Ω after a short time. This is the phenomenon behind the effective diffusivity [9, 13, 22, 27, 31, 33, 35, 37], and front speed and principle eigenvalue enhancement [1, 26, 32, 34, 39] in cellular flows.

Proof. The proof is in two steps. First, we show that for any $\delta > 0$ there exists a small $h > 0$ and a large $A_0 > 0$ so that for any $A > A_0$ we can find $h_j(A)$ with $h \le |h_j| \le 2h$, such that $0 \le \psi_A(x) \le \delta/2$ for all x on the streamline $\{x \in \mathcal{C}_j : \ \Psi(x) = h_j(A)\}$ inside the cell $\mathcal{C}_j$. Let $D_j(A)$ be the interior of those streamlines. In the second step we consider the water-pipe domain $P_A = \Omega \setminus \left(\bigcup_j D_j(A)\right)$, that is, a narrow tube around the skeleton of separatrices. Using the fact that h_j is small, we apply the maximum principle for narrow domains to conclude that the function ψ_A is smaller than δ in all of P_A and not only on its boundary. As a consequence, ψ_A is small also on the skeleton $\mathcal{D}_0$.

Step 1. We have a uniform L^2-bound for the gradient:

$$\int_\Omega |\nabla\psi_A(x)|^2 dx \le C, \tag{3.10}$$

with the constant C independent of $A > 0$, which follows from multiplying (3.9) by $\psi_A(x)$ and integrating by parts, together with the uniform L^∞-bound for $\psi_A(x)$: $\|\psi_A\|_{L^\infty} \le C$, which follows from Lemma 1.3.

Now, take $s \in (0, h/4)$ and let $F_j(h,s)$ be the domain between the two streamlines $\{\Psi(x) = 5h/4 - s\}$ and $\{\Psi(x) = 7h/4 + s\}$ inside the cell $\mathcal{C}_j$. We multiply (3.9) by $(u\cdot\nabla\psi_A)$ and integrate over F_j:

$$\begin{aligned} &\int_{F_j} |u\cdot\nabla\psi_A|^2 = \frac{1}{A}\int_{F_j}(u\cdot\nabla\psi_A)\Delta\psi_A dx = \frac{1}{A}\int_{F_j}\sum_{m,k} u_m\frac{\partial\psi_A}{\partial x_m}\frac{\partial^2\psi_A}{\partial x_k^2}dx \\ &= \frac{1}{A}\int_{\partial F_j}(u\cdot\nabla\psi_A)(n\cdot\nabla\psi_A)dl - \frac{1}{A}\int_{F_j}\frac{\partial u_m}{\partial x_k}\frac{\partial\psi_A}{\partial x_m}\frac{\partial\psi_A}{\partial x_k}dx \le \frac{C}{A}\int_{\partial F_j}|\nabla\psi_A(x)|^2dx + \frac{C}{A}. \end{aligned}$$

We used (3.10) in the last step. Averaging this estimate in $s \in (0, h/4)$ we conclude that

$$\int_{\bar{F}_j} |u\cdot\nabla\psi_A|^2 dx \le \frac{C(h)}{A},$$

where $\bar{F}_j = F_j(h,0)$ is the domain between the streamlines $\{\Psi(x) = 5h/4\}$ and $\{\Psi(x) = 7h/4\}$ inside the cell $\mathcal{C}_j$. The constant $C(h)$ may blow-up as $h \downarrow 0$ but that is not important at the moment.

It follows that there exists a value $h_j(A) \in (5h/4, 7h/4)$ so that along the streamline $L_j(A) = \{x \in \mathcal{C}_j : \ \Psi(x) = h_j(A)\}$ we have

$$\oint_{L_j(A)} |u \cdot \nabla \psi_A|^2 dl \leq \frac{C(h)}{A},$$

with a new constant $C(h)$. Therefore, the oscillation of ψ_A along $L_j(A)$ is small:

$$\mathrm{osc}_{L_j(A)} \psi_A(x) \leq \frac{C(h)}{\sqrt{A}}.$$

Hence, $\psi_A(x)$ is close to a constant $M_j(A)$ on the streamline $L_j(A)$ when A is sufficiently large. As a consequence of the gradient bound (3.10), we have $|M_j(A) - M_m(A)| \leq C\sqrt{h}$ if the cells $\mathcal{C}_j$ and $\mathcal{C}_m$ have a common piece of the boundary. As the outer boundary $\partial\Omega$, where $\psi_A(x) = 0$, is also part of some cell boundaries, it follows that $0 \leq M_j(A) \leq C\sqrt{h}$ for all cells $\mathcal{C}_j$. Therefore, we have

$$0 \leq \psi_A(x) \leq C\sqrt{h} + \frac{C(h)}{\sqrt{A}} < \frac{\delta}{2}, \text{ for } x \in L_j(A)$$

if $h \in (0, h_0)$ is sufficiently small and $A > A_0$ is large enough.

Step 2. Now, we look at the water-pipe P_A and show that solution is below δ everywhere in P_A. The function ψ_A inside P_A satisfies $0 \leq \psi_A(x) \leq \delta/2 + r_A$, where r_A is the exit time from the slightly larger domain $Q_{2h} = \Omega \setminus \left(\bigcup_j D_j(2h)\right)$:

$$\begin{aligned} &-\Delta r_A + Au \cdot \nabla r_A = 1 \text{ in } Q_{2h}, \\ &r_A = 0 \text{ on } \partial Q_{2h}. \end{aligned} \tag{3.11}$$

Now, as in the proof of Lemma 1.3 we conclude that there exists a constant $C(h)$ such that $\|r_A\|_{L^\infty(Q_{2h})} \leq C(h)$ for all $A > 0$. The same proof shows that $C(h) \to 0$ as $h \to 0$ – this happens because the principal eigenvalue of the Dirichlet Laplacian in Q_{2h} tends to infinity as $h \to 0$ while the constants $K_p(h)$ in the Poincaré inequality $\|\psi\|_{L^p(Q_{2h})} \leq K_p(h) \|\nabla\psi\|_{L^2(Q_{2h})}$, $1 < p < \infty$, satisfy $K_p \to 0$ as $h \to 0$. It follows that if we take $h > 0$ sufficiently small (independent of A) then $0 \leq r_A(x) \leq \delta/2$ for all $A > 0$. Therefore, we have $0 \leq \psi_A(x) \leq \delta$ for all $A > A_0$ in P_A and, in particular, $0 \leq \psi_A(x) \leq \delta$ on $\mathcal{D}_0$. The proof of Lemma 3.5 is now complete. □

3.5 Explosion problem in a one-cell domain

We now consider the explosion problem

$$\begin{aligned} &-\Delta \phi_A + Au \cdot \nabla \phi_A = \lambda f(\phi_A) \text{ in } \Omega, \\ &\phi_A = 0 \text{ on } \partial\Omega, \end{aligned} \tag{3.12}$$

in a domain Ω which consists of just one flow cell. Without loss of generality we assume that the single critical point $(x_0, y_0) \in \Omega$ of Ψ inside Ω is a maximum and set $H_0 = \Psi(x_0, y_0)$. Let us now formulate the version of Theorem 3.4 for a one-cell domain.

Proposition 3.6 *Let Ω be a one-cell domain and let $\lambda^*(A)$ be the explosion threshold for (3.12) and $\bar{\lambda}^*$ be the threshold for the aforementioned effective problem (3.5) posed on $[0, H_0]$. Then we have*

$$\lim_{A \to \infty} \lambda^*(A) = \bar{\lambda}^*.$$

The proof of Proposition 3.6 is in two steps. First, passing from the problem on the cell to the Freidlin problem we show that the Fredlin threshold $\bar{\lambda}^*$ is not smaller than than $\limsup_{A\to+\infty}\lambda^*(A)$. Next, we establish the opposite inequality by starting with a solution to the Freidlin problem and constructing a super-solution for (3.12). The second step is quite straightforward in the case when the boundary of Ω contains no saddles of the flow u but is somewhat more technical if $\partial\Omega$ contains such fixed points.

Passage from the cell to the Freidlin problem

We first prove that

$$\limsup_{A\to\infty}\lambda^*(A)\le\bar{\lambda}^*. \tag{3.13}$$

Assume that

$$\lambda<\limsup_{A\to\infty}\lambda^*(A). \tag{3.14}$$

We will show that then $\lambda<\bar{\lambda}^*$ by constructing a solution to the Freidlin problem (3.5) as the limit of a sequence of problems on Ω. It follows from (3.14) that there exists $\delta>0$ and a sequence $A_n\to+\infty$ such that $\lambda<(1-\delta)\lambda^*(A_n)$. Therefore, as a consequence of Proposition 2.6, the minimal positive solutions of

$$\begin{aligned}&-\Delta\phi_n+A_nu\cdot\nabla\phi_n=\lambda f(\phi_n)\text{ in }\Omega,\\&\phi_n=0\text{ on }\partial\Omega\end{aligned} \tag{3.15}$$

are uniformly bounded in $L^\infty(\Omega)\cap H_0^1(\Omega)$:

$$0\le\phi_n\le C,\quad\int|\nabla\phi_n|^2dx\le C, \tag{3.16}$$

with the constant $C>0$ independent of n. Hence, the sequence ϕ_n converges weakly in $H^1(\Omega)$ (after extracting a subsequence) and strongly in $L^2(\Omega)$ to a function $\bar{\phi}$. As the functions ϕ_n are uniformly bounded and $g(\phi)$ is smooth, the sequence $g(\phi_n)$ converges to $g(\bar{\phi})$.

We claim that $\bar{\phi}$ depends only on the variable $h=\Psi(x,y)$ and satisfies the effective Freidlin problem (3.5). The first claim follows after we divide (3.15) by A_n and let $n\to+\infty$. This leads to

$$u\cdot\nabla\bar{\phi}=0 \tag{3.17}$$

in the sense of distributions. It is convenient now to introduce the curvilinear coordinates (h,θ). The coordinates are chosen so that $h(x,y)=\Psi(x,y)$, that is, the streamlines of the flow are $\{h=\text{const}\}$, and the level lines of the coordinate $\theta=\Theta(x,y)$ are orthogonal to the flow lines: $\nabla\Theta\cdot\nabla\Psi=0$. We normalize θ so that $0\le\theta\le2\pi$ and the boundary $\partial\Omega$ is a level set: $\partial\Omega=\{h=0\}$. Then (3.17) implies that $\bar{\phi}$ depends only on the variable h. The L^∞-bound in (3.16) implies that $0\le\bar{\phi}(h)\le C$. In addition, we have

$$\int|\nabla_x\bar{\phi}|^2dx=\int|\bar{\phi}_h|^2|\nabla h|^2dx=\int_0^{H_0}|\bar{\phi}_h|^2\left(\int_0^{2\pi}\frac{|\nabla\Psi|^2}{J}d\theta\right)dh.$$

Here $J=\Psi_y\Theta_x-\Psi_x\Theta_y$ is the Jacobian of the coordinate change. Note that $\nabla\Theta=\rho\nabla^\perp\Psi$ with some scalar function $\rho>0$, so that

$$J=\rho|\nabla\Psi|^2,\quad|\nabla\Theta|=\rho|\nabla\Psi|\text{ and }dl=d\theta/|\nabla\Theta|.$$

Therefore, we have

$$\int_0^{2\pi} \frac{|\nabla\Psi|^2}{J} d\theta = \oint_{\Psi(x,y)=h} |\nabla\Psi| dl = p(h), \tag{3.18}$$

and thus we have a weighted H^1-bound

$$\int_0^{H_0} p(h)|\bar{\phi}_h|^2 dh < +\infty,$$

which follows from (3.16), and hence $\bar{\phi}(h)$ is continuous for $h < H_0$, as $p(h) \sim C(H_0 - h)$ for h close to H_0.

Next, we re-write (3.15) in the curvilinear coordinates:

$$-\frac{|\nabla\Psi|^2}{J}\frac{\partial^2\phi_n}{\partial h^2} - \frac{|\nabla\Theta|^2}{J}\frac{\partial^2\phi_n}{\partial\theta^2} - \frac{(\Delta\Psi)}{J}\frac{\partial\phi_n}{\partial h} - \frac{(\Delta\Theta)}{J}\frac{\partial\phi_n}{\partial\theta} + A_n\frac{\partial\phi_n}{\partial\theta} = \frac{1}{J}\lambda g(\phi_n), \tag{3.19}$$
$$\phi_n(H_0,\theta) = 0,\ \ \phi_n(h,\theta) \text{ is bounded for } 0 \le h \le H_0.$$

Integrating this equation in θ and passing to the limit $n \to +\infty$ we obtain the limit problem for the function $\bar{\phi}$:

$$-a(h)\bar{\phi}''(h) - b(h)\bar{\phi}'(h) = \lambda c(h) g(\bar{\phi}), \tag{3.20}$$
$$\bar{\phi}(H_0) = 0, \quad \bar{\phi}(h) \text{ is bounded for } 0 \le h \le H_0,$$

with

$$a(h) = \int_0^{2\pi} \frac{|\nabla\Psi|^2}{J} d\theta, \quad b(h) = \int_0^{2\pi} \frac{\Delta\Psi}{J} d\theta, \quad c(h) = \int_0^{2\pi} \frac{d\theta}{J}.$$

It remains only to observe that (3.20) is nothing but the effective problem (3.5). Indeed, as in (3.18) we compute that

$$c(h) = \int_0^{2\pi} \frac{d\theta}{J} = \oint_{\Psi(x,y)=h} \frac{dl}{|\nabla\Psi|} = T(h),$$

and

$$b(h) = \int_0^{2\pi} \frac{\Delta\Psi}{J} d\theta = \oint_{\Psi(x,y)=h} \frac{\Delta\Psi}{|\nabla\Psi|} dl = p'(h).$$

The last equality above follows from the fact that

$$p(h) = \oint_{\Psi(x,y)=h} |\nabla\Psi| dl = \int_{G_h} \Delta\Psi dx dy.$$

Here $G_h = \{h \le \Psi(x,y) \le H_0\}$ is the interior of the streamline $\{\Psi(x,y) = h\}$. It follows that (3.20) is, indeed, the effective problem (3.5), so that $\bar{\phi}(h)$ is a positive solution of (3.5). Therefore, in particular, we have $\lambda \le \bar{\lambda}^*$ and (3.13) holds.

Subsolution: the case with no saddles on $\partial\Omega$

We now prove that

$$\liminf_{A\to\infty} \lambda^*(A) \ge \bar{\lambda}^*. \tag{3.21}$$

Together with (3.13) this will complete the proof of Proposition 3.6. This is done as follows: we take any $\lambda_0 < \bar{\lambda}^*$ and show that $\lambda_0 \le \lambda^*(A)$ for a sufficiently large A by constructing a bounded positive super-solution to (3.12) with $\lambda = \lambda_0$. However, the singular points of $\Psi(x,y)$ cause technical difficulties in the construction of the sub-solution. Hence, we first consider the special case when $\Psi(x,y)$ has no saddles on $\partial\Omega$.

Let $\lambda_0 < \bar{\lambda}^*$ and let $\bar{\phi}(h)$ be the corresponding positive solution of (3.5) with some $\lambda \in (\lambda_0, \bar{\lambda}^*)$. An important observation is that there exists $C < +\infty$ so that for all $(x,y) \neq (x_0, y_0)$ (that is, not the maximum of the stream-function $\Psi(x,y)$), we have

$$|\Delta_{x,y}\bar{\phi}(h(x,y))| \leq C, \quad (x,y) \neq (x_0, y_0). \tag{3.22}$$

To see that we write

$$\Delta_{x,y}\bar{\phi} = |\nabla\Psi|^2\bar{\phi}''(h(x,y)) + (\Delta\Psi)\bar{\phi}'(h(x,y))$$

and note that for h close to H_0 we have $|\nabla\Psi|^2 \sim (H_0 - h)$ and $\bar{\phi}''(h) \sim 1/(H - h_0)$ so that the first term above is uniformly bounded in $(x,y) \in \Omega$.

We look for a super-solution to (3.12) in the form

$$\phi = \bar{\phi} + \eta_A,$$

where η_A is smooth and bounded. Then the uniform bound (3.22) and similar bounds on other second derivatives of $\bar{\phi}(x,y)$ show that if we have

$$-\Delta\phi + Au \cdot \nabla\phi \geq \lambda_0 g(\phi), \quad (x,y) \neq (x_0, y_0), \tag{3.23}$$

in Ω and $\phi \geq 0$ on $\partial\Omega$, then $\phi(x,y)$ is a weak super-solution to (3.12) with $\lambda = \lambda_0$ in the sense of [6] and thus $\lambda_0 \leq \lambda^*(A)$. We choose $\eta_A = \eta_A(x,y)$ as the solution of

$$\begin{aligned} &-\Delta\eta + Au \cdot \nabla\eta = \Delta_{x,y}\bar{\phi} + \lambda g(\bar{\phi}), \\ &\eta(x) = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.24}$$

Assume that we can show that

$$||\eta_A||_{L^\infty} \to 0, \text{ as } A \to \infty, \tag{3.25}$$

then (3.23) holds:

$$-\Delta\phi + Au \cdot \nabla\phi = -\Delta\bar{\phi} - \Delta\eta_A + Au \cdot \nabla\eta_A = \lambda g(\bar{\phi}) \geq \lambda_0 g\left(\bar{\phi} + \eta_A\right) = \lambda_0 g(\phi),$$

for A sufficiently large. Above we used the fact that $\lambda_0 < \lambda$ and a uniform bound for $\bar{\phi}$. Hence, $\lambda_0 \leq \lambda^*(A)$ for A sufficiently large. This will prove Proposition 3.6 in the special case when the domain Ω consists of one cell and the boundary $\partial\Omega$ contains no saddles of the stream-function Ψ.

It remains to establish (3.25). To this end consider a cut-off function $\chi(s)$ so that $0 \leq \chi(s) \leq 1$ and $\chi(s) = 1$ for $|s| < 1/2$ and $\chi(s) = 0$ for $|s| > 1$ and split

$$\Delta_{x,y}\bar{\phi} + \lambda g(\bar{\phi}) = q_1 + q_2,$$

with

$$q_1 = \left(\Delta_{x,y}\bar{\phi} + \lambda g(\bar{\phi})\right)\chi\left(\frac{H_0 - h(x,y))}{\delta}\right), \quad q_2 = \left(\Delta_{x,y}\bar{\phi} + \lambda g(\bar{\phi})\right)\left[1 - \chi\left(\frac{H_0 - h(x,y))}{\delta}\right)\right].$$

The small parameter $\delta > 0$ is to be chosen below. We define, accordingly, the functions η_j, $j = 1, 2$ as solutions of

$$\begin{aligned} &-\Delta\eta_j + Au \cdot \nabla\eta_j = q_j, \\ &\eta_j(x) = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.26}$$

so that $\eta_A = \eta_1 + \eta_2$. As q_1 is uniformly bounded in $L^\infty(\Omega)$, the function η_1 can be bounded using Lemma 1.3 as

$$\|\eta_1\|_\infty \le C\delta^\alpha$$

with some $\alpha \in (0,1)$. We may further split the function $\eta_2 = \eta_3 + \eta_4$, with the function η_3 that solves the ODE along the closed streamlines:

$$Au \cdot \nabla \eta_3 = q_2, \quad \eta_3(\theta = 0, h) = 0.$$

This equation is solvable because

$$\oint_L \left[\Delta_{x,y}\bar{h} + \lambda g(\bar{h})\right] dl = 0,$$

as this is how the Freidlin problem is obtained. The function η_3 satisfies the estimate

$$\|\eta_3\|_{C^2(\Omega)} \le \frac{F_1(\delta)}{A}, \tag{3.27}$$

with some function $F_1(\delta)$ (which may tend to infinity as $\delta \downarrow 0$). Finally, η_4 satisfies

$$\begin{aligned} &-\Delta\eta_4 + Au\cdot\nabla\eta_4 = \Delta\eta_3, \\ &\eta_4(x) = -\eta_3(x) \quad \text{on } \partial\Omega. \end{aligned} \tag{3.28}$$

Once again, Lemma 1.3 together with the C^2 estimate (3.27) on η_3 above implies that

$$\|\eta_4\|_\infty \le \frac{CF_1(\delta)}{A}.$$

Altogether we see that for any $\varepsilon > 0$ we can find find $\delta > 0$, and then find A_0 so that for any $A > A_0$

$$\|\eta_A\|_\infty \le \varepsilon. \tag{3.29}$$

This proves Proposition 3.6 in the special case when the domain Ω consists of one cell and the boundary $\partial\Omega$ contains no saddles of the stream-function Ψ.

Approximation on a smaller domain

Now, we establish the claim of Proposition 3.6 for domains Ω which consist of one cell but may have saddles of the stream-function on the boundary $\partial\Omega$. In order to avoid dealing with the singular points on the boundary of Ω in the construction a sub-solution we need to consider a slightly smaller domain $\Omega_\varepsilon = \{\varepsilon \le \Psi(x,y) \le H_0\} \subset \Omega$, with $\varepsilon > 0$ small. The domain Ω_ε has no saddles on $\partial\Omega$ and thus the conclusion of Proposition 3.6 holds for Ω_ε by what we have shown above.

Define $\lambda^*_\varepsilon(A)$ as the explosion threshold for the problem in Ω_ε:

$$\begin{aligned} &-\Delta\phi^\varepsilon_A + Au\cdot\nabla\phi^\varepsilon_A = \lambda f(\phi^\varepsilon_A) \text{ in } \Omega_\varepsilon \\ &\phi^\varepsilon_A = 0 \text{ on } \partial\Omega_\varepsilon. \end{aligned} \tag{3.30}$$

We also let $\bar{\lambda}^*_\varepsilon$ be the explosion threshold for the corresponding Freidlin problem:

$$\begin{aligned} &-\frac{1}{T(h)}\frac{d}{dh}\left(p(h)\frac{d\bar{\phi}_\varepsilon}{dh}\right) = \lambda g(\bar{\phi}_\varepsilon) \\ &\bar{\phi}_\varepsilon(\varepsilon) = 0,\ \bar{\phi}'_\varepsilon(h) \text{ is bounded for } \varepsilon \le h \le H_0, \end{aligned} \tag{3.31}$$

with $T(h)$ and $p(h)$ still given by (3.6). As we have mentioned, since Ω_ε has no saddles of Ψ on $\partial\Omega_\varepsilon$, we have

$$\lim_{A\to+\infty} \lambda^*_\varepsilon(A) = \bar\lambda_\varepsilon, \qquad (3.32)$$

for all $\varepsilon > 0$, according to the what we have already shown above.

It is clear from the definition of $\lambda^*_\varepsilon(A)$ and $\bar\lambda^*_\varepsilon$ that $\lambda^*_\varepsilon(A) \ge \lambda^*(A)$ and $\bar\lambda^*_\varepsilon \ge \bar\lambda^*$. The next two lemmas show that the passage to the limit $\varepsilon \to 0$ is harmless. The first statement concerns the Freidlin thresholds.

Lemma 3.7 *We have* $\lim_{\varepsilon\to 0} \bar\lambda^*_\varepsilon = \bar\lambda^*$, *where* $\bar\lambda^*$ *and* $\bar\lambda^*_\varepsilon$ *are the explosion thresholds of (3.5) and (3.31).*

Proof. The proof of this lemma is rather straightforward. It is clear that $\bar\lambda^*_\varepsilon \ge \bar\lambda^*$ for all $\varepsilon > 0$. On the other hand, given $\gamma \in (0,1)$, for $\lambda < (1-\gamma)\bar\lambda^*_\varepsilon$ we can find $\delta > 0$ which depends only on γ so that solution of the following problem exists:

$$\begin{aligned} &-\frac{1}{T(h)}\frac{d}{dh}\left(p(h)\frac{d\bar\phi}{dh}\right) = \lambda g(\bar\phi) \\ &\bar\phi(\varepsilon) = \delta,\ \bar\phi'(h) \text{ is bounded for } \varepsilon \le h \le H_0, \end{aligned} \qquad (3.33)$$

for all $\varepsilon > 0$. Then it is easy to verify that, for a sufficiently small $\varepsilon > 0$ (and $\delta > 0$ fixed), solutions of the iteration process

$$\begin{aligned} &-\frac{1}{T(h)}\frac{d}{dh}\left(p(h)\frac{d\bar\phi_n}{dh}\right) = \lambda g(\bar\phi_{n-1}) \\ &\bar\phi_n(0) = 0,\ \bar\phi_n'(h) \text{ is bounded for } 0 \le h \le H_0, \end{aligned} \qquad (3.34)$$

with $\phi_0 = 0$ are increasing in n, and satisfy $\bar\phi_n(h) \le \delta$ for $0 \le h \le \varepsilon$ and $\bar\phi_n(x) \le \bar\phi$ for $\varepsilon \le h \le H_0$. Thus, the sequence $\bar\phi_n(h)$ converges as $n \to +\infty$ to a bounded solution of (3.5) so that $\lambda \le \bar\lambda$. □

The next lemma shows that $\lambda^*_\varepsilon(A)$ is close to $\lambda^*(A)$ for ε small.

Lemma 3.8 *For any* $\gamma > 0$ *there exists* $\varepsilon_0 > 0$ *and* A_0 *so that for all* $\varepsilon \in (0, \varepsilon_0)$ *and* $A > A_0$ *we have* $(1-\gamma/4)\lambda^*_\varepsilon(A) \le \lambda^*(A)$.

This lemma is sufficient to show that

$$\bar\lambda^* \le \liminf_{A\to+\infty} \lambda^*(A) \qquad (3.35)$$

and thus finish the proof of Proposition 3.6 for all one-cell domains, as we have already established (3.13). In order to see that (3.35) holds, take $\gamma \in (0,1)$ and find ε_0 and A_0 as in Lemma 3.8. Consider any $\lambda < (1-\gamma)\bar\lambda^*$. Then Lemma 3.7 implies that there exists $\varepsilon_1 < \varepsilon_0$ so that $\lambda < (1-\gamma/2)\bar\lambda^*_{\varepsilon_1}$. Now, (3.32) implies that we can find A_1 so that $\lambda < (1-\gamma/4)\lambda^*_{\varepsilon_1}(A)$ for all $A > A_1$. As $\varepsilon_1 < \varepsilon_0$ we may use Lemma 3.8 to conclude that $\lambda < \lambda^*(A)$ for all $A > A_0 + A_1$. Therefore, (3.35) holds. This finishes the proof of Proposition 3.6. □

The proof of Lemma 3.8

The proof of Lemma 3.8 is based on the iteration argument and stratification Lemma 3.5. We start with $\lambda < (1-\gamma)\lambda^*_\varepsilon(A)$ for all $A \ge A_0$, $\varepsilon \le \varepsilon_0$ and construct a solution of the explosion problem on the whole domain Ω by the iteration procedure. Set $\phi_0 = 0$ and let ϕ_n be the solution of

$$\begin{aligned} &-\Delta\phi_n + Au\cdot\nabla\phi_n = \lambda g(\phi_{n-1}) \text{ in } \Omega, \\ &\phi_n = 0 \text{ on } \partial\Omega. \end{aligned} \qquad (3.36)$$

We claim that the sequence $\phi_n(x)$ is increasing in n, pointwise in x, for each A and there exists A_0 so that for $A > A_0$ it has a uniformly bounded limit $\bar{\phi}(x) \in L^\infty(\Omega)$ which satisfies (3.12).

Pointwise monotonicity of $\phi_n(x)$ in n is standard: the difference $\eta_1(x) = \phi_1(x) - \phi_0(x) = \phi_1(x)$ satisfies

$$\begin{aligned} &-\Delta\eta_1 + Au\cdot\nabla\eta_1 = \lambda g(0) > 0 \text{ in } \Omega, \\ &\eta_1 = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.37}$$

Hence, we have $\eta_1(x) > 0$ in Ω and $\phi_1(x) > \phi_0(x)$. Let us assume that $\eta_n(x) = \phi_n(x) - \phi_{n-1}(x) \geq 0$ in Ω. As the nonlinearity $g(s)$ is increasing in s, the function $\eta_{n+1}(x)$ is the solution of

$$\begin{aligned} &-\Delta\eta_{n+1} + Au\cdot\nabla\eta_{n+1} = \lambda[g(\phi_n) - g(\phi_{n-1})] \geq 0 \text{ in } \Omega, \\ &\eta_1 = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.38}$$

Therefore, we have $\eta_{n+1}(x) > 0$ inside Ω and thus the sequence $\phi_n(x)$ is increasing in n.

We need to show that the sequence $\phi_n(x)$ is uniformly bounded from above. We recall that $\lambda < (1-\gamma)\lambda^*_\varepsilon(A)$ and thus the minimal positive solution of (3.30) satisfies $0 \leq \phi^\varepsilon_A \leq C(\gamma)$ with the constant $C(\gamma)$ independent ε and A. Hence, we may find $\delta > 0$ which depends only on $\gamma > 0$ but not on ε or A so that solution of the following problem exists:

$$\begin{aligned} &-\Delta\zeta + Au\cdot\nabla\zeta = \lambda g(\zeta) \text{ in } \Omega_\varepsilon, \\ &\zeta = \delta \text{ on } \partial\Omega_\varepsilon. \end{aligned} \tag{3.39}$$

To see that such $\delta > 0$ exists, let $\psi(x)$ satisfy (3.30) with $\lambda' = (1-\gamma/2)\lambda^*_\varepsilon(A)$ and set $r(x) = \psi(x)+\delta$, then $r(x)$ satisfies

$$\begin{aligned} &-\Delta r + Au\cdot\nabla r = \lambda' g(r-\delta) \text{ in } \Omega_\varepsilon \\ &r = \delta \text{ on } \partial\Omega_\varepsilon. \end{aligned} \tag{3.40}$$

It is a super-solution for (3.39) if we ensure that $\lambda' g(r-\delta) > \lambda g(r)$, or, equivalently, $\delta \in (0,1)$ is taken so small that we have for all $x \in \Omega$:

$$\frac{g(\psi(x))}{g(\psi(x)+\delta)} \geq \frac{1-\gamma}{1-\gamma/2}. \tag{3.41}$$

The function $\psi(x)$ obeys a uniform bound $\|\psi\|_{L^\infty(\Omega_\varepsilon)} \leq K(\gamma)$ with $K(\gamma)$ independent of ε and A. Let $M = \sup_{0\leq s\leq K(\gamma)+1} g'(s)$, then (3.41) is guaranteed if we have

$$\frac{g(s)}{g(s)+M\delta} \geq \frac{1-\gamma}{1-\gamma/2}$$

for all $s \in [0, K(\gamma)]$. A direct computation shows that this is possible if we take

$$\delta < \delta_0 = \frac{\gamma g(0)}{M(1-\gamma)}.$$

Under this assumption $r(x)$ provides a super-solution for (3.39) and thus a positive solution of this problem can be constructed by the standard iteration procedure.

In order to show that the sequence $\phi_m(x)$ is bounded we use $\zeta(x)$, the minimal positive solution of (3.39) and we need the following analog of the stratification Lemma 3.5.

Lemma 3.9 *Fix $M > 0$. Let $q_A(x)$ be solution of*

$$-\Delta q_A + Au \cdot \nabla q_A = M \text{ in } \Omega, \qquad (3.42)$$
$$q_A = 0 \text{ on } \partial\Omega.$$

Then, given any $\delta > 0$ there exist $A_0 > 0$ and $\varepsilon_0 > 0$ so that for all solutions of (3.42) we have $0 \le q_A(x) \le \delta$ in $G_\varepsilon = \Omega \setminus \Omega_\varepsilon$ for all $A > A_0$ and $\varepsilon < \varepsilon_0$.

We do not present the proof of this lemma as it is essentially contained in that of Lemma 3.5.

We choose $M > 0$ so that $\lambda g(\zeta(x)) \le M$ for all $x \in \Omega_\varepsilon$, where $\zeta(x)$ is the minimal positive solution of (3.39). We may also take $A > 0$ sufficiently large, as in Lemma 3.9. We claim that then we will have, for all $m \ge 1$, (i) $0 \le \phi_m(x) \le \delta$ in $G_\varepsilon = \Omega \setminus \Omega_\varepsilon$, and (ii) $0 \le \phi_m(x) \le \zeta(x)$ for all $x \in \Omega_\varepsilon$.

Let us prove this by induction. The function $\phi_1(x)$ satisfies

$$-\Delta\phi_1 + Au \cdot \nabla\phi_1 = \lambda g(0) \text{ in } \Omega \qquad (3.43)$$
$$\phi_1 = 0 \text{ on } \partial\Omega.$$

Our choice of M ensures that the right side in (3.43) is bounded above by M. Thus, if the amplitude A is sufficiently large we have $0 \le \phi_1(x) \le \delta$ in the tube G_ε and in particular on $\partial\Omega_\varepsilon$. Therefore, the difference $s_1(x) = \zeta(x) - \phi_1(x)$ satisfies

$$-\Delta s_1 + Au \cdot \nabla s_1 = \lambda[g(q_j) - g(0)] \ge 0 \text{ in } \Omega_\varepsilon, \qquad (3.44)$$
$$s_1 \ge 0 \text{ on } \partial\Omega_\varepsilon.$$

It follows that $s_1(x) \ge 0$ and $0 \le \phi_1(x) \le \zeta(x)$ in Ω_ε so that our claim holds for $n = 1$. Assume now that (i) and (ii) hold for $\phi_{n-1}(x)$. It follows from the induction assumption that

$$\lambda g(\phi_{n-1}(x)) \le \lambda g(\zeta(x)) \le M \text{ in } \Omega_\varepsilon,$$

and $\lambda g(\phi_{n-1}(x)) \le \lambda g(\delta) \le M$ in G_ε. Therefore, for $\phi_n(x)$ we have

$$-\Delta\phi_n + Au \cdot \nabla\phi_n = \lambda g(\phi_{n-1}) \le M, \text{ in } \Omega \qquad (3.45)$$
$$\phi_n = 0 \text{ on } \partial\Omega.$$

Hence, by Lemma 3.9 we have $0 \le \phi_m(x) \le \delta$ in G_ε. On the other hand, the difference $s_m(x) = \zeta(x) - \phi_m(x)$ inside $\partial\Omega_\varepsilon$ obeys

$$-\Delta s_m + Au \cdot \nabla s_m = \lambda[g(\zeta) - g(\phi_{m-1})] \ge 0 \text{ in } \Omega_\varepsilon, \qquad (3.46)$$
$$s_m \ge 0 \text{ on } \partial\Omega_\varepsilon,$$

and thus $s_m(x) \ge 0$, so that $0 \le \phi_m(x) \le \zeta(x)$ in Ω_ε. Therefore, the sequence ϕ_m is increasing and uniformly bounded from above. The limit $\bar\phi$ is a positive solution of (3.12) and thus $\lambda \le \lambda^*(A)$. This finishes the proof of Lemma 3.8. □

3.6 General two-dimensional cellular flows

We now look at the explosion problem

$$-\Delta\phi^A + Au \cdot \nabla\phi^A = \lambda g(\phi^A) \text{ in } \Omega, \qquad (3.47)$$
$$\phi^A = 0 \text{ on } \partial\Omega,$$

in a two-dimensional domain $\Omega \subset \mathbb{R}^2$ with a cellular flow u which may contain more than one cell in Ω and complete the proof of Theorem 3.4. Let $\lambda_j^*(A)$ be the explosion threshold for the problem inside each cell:

$$-\Delta\phi + Au \cdot \nabla\phi = \lambda g(\phi) \text{ in } \mathcal{C}_j, \qquad (3.48)$$
$$\phi^A = 0 \text{ on } \mathcal{C}_j.$$

We already know that

$$\lim_{A\to+\infty} \lambda_j^*(A) = \bar{\lambda}_j^*,$$

from Proposition 3.6 and, of course, $\lambda_j^*(A) \geq \lambda^*(A)$ for all j. Hence, all we need to verify for the proof of Theorem 3.4 is that for any $\lambda < \lim_{A\to+\infty} \lambda_j^*(A)$ solution of the problem (3.47) on the whole domain Ω exists.

The proof is quite similar to the last part of the proof of Proposition 3.6: we construct the solution of (3.47) by the iteration procedure. Set

$$\lambda_0 = \min_j \left[\lim_{A\to+\infty} \lambda_j^*(A)\right]$$

and take $\gamma > 0$ fixed. Consider any $\lambda \in (0, (1-\gamma)\lambda_0)$, start the iteration process with $\phi_0 = 0$ and define ϕ_m as the solution of

$$-\Delta\phi_m^A + Au \cdot \nabla\phi_m^A = \lambda g(\phi_{m-1}^A) \text{ in } \Omega, \qquad (3.49)$$
$$\phi_m^A = 0 \text{ on } \partial\Omega.$$

Lemma 3.10 *The sequence $\phi_m^A(x)$ is increasing in m, pointwise in x, for each $A > A_0$ and has a uniformly bounded limit $\bar{\phi}_A(x) \in L^\infty(\Omega)$ which is a solution of (3.47).*

The proof of Lemma 3.10.

The proof is quite analogous to that of Lemma 3.8 except that we use Lemma 3.5 where Lemma 3.9 was used before. The first increment $\eta_1(x) = \phi_1^A(x) - \phi_0^A(x) = \phi_1^A(x)$ satisfies

$$-\Delta\eta_1 + Au \cdot \nabla\eta_1 = \lambda g(0) > 0 \text{ in } \Omega, \qquad (3.50)$$
$$\eta_1 = 0 \text{ on } \partial\Omega.$$

Hence, we have $\eta_1(x) > 0$ in Ω and $\phi_1(x) > \phi_0(x)$. Let us assume that $\eta_n(x) = \phi_n^A(x) - \phi_{n-1}^A(x) \geq 0$ in Ω. As the nonlinearity $g(s)$ is increasing in s, the function $\eta_{n+1}(x)$ is the solution of

$$-\Delta\eta_{n+1} + Au \cdot \nabla\eta_{n+1} = \lambda[g(\phi_n) - g(\phi_{n-1})] \geq 0 \text{ in } \Omega, \qquad (3.51)$$
$$\eta_1 = 0 \text{ on } \partial\Omega.$$

Therefore, we have $\eta_{n+1}(x) > 0$ inside Ω and thus the sequence $\phi_n^A(x)$ is increasing in n. We need to show that it is uniformly bounded from above. As in the proof of Lemma 3.8, we can find $\delta > 0$ so that solution of the following explosion problem exists on each cell $\mathcal{C}_j$ for all $A > A_0$, $\delta \in (0, \delta_0)$ and $\lambda < (1-\gamma)\lambda_0$:

$$-\Delta q_j + Au \cdot \nabla q_j = \lambda g(q_j) \text{ in } \mathcal{C}_j \qquad (3.52)$$
$$q_j = \delta \text{ on } \partial\mathcal{C}_j,$$

and, moreover, $0 \le q_j(x) \le K(\gamma)$ with the constant $K(\gamma)$ which does not depend on the flow amplitude A. This is shown by exactly the same argument we used to show the existence of a positive solution for (3.39).

As $q_j(x)$ are uniformly bounded by $K_j(\gamma)$, we may choose $M > 0$ so that $\lambda g(q_j(x)) \le M$ for all j and all $x \in \Omega$. We also take $A > 0$ sufficiently large, as in Lemma 3.5 (but with the right side of (3.9) replaced by the constant M rather than 1). We claim that the following bounds will be preserved by the iteration procedure: for all $m \ge 1$, (i) $0 \le \phi_m(x) \le \delta$ on the skeleton of separatrices $\mathcal{D}_0$, and (ii) $0 \le \phi_m(x) \le q_j(x)$ for all $x \in \mathcal{C}_j$. Again, we check this by induction. The function $\phi_1^A(x)$ satisfies

$$\begin{aligned} &-\Delta\phi_1^A + Au\cdot\nabla\phi_1^A = \lambda g(0), \text{ in } \Omega, \\ &\phi_1^A = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.53}$$

The right side in (3.53) is bounded above by M and hence, according to Lemma 3.5, for the amplitude A sufficiently large we have $0 \le \phi_1^A(x) \le \delta$ on the skeleton $\mathcal{D}_0$. Therefore, the first increment $s_1(x) = q_j(x) - \phi_1(x)$ satisfies

$$\begin{aligned} &-\Delta s_1 + Au\cdot\nabla s_1 = \lambda[g(q_j) - g(0)] \ge 0 \text{ in } \mathcal{C}_j, \\ &s_1 \ge 0 \text{ on } \partial\mathcal{C}_j, \end{aligned} \tag{3.54}$$

and thus $s_1(x) \ge 0$ and $0 \le \phi_1^A(x) \le q_j(x)$ in $\mathcal{C}_j$. Let us now assume that (i) and (ii) are true for $\phi_{m-1}(x)$ and show that they hold for $\phi_m(x)$ – the argument is exactly as for $m = 1$. By the induction assumption we have

$$\lambda g(\phi_{m-1}(x)) \le \lambda g(q_j(x)) \le M \text{ in the cell } \mathcal{C}_j,$$

and thus $\phi_m(x)$ satisfies

$$\begin{aligned} &-\Delta\phi_m^A + Au\cdot\nabla\phi_m^A = \lambda g(\phi_{m-1}^A) \le \lambda g(q_j(x)) \le M, \text{ in } \Omega, \\ &\phi_m^A = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.55}$$

Hence, by Lemma 3.5 we have $0 \le \phi_m^\varepsilon(x) \le \delta$ on $\mathcal{D}_0$. Now, the difference $s_m(x) = q_j(x) - \phi_m(x)$ inside $\mathcal{C}_j$ satisfies

$$\begin{aligned} &-\Delta s_m + Au\cdot\nabla s_m = \lambda[g(q_j) - g(\phi_{m-1}^\varepsilon)] \ge 0 \text{ in } \mathcal{C}_j, \\ &s_m \ge 0 \text{ on } \partial\mathcal{C}_j, \end{aligned} \tag{3.56}$$

and thus $s_m(x) \ge 0$, so that $0 \le \phi_m^A(x) \le q_j(x)$ in $\mathcal{C}_j$. Therefore, the sequence ϕ_m^A is increasing and uniformly bounded from above. The limit $\bar\phi^A(x)$ is a positive solution of (3.47). This completes the proof of Lemma 3.10 and thus that of Theorem 3.4. □

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