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GLOBAL REGULARITY FOR 1D EULERIAN DYNAMICS WITH SINGULAR INTERACTION FORCES*

ALEXANDER KISELEV† AND CHANGHUI TAN‡

Abstract. The Euler–Poisson-alignment (EPA) system appears in mathematical biology and is used to model, in a hydrodynamic limit, a set of agents interacting through mutual attraction/repulsion as well as alignment forces. We consider one-dimensional EPA system with a class of singular alignment terms as well as natural attraction/repulsion terms. The singularity of the alignment kernel produces an interesting effect regularizing the solutions of the equation and leading to global regularity for wide range of initial data. This was recently observed in [Do et al., *Arch. Ration. Mech. Anal.*, 228 (2018), pp. 1–37]. Our goal in this paper is to generalize the result and to incorporate the attractive/repulsive potential. We prove that global regularity persists for these more general models.

Key words. Eulerian dynamics, singular interactions, nonlinear dissipation, Euler–Poisson-alignment system, global regularity

AMS subject classifications. 35Q35, 35Q92

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1. Introduction and statement of main results.

We consider the following one-dimensional (1D) system of pressureless Euler equations with nonlocal interaction forces

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

$$\partial_t u + u\partial_x u = \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t))\rho(y,t)dy - \partial_x K \star \rho, \tag{1.2}$$

subject to initial density and velocity

$$(\rho(\cdot,t), u(\cdot,t))|_{t=0} = (\rho_0, u_0). \tag{1.3}$$

The term on the right-hand side of (1.2) consists of two parts: an *alignment* interaction with communication weight ψ and an *attraction-repulsion* interaction through a potential K.

1.1. Self-organized dynamics with three-zone interactions.

System (1.1)–(1.2) arises from many contexts in mathematical physics and biology. In particular, it serves as a macroscopic system in modeling collective behaviors of complex biological systems. The corresponding agent-based model has the form

$$\dot{x}_i = v_i, \quad m\dot{v}_i = \frac{1}{N}\sum_{j=1}^N \psi(x_i - x_j)(v_j - v_i) - \frac{1}{N}\sum_{j=1}^N \nabla_{x_i} K(x_i - x_j), \tag{1.4}$$

where $(x_i, v_i)_{i=1}^N$ represent the position and velocity of agent i. The dynamics is gov-

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†Department of Mathematics, Duke University, Durham, NC 27708 (kiselev@math.duke.edu).
‡Department of Mathematics, University of South Carolina, Columbia, SC 29208 (tan@math.sc.edu).

erned by Newton's second law, with the interaction force modeled under a celebrated "three-zone" framework proposed in [17], including long-range attraction, short-range repulsion, and midrange alignment.

The first part of the force describes the alignment interaction, where ψ characterizes the strength of the velocity alignment between two agents. Naturally, it is a decreasing function of the distance between agents. Such alignment force has been proposed by Cucker and Smale in [4]. The corresponding dynamics enjoys the flocking property [9], which is a common phenomenon observed in animal groups.

The second part of the force represents the attraction-repulsion interaction. The sign of the force $-\nabla K$ determines whether the interaction is attractive or repulsive. This type of potential driven interaction force is widely considered in many physical and biological models, e.g., [6, 15].

Starting from the agent-based model (1.4), one can derive a kinetic representation of the system that describes the mean-field behavior as $N \to \infty$; see [2, 10, 21]. Then, a variety of hydrodynamics limits can be obtained that capture the macroscopic behaviors in different regimes [8, 11, 16]. In particular, if we consider the mono-kinetic regime, the corresponding macroscopic system becomes (1.1)–(1.2).

1.2. Global regularity versus finite time blowup. We are interested in the global existence and regularity for the solution of the system (1.1)–(1.2).

Let us start with the case with no interaction forces, namely, $\psi = K \equiv 0$. The system can be recognized as the pressureless Euler system. In particular, (1.2) becomes the classical inviscid Burgers equation, where smooth data forms shock discontinuity in finite time due to nonlinear convection $u\partial_x u$. Together with (1.1), it is well-known that the solution generates singular shocks in finite time: $\rho(x,t) \to \infty$ at the position and time when shock occurs.

With alignment force $\psi \geq 0$ and $K \equiv 0$, the system is called the *Euler-alignment system*. When ψ is Lipschitz, the system has been studied in [1, 20], where it is discovered that the alignment force tends to regularize the solution and prevent finite time blowup, but only for some initial data. This is the so-called *critical threshold phenomenon*: for subcritical initial data, the alignment force beats the nonlinear convection, and the solution is globally regular; while for supercritical initial data, the convection wins and the solution admits a finite time blowup.

Another interesting and natural setting is when ψ is singular, taking the form

$$\psi(x) = \frac{c_\alpha}{|x|^{1+\alpha}}, \quad \alpha > 0, \tag{1.5}$$

with c_α be a positive constant. The range $0 < \alpha \leq 2$ is most natural, and the case $0 < \alpha < 1$ is most interesting for the reasons explained later in this subsection. The Euler-alignment system corresponding to the choice (1.5) is studied in [5] for the periodic case.

Without loss of generality, we set the scale and let $\mathbb{T} = [-1/2, 1/2]$ be the periodic domain of size 1. The singular alignment force can be equivalently expressed as

$$\int_{\mathbb{T}} \psi_\alpha(y)(u(x+y,t) - u(x,t))\rho(x+y,t)dy \tag{1.6}$$

with the periodic influence function ψ_α defined as

$$\psi_\alpha(x) = \sum_{m\in\mathbb{Z}} \frac{c_\alpha}{|x+m|^{1+\alpha}} \quad \forall\, x \in \mathbb{T}\backslash\{0\}. \tag{1.7}$$

Clearly, ψ_α is singular at $x = 0$. Moreover, it has a positive lower bound

$$\psi_m = \psi_m(\alpha) := \min_{x\in\mathbb{T}} \psi_\alpha(x) = \psi_\alpha\left(\frac{1}{2}\right) > 0. \tag{1.8}$$

This leads to the following fractional Euler-alignment system

$$\partial_t\rho + \partial_x(\rho u) = 0, \qquad x \in \mathbb{T}, \tag{1.9}$$

$$\partial_t u + u\partial_x u = \int_{\mathbb{T}} \psi_\alpha(y)(u(x+y,t) - u(x,t))\rho(x+y,t)dy. \tag{1.10}$$

It is shown in [5] that system (1.9)–(1.10) has a global smooth solution for all smooth initial data with $\rho_0 > 0$. This result is most interesting for the case $0 < \alpha < 1$: if we set $\rho \equiv 1$ in (1.10), we get a Burgers equation with fractional dissipation. It is well-known that in this case, there exist initial data leading to finite time blowup (when $0 < \alpha < 1$; the $1 \le \alpha \le 2$ range leads to global regularity). However, it turns out that in the nonlinear dissipation/alignment case described by (1.10), the singularity in the influence function and density modulation dominate the nonlinear convection for all initial data. This also contrasts with the case of Lipschitz regular influence function ψ, where one has critical threshold in the phase space separating initial data leading to finite time blowup and to global regularity.

1.3. The Euler–Poisson-alignment system. Now, we take into account the attraction-repulsion force, namely, $K \not\equiv 0$. We shall begin with a particular potential

$$\mathcal{N}(x) = \frac{k|x|}{2}. \tag{1.11}$$

The potential is the 1*D Newtonian potential*, and it is the kernel for the 1D Poisson equation, namely,

$$\partial_{xx}^2 \mathcal{N} \star \rho = k\rho.$$

When $k > 0$, the Newtonian force $\partial_{xx}^2\mathcal{N} \star \rho$ is attractive, and when $k < 0$, the Newtonian force is repulsive. We call the corresponding system the Euler–Poisson-alignment (EPA) system. It has the form

$$\partial_t\rho + \partial_x(\rho u) = 0, \tag{1.12}$$

$$\partial_t u + u\partial_x u = -\partial_x\phi + \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t))\rho(y,t)dy, \tag{1.13}$$

where the stream function $\phi = \mathcal{N} \star \rho$ satisfies the Poisson equation

$$\partial_{xx}^2\phi = k\rho. \tag{1.14}$$

When there is no alignment force $\psi \equiv 0$, the system coincides with the 1D pressureless Euler–Poisson equation, which has been extensively studied in [7]. The result is as follows: when $k > 0$, the attraction force together with convection drives the solution of the Euler–Poisson equation to a finite time blowup for all smooth initial data; when $k < 0$, the repulsive force competes with the convection, and there exists a critical threshold on initial conditions which separates global regularity and finite time blowup.

The EPA system (1.12)–(1.14) is studied in [1], in the case when ψ is Lipschitz. When $k < 0$, a larger subcritical region of initial data is obtained that ensures global regularity. This implies that the alignment force helps repulsive potential to compete with the convection. However, it is also shown that when $k > 0$, the alignment force is too weak to compete with convection and attractive potential, so all smooth initial data lead to finite time blowup.

Our first result concerns EPA system with singular alignment force, where the influence function has the form (1.5). The main goal is to understand whether the singular alignment can still regularize the solution when the Newtonian force is present.

We shall study the system in the periodic setting. The 1D periodic Newtonian potential reads

$$\mathcal{N}(x) = -\frac{k}{2}\left(\frac{1}{2} - |x|\right)^2 \quad \forall\, x \in \mathbb{T}. \tag{1.15}$$

It is the kernel of the Poisson equation with background, namely,

$$\partial_x^2(\mathcal{N} * \rho) = k(\rho - \bar{\rho}),$$

where $\bar{\rho}$ is the average density

$$\bar{\rho} = \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} \rho(x,t)dx = \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} \rho_0(x)dx. \tag{1.16}$$

Note that $\bar{\rho}$ is conserved in time due to conservation of mass by evolution. The stream function ϕ in (1.13) satisfies the Poisson equation with constant background

$$\partial_{xx}^2\phi = k(\rho - \bar{\rho}). \tag{1.17}$$

The presence of the background $\bar{\rho}$ could change the behavior of the solution. For the Euler–Poisson equation in periodic domain, namely, (1.12)–(1.13), (1.17) with $\psi \equiv 0$, it is pointed out in [7] that the background has the tendency to balance both the convection and attractive forces. So for the attractive case $k > 0$, instead of finite time blowup for all initial data, a critical threshold is obtained.

Through similar techniques as in [1], one can derive critical thresholds for the EPA system (1.12)–(1.13), (1.17) with bounded Lipschitz influence function ψ.

The EPA system with singular alignment force (1.6) and potential (1.15) reads

$$\partial_t\rho + \partial_x(\rho u) = 0, \tag{1.18}$$

$$\partial_t u + u\partial_x u = -\partial_x\phi + \int_{\mathbb{T}} \psi_\alpha(y)(u(x+y,t) - u(x,t))\rho(x+y,t)dy, \tag{1.19}$$

$$\partial_{xx}^2\phi = k(\rho - \bar{\rho}). \tag{1.20}$$

The following theorem shows that the singular alignment force dominates the Poisson force, and global regularity is obtained for all initial data.

THEOREM 1.1 (global regularity). *For $\alpha \in (0,1)$, the fractional EPA system* (1.18)–(1.20) *with smooth periodic initial data (ρ_0, u_0) such that $\rho_0 > 0$ has a unique smooth solution.*

Remark 1.2. The proof can be extended to the range $\alpha \geq 1$ through similar arguments. Such a scenario has also been studied in [18], through a different approach. We focus on the $0 < \alpha < 1$ case in the rest of the paper.

We note that the proof of global regularity in [5] is based, in particular, on rather precise algebraic structures that we will discuss below. Even though the interaction force we are adding is formally subcritical, it is far from obvious that the fairly intricate arguments of [5] survive such perturbation.

1.4. Euler dynamics with general three-zone interactions. The results on the EPA system can be extended to systems with more general interaction forces.

In [1], critical thresholds are obtained for the system (1.1)–(1.2), with Lipschitz influence function ψ and regular potential $K \in W^{2,\infty}$.

In this paper, we will also consider the case of more general singular influence function ψ. More precisely, we assume that $\psi \geq \psi_m > 0$, and ψ can be decomposed into two parts

$$\psi = c\psi_\alpha + \psi_L, \tag{1.21}$$

where $c > 0$, ψ_α is defined in (1.7), and ψ_L is a bounded Lipschitz function.

THEOREM 1.3. *Consider system* (1.1)–(1.2) *in the periodic setup*

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad x \in \mathbb{T}, \quad t > 0, \tag{1.22}$$

$$\partial_t u + u\partial_x u = \int_{\mathbb{T}} \psi(y)(u(x+y,t) - u(x,t))\rho(x+y,t)dy - \partial_x K \star \rho \tag{1.23}$$

with smooth initial data (ρ_0, u_0) *such that* $\rho_0 > 0$. *Assume* ψ *is singular in the sense of* (1.21)*, and* K *is a linear combination of Newtonian potential* (1.15) *and regular* $W^{2,\infty}(\mathbb{T})$ *potential.*

Then, the system has a unique global smooth solution.

We summarize the global behaviors of Euler equations with nonlocal interaction (1.1)–(1.2) under different choices of interaction forces.

Potential	Alignment	Name	Domain	Behaviors
No	No	Euler	$\mathbb{R}$ or $\mathbb{T}$	Finite time blowup
	Lipschitz	Euler-alignment	$\mathbb{R}$ or $\mathbb{T}$	Critical threshold [1, 20]
	Singular	Fractional EA	$\mathbb{T}$	Global regularity [5]
Newtonian	No	Euler–Poisson	$\mathbb{R}$	Finite time blowup [7]
			$\mathbb{T}$	Critical threshold [7]
	Lipschitz	EPA	$\mathbb{R}$	Finite time blowup (attractive) Critical threshold (repulsive) [1]
			$\mathbb{T}$	Critical threshold
	Singular	Fractional EPA	$\mathbb{T}$	Global regularity (Theorem 1.1)
General	Lipschitz	Euler-3Zone	$\mathbb{R}$ or $\mathbb{T}$	Critical thresholds [1]
	Singular	Singular 3Zone	$\mathbb{T}$	Global regularity (Theorem 1.3)

2. The Euler–Poisson-alignment system. In this section, we consider EPA system (1.18)–(1.20) with singular alignment force (1.6).

Following the idea in [5], we let

$$G = \partial_x u - \Lambda^\alpha \rho, \tag{2.1}$$

where $\Lambda^\alpha \rho$ is the fractional Laplacian operator, defined as

$$\Lambda^\alpha \rho = c_\alpha \int_{\mathbb{R}} \frac{\rho(x) - \rho(y)}{|x-y|^{1+\alpha}} dy \quad \text{with} \quad c_\alpha = \frac{2^\alpha \Gamma((\alpha+1)/2)}{\sqrt{\pi}|\Gamma(-\alpha/2)|}.$$

We calculate the dynamics of G using (1.18) and (1.19):

$$\begin{aligned}\partial_t G = \partial_t\partial_x u - \partial_t\Lambda^\alpha\rho &= -\partial_x(u\partial_x u) - k(\rho-\bar\rho) + \partial_x\left(-\Lambda^\alpha(\rho u) + u\Lambda^\alpha\rho\right) + \Lambda^\alpha\partial_x(\rho u)\\ &= -u\partial_x(\partial_x u - \Lambda^\alpha\rho) - \partial_x u(\partial_x u - \Lambda^\alpha\rho) - k(\rho-\bar\rho) = -\partial_x(Gu) - k(\rho-\bar\rho).\end{aligned}$$

So, we can rewrite the dynamics in terms of (ρ, G) as

$$\partial_t\rho + \partial_x(\rho u) = 0, \tag{2.2}$$

$$\partial_t G + \partial_x(Gu) = -k(\rho-\bar\rho), \tag{2.3}$$

$$\partial_x u = \Lambda^\alpha\rho + G. \tag{2.4}$$

The velocity u can be recovered as

$$u(x,t) = \Lambda^\alpha\partial_x^{-1}(\rho(x,t)-\bar\rho) + \partial_x^{-1}G(x,t) + I_0(t), \tag{2.5}$$

where I_0 can be determined by conservation of momentum.

$$\int_{\mathbb{T}}\rho(x,t)u(x,t)dx = \int_{\mathbb{T}}\rho_0(x)u_0(x)dx. \tag{2.6}$$

See [5] for a detailed discussion.

2.1. A priori bounds. We first show an upper and lower bounds on density ρ for all finite times. For $k = 0$, a uniform in time bound is obtained in [5]. With the Newtonian potential, especially when $k > 0$, the attractive force definitely helps density concentration. Hence, the upper bound on ρ can be expected to grow in time. However, the bound we obtain in this section indicates that there is no finite time singular concentration on density, thanks to the singular alignment force.

Let $F = G/\rho$. We can rewrite (2.2) as

$$(\partial_t + u\partial_x)\rho = -\rho\Lambda^\alpha\rho - \rho^2 F. \tag{2.7}$$

The first step is to obtain a bound on F. We calculate

$$\partial_t F = \frac{\rho\partial_t G - G\partial_t\rho}{\rho^2} = \frac{\rho(-\partial_x(Gu) - k(\rho-\bar\rho)) - G(-\partial_x(\rho u))}{\rho^2} = -u\partial_x F - \frac{k(\rho-\bar\rho)}{\rho}.$$

This implies that

$$(\partial_t + u\partial_x)F = -k\left(1 - \frac{\bar\rho}{\rho}\right). \tag{2.8}$$

Denote $X(x,t)$ the trajectory of the characteristic path starting at x, namely,

$$\frac{d}{dt}X(x,t) = u(X(x,t),t), \quad X(x,0) = x. \tag{2.9}$$

Then, we can solve for F along the characteristic path

$$F(X(x,t),t) = F_0(x) - kt + \int_0^t \frac{k\bar\rho}{\rho(X(x,s),s)}ds. \tag{2.10}$$

Define $\rho_m(t)$ as the lower bound of ρ on time interval $[0,t]$

$$\rho_m(t) = \min_{s\in[0,T]}\min_{x\in\mathbb{T}}\rho(x,s). \tag{2.11}$$

Then, we get a bound on F from (2.10):

$$\|F(\cdot,t)\|_{L^\infty} \leq \|F_0\|_{L^\infty} + |k|t + |k|\bar{\rho}\int_0^t \frac{1}{\rho_m(s)}ds. \tag{2.12}$$

Therefore, in order to control F in L^∞, we need a lower bound estimate on the density.

THEOREM 2.1 (lower bound on density). *Let (ρ,u) be a strong solution to EPA system* (1.18)–(1.20) *with smooth periodic initial conditions (ρ_0,u_0) such that $\rho_m(0) > 0$. Then, there exist two positive constants A_m and C_m, depending only on the initial conditions, such that for any $t \geq 0$,*

$$\rho_m(t) \geq C_m e^{-A_m t}. \tag{2.13}$$

Proof. We depart from (2.7) and estimate $\Lambda^\alpha\rho$ and F. For a fixed time t, denote $\underline{x}$ a point where ρ attains its minimum. Note that $\underline{x}$ depends on t and it is not necessarily unique. The estimates below apply at any such point. We have

$$\begin{aligned}-\Lambda^\alpha\rho(\underline{x},t) &= c_\alpha\int_{-\infty}^{\infty}\frac{\rho(\underline{x}+y,t)-\rho(\underline{x},t)}{|y|^{1+\alpha}}dy = \int_{\mathbb{T}}\psi_\alpha(y)\big(\rho(\underline{x}+y,t)-\rho(\underline{x},t)\big)dy\\ &\geq \psi_m\int_{\mathbb{T}}\big(\rho(\underline{x}+y,t)-\rho(\underline{x},t)\big)dy = \psi_m\big(\bar{\rho}-\rho(\underline{x},t)\big).\end{aligned} \tag{2.14}$$

Here, we recall that ψ_m is the positive lower bound of ψ_α defined in (1.8).

Combining (2.12) and (2.14), we obtain

$$\partial_t\rho(\underline{x},t) \geq \big(\psi_m\bar{\rho}\big)\rho(\underline{x},t) - \left[\psi_m + \|F_0\|_{L^\infty} + |k|t + |k|\bar{\rho}\int_0^t\frac{1}{\rho_m(s)}ds\right]\rho(\underline{x},t)^2. \tag{2.15}$$

We prove (2.13) by contradiction. For $t=0$, the bound (2.13) holds if we let $C_m \leq \rho_m(0)$. Suppose (2.13) does not hold for all $t \geq 0$. Then, there exists a finite time $t_0 > 0$ so that the inequality is violated for the first time at $t = t_0+$. Pick any $\underline{x} = \underline{x}(t_0)$. Due to continuity of ρ, we know

$$\rho_m(t_0) = \rho(\underline{x},t_0) = C_m e^{-A_m t_0}. \tag{2.16}$$

Plug in (2.16) to (2.15) and use the fact that (2.13) holds for all $t \in [0,t_0]$. We get

$$\begin{aligned}\partial_t\rho(\underline{x},t_0) \geq\ & \rho_m(t_0)\left[\big(\psi_m\bar{\rho}\big) - \left(\psi_m + \|F_0\|_{L^\infty} + |k|t + |k|\bar{\rho}\int_0^{t_0}\frac{1}{\rho_m(s)}ds\right)\rho_m(t_0)\right]\\ \geq\ & \rho_m(t_0)\left[\big(\psi_m\bar{\rho}\big) - \left(\psi_m + \|F_0\|_{L^\infty} + |k|t_0 + \frac{|k|\bar{\rho}}{A_mC_m}(e^{A_mt_0}-1)\right)C_me^{-A_mt_0}\right]\\ \geq\ & \rho_m(t_0)\left[\left(\psi_m\bar{\rho} - \frac{|k|\bar{\rho}}{A_m}\right) - \left(\psi_m + \|F_0\|_{L^\infty} + |k|t_0 - \frac{|k|\bar{\rho}}{A_mC_m}\right)C_me^{-A_mt_0}\right]\\ \geq\ & \rho_m(t_0)\left[\left(\psi_m\bar{\rho} - \frac{|k|\bar{\rho}}{A_m} - \frac{|k|C_m}{eA_m}\right) + \left(\frac{|k|\bar{\rho}}{A_m} - C_m(\psi_m + \|F_0\|_{L^\infty})\right)e^{-A_mt_0}\right].\end{aligned}$$

The right-hand side is positive if we pick A_m large enough and C_m small enough. For instance, we can pick

$$A_m = \frac{|k|}{\psi_m}(1+\epsilon), \quad C_m = \min\{\rho_m(0), \epsilon e\bar{\rho}\} \tag{2.17}$$

for any $\epsilon \in (0, \epsilon_*)$, where $\epsilon_* = \frac{1}{2}(\sqrt{1 + \frac{4\psi_m}{e(\psi_m + \|F_0\|_{L^\infty})}} - 1)$. With this choice of A_m and C_m, we get $\partial_t \rho(\underline{x}, t_0) > 0$.

Now we obtain that $\rho(\underline{x}) < C_m e^{-A_m t_0} < C_m e^{-A_m t}$ for some $t < t_0$. This contradicts our choice of t_0. ∎

Remark 2.2. The bound (2.13) with decay rate (2.17) is not necessarily sharp, but is enough for our purpose, as it eliminates the possibility of finite time creation of a vacuum. One important observation is that for $k = 0$, we get $A_m = 0$. In this case, the lower bound is uniform in time.

Applying the lower bound (2.13) to (2.12), we immediately derive a bound on F

$$\|F(\cdot, t)\|_{L^\infty} \leq \|F_0\|_{L^\infty} + |k|t + \frac{|k|\bar{\rho}}{A_m C_m} e^{A_m t} =: F_M(t). \tag{2.18}$$

Now, we are ready to obtain an upper bound on density ρ.

THEOREM 2.3 (upper bound on density). *Let (ρ, u) be a strong solution to EPA system* (1.18)–(1.19) *with smooth periodic initial conditions (ρ_0, u_0) such that $\rho_m(0) > 0$. Then, there exist two positive constants A_M and C_M, depending only on the initial conditions, such that for any $t \geq 0$ and $x \in \mathbb{T}$,*

$$\rho(x, t) \leq \rho_M(t) := C_M e^{A_M t}. \tag{2.19}$$

Proof. We again depart from (2.7) and start with a lower bound estimate on $\Lambda^\alpha \rho$. For a fixed time t, denote $\bar{x}$ a point where ρ attains its maximum. Applying the nonlinear maximum principle by Constantin and Vicol [3], one can estimate

$$\Lambda^\alpha \rho(\bar{x}, t) \geq C_1 \rho(\bar{x}, t)^{1+\alpha} \tag{2.20}$$

if $\rho(\bar{x}, t) \geq 3\bar{\rho}$. The constant C_1 only depends on initial conditions. One can consult [5] for more details of the estimate.

Plugging the estimates (2.18) and (2.20) into (2.7), we obtain

$$\partial_t \rho(\bar{x}, t) \leq -C_1 \rho(\bar{x}, t)^{2+\alpha} + F_M(t) \rho(\bar{x}, t)^2. \tag{2.21}$$

It follows that $\partial_t \rho(\bar{x}, t) < 0$ if $\rho(\bar{x}, t) > (F_M/C_1)^{1/\alpha}$. Therefore,

$$\rho(x, t) \leq \rho(\bar{x}, t) \leq \max\left\{ \|\rho_0\|_{L^\infty}, 3\bar{\rho}, \left(\frac{F_M(t)}{C_1}\right)^{1/\alpha} \right\}, \tag{2.22}$$

and (2.19) holds with

$$A_M = \frac{A_m}{\alpha},\ C_M = \max\left\{ \max_{x \in \mathbb{T}} \rho_0(x), 3\bar{\rho}, \left[\frac{1}{C_1}\left(\|F_0\|_{L^\infty} + \frac{|k|}{eA_m} + \frac{|k|\bar{\rho}}{A_m C_m}\right)\right]^{1/\alpha} \right\}. \quad ∎$$

2.2. Local wellposedness. With the a priori bounds, we state a local wellposedness result for the fractional EPA system (1.18)-(1.19), as well as a Beale–Kato–Majda type necessary and sufficient condition to guarantee global wellposedness. The local wellposedness theory has been presented in detail in [5] for the fractional Euler-alignment system. We will show that the presence of the Poisson force does not seriously affect the argument, no matter whether it is attractive or repulsive. We will only sketch the proof, indicating changes necessary.

THEOREM 2.4 (local wellposedness). *Consider EPA system* (1.18)–(1.19) *with initial conditions* ρ_0 *and* u_0 *that satisfy*

$$\rho_0 \in H^s(\mathbb{T}), \quad \min_{x\in\mathbb{T}} \rho_0(x) > 0, \quad \partial_x u_0 - \Lambda^\alpha \rho_0 \in H^{s-\frac{\alpha}{2}}(\mathbb{T}) \tag{2.23}$$

with a sufficiently large even integer $s > 0$. *Then, there exists* $T_0 > 0$ *such that the EPA system has a unique strong solution* $\rho(x,t), u(x,t)$ *on* $[0,T_0]$ *with*

$$\rho \in C([0,T_0], H^s(\mathbb{T})) \cap L^2([0,T_0], H^{s+\frac{\alpha}{2}}(\mathbb{T})), \quad u \in C([0,T_0], H^{s+1-\alpha}(\mathbb{T})). \tag{2.24}$$

Moreover, a necessary and sufficient condition for the solution to exist on a time interval $[0,T]$ *is*

$$\int_0^T \|\partial_x \rho(\cdot,t)\|_{L^\infty}^2 dt < \infty. \tag{2.25}$$

Proof. We follow the proof in [5] and rewrite (2.2) and (2.3) in terms of (θ, G), where $\theta = \rho - \bar\rho$.

$$\partial_t \theta + \partial_x(\theta u) = -\bar\rho \partial_x u, \tag{2.26}$$

$$\partial_t G + \partial_x(G u) = -k\theta. \tag{2.27}$$

The velocity u is defined in (2.5).

Given any $T > 0$, we will obtain a differential inequality on

$$Y(t) := 1 + \|\theta(\cdot,t)\|_{H^s}^2 + \|G(\cdot,t)\|_{H^{s-\frac{\alpha}{2}}}^2 \tag{2.28}$$

for all $t \in [0,T]$.

Through a commutator estimate [5, equation (3.23)], one can get

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{H^s}^2 \le C\left(1 + \frac{1}{\rho_m}\right)(1 + \|\partial_x\theta\|_{L^\infty}^2 + \|G\|_{L^\infty})Y(t) - \frac{\rho_m}{3}\|\theta\|_{H^{s+\frac{\alpha}{2}}}^2, \tag{2.29}$$

where $\rho_m(t)$ has a positive lower bound for $t \in [0,T]$ due to Theorem 2.1. Also, $\|G(\cdot,t)\|_{L^\infty}$ is bounded for $t \in [0,T]$ as $G = F\rho$ and both F and ρ are bounded; see (2.18) and (2.19), respectively.

We also compute

$$\frac{1}{2}\frac{d}{dt}\|G\|_{\dot H^{s-\frac{\alpha}{2}}}^2 = -\int_{\mathbb{T}} (\Lambda^{s-\frac{\alpha}{2}} G)\cdot(\Lambda^{s-\frac{\alpha}{2}}\partial_x(Gu))dx - k\int_{\mathbb{T}} (\Lambda^{s-\frac{\alpha}{2}} G)\cdot(\Lambda^{s-\frac{\alpha}{2}}\theta)dx = \mathrm{I} + \mathrm{II}. \tag{2.30}$$

The first term can be controlled by the following estimate [5, equation (3.25)]:

$$|\mathrm{I}| \le \frac{\rho_m}{6}\|\theta\|_{H^{s+\frac{\alpha}{2}}}^2 + C\left(1 + \frac{1}{\rho_m}\|G\|_{L^\infty}^2 + \|\partial_x\theta\|_{L^\infty} + \|G\|_{L^\infty}\right)\|G\|_{H^{s-\frac{\alpha}{2}}}^2. \tag{2.31}$$

The II term encodes the contribution of the attractive-repulsive potential. We have the following estimate:

$$|\mathrm{II}| \le |k|\|G\|_{\dot H^{s-\frac{\alpha}{2}}}\|\theta\|_{\dot H^{s-\frac{\alpha}{2}}} \le C|k|\|G\|_{\dot H^{s-\frac{\alpha}{2}}}\|\theta\|_{H^s} \le C|k|Y(t). \tag{2.32}$$

Combine (2.29), (2.31), and (2.32), we get

$$\frac{d}{dt}Y(t) \le C(1 + \|\partial_x\theta(\cdot,t)\|_{L^\infty}^2)Y(t) - \frac{\rho_m(t)}{6}\|\theta\|_{H^{s+\frac{\alpha}{2}}}^2, \tag{2.33}$$

where C is a positive constant which might depend on T.

Applying Gronwall's inequality, we get

(2.34)
$$Y(t)+\frac{1}{6}\min_{t\in[0,T]}\rho_m(t)\|\theta\|^2_{L^2([0,T];H^{s+\frac{\alpha}{2}}(\mathbb{T}))}\leq Y(0)\exp\left[C(T)\int_0^T(1+\|\partial_x\theta(\cdot,s)\|^2_{L^\infty})ds\right]$$

for all $t\in[0,T]$. The right-hand side is bounded as long as condition (2.25) is satisfied. Therefore,

$$\theta\in C([0,T],H^s(\mathbb{T}))\cap L^2([0,T],H^{s+\frac{\alpha}{2}}(\mathbb{T})),\quad G\in C([0,T],H^{s-\frac{\alpha}{2}}(\mathbb{T})).$$

This directly implies the regularity conditions on ρ in (2.24). The regularity conditions on u can also be easily obtained from (2.5). ☐

2.3. Global wellposedness. In this section, we prove that the Beale–Kato–Majda type condition (2.25) holds for any finite time T. This will imply global wellposedness of the fractional EPA system and hence finish the proof of Theorem 1.1. Throughout the section, we fix a time $T>0$ (which is arbitrary).

To derive a uniform L^∞ bound on $\partial_x\rho$, we argue that $\rho(\cdot,t)$ will obey a certain modulus of continuity for $t\in[0,T]$. Such a method has been successfully used to obtain global regularity for a 2D quasi-geostrophic equation with critical dissipation [14], fractal Burgers equation [13], as well as fractional Euler-alignment system [5]. In all these examples, the solution has a certain scaling invariance property. Unfortunately, such a property is not available for the fractional EPA system (1.18)–(1.19). We note that the modulus method has been applied to subcritical perturbations destroying scaling before (e.g., [12]). The argument in [12], however, relies on the specific structure of the perturbation, and cannot be readily ported to other settings. A novel feature compared to both [12] and [5] will be dependence of the modulus on time. This feature is linked to the possible decay of ρ_m and growth of $\|\rho\|_{L^\infty}$ and appears to be an intrinsic property of the problem.

We use the same family of moduli of continuity as in [5],

$$\omega(\xi)=\begin{cases}\xi-\xi^{1+\alpha/2}, & 0\leq\xi<\delta\leq 1,\\ \gamma\log(\xi/\delta)+\delta-\delta^{1+\alpha/2}, & \xi\geq\delta,\end{cases}\tag{2.35}$$

where γ,δ are small constants to be determined. Set $\omega_B(\xi)=\omega(B\xi)$, where B is a large constant to be determined as well. Due to lack of scaling invariance, we will work directly on ω_B.

$$\omega_B(\xi)=\begin{cases}B\xi-(B\xi)^{1+\alpha/2}, & 0\leq\xi<B^{-1}\delta,\\ \gamma\log\frac{B\xi}{\delta}+\delta-\delta^{1+\alpha/2}, & \xi\geq B^{-1}\delta.\end{cases}\tag{2.36}$$

We say that a function f obeys modulus of continuity ω if

$$|f(x)-f(y)|<\omega(|x-y|)\quad\forall\ x,y\in\mathbb{T}.\tag{2.37}$$

Our plan is to find a ω_B such that $\rho(\cdot,t)$ obeys ω_B for all $t\in[0,T]$. To construct ω_B, we will first choose δ and γ which depend on initial conditions and T, but not on B. Then, we will choose B that depend on T,δ,γ as well as initial conditions.

First, we would like to make sure that ρ_0 obeys ω_B.

LEMMA 2.5. *Let $\rho_0\in C^1(\mathbb{T})$. Then, ρ_0 obeys ω_B if*

$$\delta<\frac{2\|\rho_0\|_{L^\infty}}{\|\partial_x\rho_0\|_{L^\infty}},\quad B>\frac{\delta\|\partial_x\rho_0\|_{L^\infty}}{2\|\rho_0\|_{L^\infty}}\exp\left(\frac{2\|\rho_0\|_{L^\infty}}{\gamma}\right).\tag{2.38}$$

Proof. We start with an elementary inequality

$$|\rho_0(x) - \rho_0(y)| \leq \min\{2\|\rho_0\|_{L^\infty}, \|\partial_x\rho_0\|_{L^\infty}|x-y|\}. \tag{2.39}$$

As ω_B is concave and monotone increasing, the right-hand side of (2.39) is bounded by $\omega_B(|x-y|)$ if

$$\omega_B\left(\frac{2\|\rho_0\|_{L^\infty}}{\|\partial_x\rho_0\|_{L^\infty}}\right) > 2\|\rho_0\|_{L^\infty}. \tag{2.40}$$

Since $\omega_B(\xi) \to +\infty$ as $B \to +\infty$, (2.40) is satisfied by taking B large enough. Indeed, if δ and B satisfy (2.38), then

$$\omega_B\left(\frac{2\|\rho_0\|_{L^\infty}}{\|\partial_x\rho_0\|_{L^\infty}}\right) > \gamma\log\left(\frac{2B\|\rho_0\|_{L^\infty}}{\delta\|\partial_x\rho_0\|_{L^\infty}}\right) > 2\|\rho_0\|_{L^\infty}. \qquad \square$$

The following lemma describes the only possible breakthrough scenario for the modulus.

LEMMA 2.6. *Suppose ρ_0 obeys a modulus of continuity ω_B as in* (2.36)*. If the solution $\rho(x,t)$ violates ω_B at some positive time, then there must exist $t_1 > 0$ and $x_1 \neq y_1$ such that*

$$\rho(x_1,t_1)-\rho(y_1,t_1) = \omega_B(|x_1-y_1|), \text{ and } \rho(\cdot,t) \text{ obeys } \omega_B \text{ for every } 0 \leq t < t_1. \tag{2.41}$$

The main point of the lemma is the existence of two distinct points where the solution touches the modulus (as opposed to a single point x with $|\partial_x\rho(x)| = \omega_B'(0) = B$). This property is a consequence of $\omega_B''(0) = -\infty$; see [14] for more details.

We will show that in the breakthrough scenario as above,

$$\partial_t(\rho(x_1,t_1) - \rho(y_1,t_1)) < 0 \quad \forall\ t_1 \in (0,T], \tag{2.42}$$

achieving a contradiction with the choice of time t_1—and thus showing that the modulus ω_B cannot be broken. Together with Lemma 2.5 this implies that $\rho(\cdot,t)$ obeys ω_B for all $t \in [0,T]$. Therefore,

$$\|\partial_x\rho(\cdot,t)\|_{L^\infty} \leq \omega_B'(0) = B \quad \forall\ t \in [0,T]. \tag{2.43}$$

This proves global regularity of the fractional EPA system and ends the proof of Theorem 1.1.

The rest of the section is devoted to the proof of (2.42). We fix t_1 and drop the time variable for simplicity. Let $\xi = |x_1 - y_1|$. Then

$$\begin{aligned}
\partial_t(\rho(x_1) - \rho(y_1)) &= -\,\partial_x(\rho(x_1)u(x_1)) + \partial_x(\rho(y_1)u(y_1)) \\
&= -\,\big(u(x_1)\partial_x\rho(x_1) - u(y_1)\partial_x\rho(y_1)\big) - \big(\rho(x_1)-\rho(y_1)\big)\partial_x u(x_1) \\
&\quad -\rho(y_1)\big(\partial_x u(x_1) - \partial_x u(y_1)\big) \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{2.44}$$

Decompose u into two parts $u = u_1 + u_2$, where

$$u_1(x) = \Lambda^\alpha\partial_x^{-1}(\rho(x) - \bar\rho), \quad u_2(x) = \partial_x^{-1}G(x) + I_0. \tag{2.45}$$

Then, we can write (2.44) as

$$\partial_t(\rho(x_1) - \rho(y_1)) = \mathrm{I}_1 + \mathrm{II}_1 + \mathrm{III}_1 + \mathrm{I}_2 + \mathrm{II}_2 + \mathrm{III}_2, \tag{2.46}$$

where $\mathrm{I}_1, \mathrm{II}_1, \mathrm{III}_1$ represent the contributions from u_1, and $\mathrm{I}_2, \mathrm{II}_2, \mathrm{III}_2$ represent the contribution from u_2.

2.3.1. Estimates on $\mathrm{I}_1, \mathrm{II}_1$, and III_1. We proceed with an argument parallel to [5]. Let us recall the result. The following quantities play a role in the proof:

$$\Omega(\xi) = c_{1,\alpha}\left(\int_0^\xi \frac{\omega(\eta)}{\eta^\alpha}\,d\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^{1+\alpha}}\,d\eta\right);$$

$$A(\xi) = c_{2,\alpha}\int_{\mathbb{R}} \frac{\omega(\xi)-\omega(|\xi-\eta|)}{|\eta|^{1+\alpha}}\,d\eta;$$

$$D(\xi) = c_{3,\alpha}\left(\int_0^{\xi/2} \frac{2\omega(\xi)-\omega(\xi+2\eta)-\omega(\xi-2\eta)}{\eta^{1+\alpha}}\,d\eta\right.$$
$$\left. + \int_{\xi/2}^\infty \frac{2\omega(\xi)-\omega(\xi+2\eta)+\omega(2\eta-\xi)}{\eta^{1+\alpha}}\,d\eta\right).$$

LEMMA 2.7 (see [5, Lemmas 4.4 and 4.5]). *Let $\rho(\cdot,t)$ obey the modulus of continuity ω as in* (2.35) *for $0\le t<t_1\le T$, and let x_1, y_1 be the breakthrough points at the first breakthrough time t_1. Suppose δ and γ are small constants such that*

$$\delta<1, \quad \gamma\le\frac{\delta-\delta^{1+\alpha/2}}{2\log 2}. \tag{2.47}$$

Then, there exist positive constants C_{I}, C_{II}, and C_{III}, which may only depend on α, such that

$$|\mathrm{I}_1|\le\omega'(\xi)\Omega(\xi), \quad \textit{where } \Omega(\xi)\le\begin{cases} C_{\mathrm{I}}\xi, & 0<\xi<\delta,\\ C_{\mathrm{I}}\xi^{1-\alpha}\omega(\xi), & \xi\ge\delta.\end{cases} \tag{2.48}$$

$$\mathrm{II}_1\le\omega(\xi)A(\xi), \quad \textit{where } A(\xi)\le\begin{cases} C_{\mathrm{II}}, & 0<\xi<\delta,\\ C_{\mathrm{II}}\gamma\xi^{-\alpha}, & \xi\ge\delta.\end{cases} \tag{2.49}$$

$$\mathrm{III}_1\le-\rho_m D(\xi), \quad \textit{where } D(\xi)\ge\begin{cases} C_{\mathrm{III}}\xi^{1-\alpha/2}, & 0<\xi<\delta,\\ C_{\mathrm{III}}\omega(\xi)\xi^{-\alpha}, & \xi\ge\delta.\end{cases} \tag{2.50}$$

Applying the proof of Lemma 2.7 to the modulus of continuity ω_B, we get the following estimates.

LEMMA 2.8. *Let $\rho(\cdot,t)$ obey the modulus of continuity ω_B as in* (2.36) *for $0\le t<t_1\le T$, and let x_1, y_1 be the breakthrough points at the first breakthrough time t_1, as in* (2.41)*. Suppose δ and γ are small constants satisfying* (2.47)*. Then there exist positive constants C_2 and C_3, which may only depend on α, such that*

$$|\mathrm{I}_1|,\ \mathrm{II}_1\le\begin{cases} C_2B^{1+\alpha}\xi, & 0<\xi<B^{-1}\delta,\\ C_2\gamma\omega_B(\xi)\xi^{-\alpha}, & \xi\ge B^{-1}\delta,\end{cases} \tag{2.51}$$

and

$$\mathrm{III}_1\le-\rho_m D_B(\xi), \quad D_B(\xi):=\begin{cases} C_3B^{1+\alpha/2}\xi^{1-\alpha/2}, & 0<\xi<B^{-1}\delta,\\ C_3\omega_B(\xi)\xi^{-\alpha}, & \xi\ge B^{-1}\delta.\end{cases} \tag{2.52}$$

Proof. Through the same proof of Lemma 2.7 and replacing ω by ω_B, one can obtain the following estimates similar to (2.48), (2.49), and (2.50):

$$|\mathrm{I}_1| \le \omega_B'(\xi)\Omega_B(\xi), \quad \mathrm{II}_1 \le \omega_B(\xi)A_B(\xi), \quad \mathrm{III}_1 \le -\rho_m D_B(\xi).$$

Here ω_B is defined in (2.36), and

$$\Omega_B(\xi) = B^{\alpha-1}\Omega(B\xi), \quad A_B(\xi) = B^\alpha A(B\xi), \quad D_B(\xi) = B^\alpha D(B\xi).$$

This directly implies (2.51) and (2.52) with $C_2 = \max\{C_{\mathrm{I}}, C_{\mathrm{II}}\}$ and $C_3 = C_{\mathrm{III}}$. ◻

If we pick δ small enough so that

$$\delta < \left(\frac{C_3}{4C_2}\rho_m(T)\right)^{2/\alpha}, \tag{2.53}$$

then

$$C_2 B^{1+\alpha}\xi \le C_2 B^{1+\alpha/2}\xi^{1-\alpha/2}\delta^{\alpha/2} \le \frac{1}{4}\rho_m D_B(\xi) \quad \forall\, \xi \in (0, B^{-1}\delta).$$

Also, pick γ small enough so that

$$\gamma < \frac{C_3}{4C_2}\rho_m(T), \tag{2.54}$$

then

$$C_2\gamma\omega_B(\xi)\xi^{-\alpha} \le \frac{1}{4}\rho_m D_B(\xi) \quad \forall\, \xi \ge B^{-1}\delta.$$

Therefore, we have

$$\mathrm{I}_1 + \mathrm{II}_1 + \mathrm{III}_1 \le -\frac{1}{2}\rho_m D_B(\xi). \tag{2.55}$$

It remains to control $\mathrm{I}_2, \mathrm{II}_2$, and III_2.

2.3.2. Estimates on $\mathbf{I_2}$.

LEMMA 2.9. *Let $\rho(\cdot, t)$ obey the modulus of continuity ω_B as in* (2.36) *for $0 \le t < t_1 \le T$, and let x_1, y_1 be the breakthrough points at the first breakthrough time t_1, as in* (2.41). *Suppose δ and γ satisfy* (2.47), *and in addition*

$$\delta < \left(\frac{\rho_m(T)C_3}{6\rho_M(T)F_M(T)}\right)^{2/\alpha}, \gamma < \alpha(\delta - \delta^{1+\alpha/2}), B > \max\left\{1, 2\delta \exp\left(\frac{6\rho_M(T)F_M(T)}{C_3\rho_m(T)}\right)\right\}. \tag{2.56}$$

Then,

$$|I_2| \le \frac{1}{6}\rho_m D_B(\xi). \tag{2.57}$$

Proof. We recall

$$I_2 = -(u_2(x_1)\partial_x\rho(x_1) - u(y_1)\partial_x\rho(y_1).$$

It corresponds to the drift term $u_2\partial_x\rho$. Following from the estimate in [14, section 4], we obtain

$$|I_2| \le \Omega_2(\xi)\omega_B'(\xi),$$

where $\Omega_2(\xi)$ is a modulus of continuity of u_2, which can be further bounded by

$$\Omega_2(\xi) \le \|\partial_x u_2\|_{L^\infty}\xi.$$

Then, we have the estimate

$$|I_2| \le \|\partial_x u_2\|_{L^\infty}\xi\omega_B'(\xi) = \|G\|_{L^\infty}\xi\omega_B'(\xi) \le \rho_M(T)F_M(T)\xi\omega_B'(\xi).$$

For $\xi \in (0, B^{-1}\delta)$, $\omega_B'(\xi) < B$. So,

$$|I_2| \le \rho_M(T)F_M(T)B\xi \le \frac{1}{6}\rho_m D_B(\xi), \tag{2.58}$$

provided that δ is small enough, satisfying (2.56), and $B > 1$.

For $\xi \ge B^{-1}\delta$, since ρ is periodic and ω_B is increasing, the breakthrough cannot happen first at $\xi > 1/2$. So we only need to consider $\xi \in (B^{-1}\delta, 1/2]$. As $\omega_B'(\xi) = \frac{\gamma}{\xi}$ in this range, we get

$$|I_2| \le \rho_M(T)F_M(T)\gamma. \tag{2.59}$$

On the other hand, compute

$$\frac{d}{d\xi}D_B(\xi) = C_3\xi^{-\alpha-1}(-\alpha\omega_B(\xi) + \gamma) \le C_3\xi^{-\alpha-1}(-\alpha(\delta - \delta^{1+\alpha/2}) + \gamma) < 0$$

for all $\xi \ge B^{-1}\delta$, provided that γ is small enough, satisfying (2.56). Therefore,

$$\min_{B^{-1}\delta\le\xi\le1/2} D_B(\xi) = D_B(1/2) \ge C_3\gamma\log\left(\frac{B}{2\delta}\right). \tag{2.60}$$

Combining (2.59), (2.60), and the assumption on B in (2.56), we conclude

$$|I_2| \le \rho_M(T)F_M(T)\gamma \le \frac{C_3}{6}\gamma\rho_m(T)\log\left(\frac{B}{2\delta}\right) \le \frac{1}{6}\rho_m D_B(\xi). \tag{2.61}$$

∎

2.3.3. Estimates on II_2 and III_2. The estimates on II_2 and III_2 are more subtle. To proceed, it is convenient to decompose $\mathrm{II}_2 + \mathrm{III}_2$ in an alternative way:

$$\begin{aligned}\mathrm{II}_2 + \mathrm{III}_2 &= -\big(\rho(x_1)\partial_x u_2(x_1) - \rho(y_1)\partial_x u_2(y_1)\big) = -\big(\rho(x_1)^2F(x_1) - \rho(y_1)^2F(y_1)\big)\\ &= -\big(\rho(x_1)^2 - \rho(y_1)^2\big)F(x_1) - \rho(y_1)^2\big(F(x_1) - F(y_1)\big) = \mathrm{IV} + \mathrm{V}.\end{aligned} \tag{2.62}$$

We first consider the case when $\xi < B^{-1}\delta$. For IV, the estimate is similar to (2.58)

$$|\mathrm{IV}| = \omega_B(\xi)(\rho(x_1) + \rho(y_1))|F(x_1)| \le 2\rho_M F_M B\xi \le \frac{1}{6}\rho_m D_B(\xi), \tag{2.63}$$

where the last inequality holds if δ is picked to be small enough, satisfying

$$\delta < \left(\frac{C_3\rho_m(T)}{12\rho_M(T)F_M(T)}\right)^{2/\alpha}. \tag{2.64}$$

For V, we need the following lemma.

Lemma 2.10. *Let $\rho(\cdot,t)$ obey the modulus of continuity ω_B with any $B>1$ as in (2.36) for $0\le t<t_1\le T$. Then, there exists a constant $C_F=C_F(T)$ such that*

$$|F(x,t)-F(y,t)|\le C_F(T)B|x-y| \quad \forall\ x,y\in\mathbb{T}, \quad \forall\ t\in[0,t_1]. \tag{2.65}$$

Proof. Recall the dynamics of F

$$\partial_t F+u\partial_x F=-k\left(1-\frac{\bar\rho}{\rho}\right). \tag{2.66}$$

Let $f=\partial_x F$. Differentiate (2.66) with respect to x and get

$$\partial_t f+\partial_x(uf)=-k\bar\rho\frac{\partial_x\rho}{\rho^2}. \tag{2.67}$$

Let $q=f/\rho$. Using (1.18) and (2.67), we obtain

$$\partial_t q+u\partial_x q=-k\bar\rho\frac{\partial_x\rho}{\rho^3}. \tag{2.68}$$

It follows that

$$q(X(x,t),t)=q_0(x)-k\bar\rho\int_0^t\frac{\partial_x\rho(X(x,s),s)}{\rho(X(x,s),s)^3}ds,$$

where X is the trajectory of the characteristic path defined in (2.9). Then, since for $t\le t_1$, $\rho(\cdot,t)$ obeys ω_B, we obtain the following estimate:

$$\|q(\cdot,t)\|_{L^\infty}\le\|q_0\|_{L^\infty}+|k|\bar\rho\int_0^t\frac{B}{\rho_m(s)^3}ds\le C'(T)B, \tag{2.69}$$

where the finite constant C' depends on T and initial data. This implies

$$|F(x)-F(y)|\le\|f\|_{L^\infty}|x-y|\le\rho_M(T)C'(T)B|x-y|=:C_F(T)B|x-y|. \qquad \square$$

Applying the estimate (2.65) at the breakthrough points and using the upper bound on ρ (2.19), we get

$$|\mathrm{V}|\le\rho_M(T)^2C_F(T)B\xi<\frac{1}{6}\rho_m D_B(\xi), \tag{2.70}$$

where the second inequality holds by picking sufficiently small δ, satisfying

$$\delta<\left(\frac{C_3\rho_m(T)}{6\rho_M(T)^2C_F(T)}\right)^{2/\alpha}, \tag{2.71}$$

similar to the estimate in (2.58).

Combining (2.55), (2.58), (2.63), and (2.70), we conclude that

$$\partial_t(\rho(x_1)-\rho(y_1))<0 \quad \forall\ \xi=|x_1-y_1|<B^{-1}\delta.$$

Finally, we estimate $\mathrm{II}_2+\mathrm{III}_2$ for $\xi\in[B^{-1}\delta,1/2]$. As ρ and F are bounded, it is clear that

$$|\mathrm{II}_2+\mathrm{III}_2|\le 2\rho_M(T)^2F_M(T)<\frac{1}{3}\rho_m D_B(\xi). \tag{2.72}$$

The second inequality holds by picking B large enough. This is due to the fact that $D_B(\xi)$ is an increasing in B with $\lim_{B\to\infty} D_B(\xi) = \infty$. More precisely, using the bound (2.60), it suffices to pick

$$B > 2\delta \exp\left(\frac{6\rho_M(T)^2 F_M(T)}{C_3\gamma\rho_m(T)}\right). \tag{2.73}$$

Combining (2.55), (2.61), and (2.72), we conclude that

$$\partial_t(\rho(x_1) - \rho(y_1)) < 0 \quad \forall\, \xi = |x_1 - y_1| \in [B^{-1}\delta, 1/2].$$

Let us summarize the procedure on the construction of the modulus of continuity ω_B. First, we fix a time T. Then, we pick a small parameter δ satisfying (2.38), (2.47), (2.53), (2.56), (2.64), and (2.71):

$$\delta < \min\left\{1, \frac{2\|\rho_0\|_{L^\infty}}{\|\partial_x\rho_0\|_{L^\infty}}, \left(\frac{C_3\rho_m(T)}{\max\{4C_2, 12\rho_M(T)F_M(T), 6\rho_M(T)^2C_F(T)\}}\right)^{2/\alpha}\right\}. \tag{2.74}$$

Next, we pick a small parameter γ satisfying (2.47), (2.54), and (2.56):

$$\gamma < \min\left\{\frac{C_3}{4C_2}\rho_m(T),\ \min\left(\frac{1}{2\log 2}, \alpha\right)\cdot(\delta - \delta^{1+\alpha/2})\right\}. \tag{2.75}$$

Finally, we pick a large parameter B satisfying (2.38), (2.56), and (2.73):

$$B > \max\left\{1, \frac{\delta\|\partial_x\rho_0\|_{L^\infty}}{2\|\rho_0\|_{L^\infty}}\exp\left(\frac{2\|\rho_0\|_{L^\infty}}{\gamma}\right), 2\delta\exp\left(\frac{6\rho_M(T)^2F_M(T)}{C_3\gamma\rho_m(T)}\right)\right\}. \tag{2.76}$$

Here we assume, without loss of generality, that $\gamma \le 1$ and $\rho_M(T) \ge 1$ to simplify the expression.

With this choice of ω_B, we have shown that $\rho(\cdot, t)$ obeys ω_B for all $t \in [0, T]$. Hence,

$$\|\partial_x\rho(\cdot, t)\|_{L^\infty} \le B \quad \forall\, t \in [0, T].$$

Therefore, condition (2.25) is satisfied, and we obtain global regularity of the system.

We end this section by the following remark.

Remark 2.11. When $k = 0$, all the quantities ρ_m, ρ_M, F_M, and C_F do not depend on T. As a consequence, δ, γ, and B do not depend on T either. Therefore, $\|\partial_x\rho(\cdot, t)\|_{L^\infty} \le B$ for any $t \ge 0$. This estimate improves the result obtained in [5], where the bound on $\partial_x\rho$ could grow in time. We note that stationary in time bound on $\partial_x\rho$ for the Euler-alignment model (without Poisson forcing) has been derived in [19] by a different argument.

For $k \ne 0$, with the singular attractive or repulsive force, our estimate on ρ_m and ρ_M is not uniform in time. We are able to obtain time-dependent bounds (2.13) and (2.19), where ρ_m can decay exponentially in time, and ρ_M (and F_M, C_F) can grow exponentially in time. From (2.74) and (2.75), we see that δ and γ decay exponentially in time. Finally, from (2.76), B grows double exponentially in time. Therefore, we obtain a double exponential in time bound on $\|\partial_x\rho(\cdot, t)\|_{L^\infty}$. It is not clear whether such a bound is optimal. We will leave it for future investigation.

3. Euler dynamics with general three-zone interactions. In this section, we extend our global regularity result for EPA system to more general Euler dynamics with three-zone interactions. Recall the *Euler-3Zone system* under periodic setup

$$\partial_t\rho + \partial_x(\rho u) = 0, \quad x \in \mathbb{T},\ t > 0, \tag{3.1}$$

$$\partial_t u + u\partial_x u = \int_{\mathbb{T}} \psi(y)(u(x+y,t) - u(x,t))\rho(y,t)dy - \partial_x K \star \rho. \tag{3.2}$$

We will discuss the global wellposedness of the system with more general singular influence function ψ and interaction potential K.

3.1. General singular influence function. Consider a general influence function ψ which is positive

$$\psi_m := \min_{x\in\mathbb{T}} \psi(x) > 0 \tag{3.3}$$

and singular at origin. Recall the decomposition (1.21): we will consider the class of functions where one can decompose ψ into two parts

$$\psi = c\psi_\alpha + \psi_L. \tag{3.4}$$

Here ψ_α is the singular power defined in (1.7), and ψ_L is bounded and Lipschitz. In this case, let

$$G = \partial_x u - c\Lambda^\alpha \rho + \psi_L \star \rho. \tag{3.5}$$

Then, the dynamics of G reads

$$\begin{aligned}
\partial_t G &= \partial_t\partial_x u - c\partial_t \Lambda^\alpha\rho + \psi_L \star \partial_t\rho \\
&= -\partial_x(u\partial_x u) + c\partial_x\big(-\Lambda^\alpha(\rho u) + u\Lambda^\alpha\rho\big) - \partial_x\big(\psi_L\star(\rho u) - u(\psi_L\star\rho)\big) \\
&\quad - \partial^2_{xx}K\star\rho + c\Lambda^\alpha\partial_x(\rho u) - \psi_L\star\partial_x(\rho u) \\
&= -\partial_x\big(u(\partial_x u - c\Lambda^\alpha\rho + \psi_L\star\rho)\big) - \partial^2_{xx}K\star\rho = -\partial_x(Gu) - \partial^2_{xx}K\star\rho.
\end{aligned}$$

Therefore, (ρ, G) still satisfy (2.2) and (2.3),

$$\partial_t\rho + \partial_x(\rho u) = 0, \quad \partial_t G + \partial_x(Gu) = -\partial^2_{xx}K\star\rho, \tag{3.6}$$

with a different relation

$$\partial_x u = \Lambda^\alpha\rho + G - \psi_L\star\rho. \tag{3.7}$$

Then the velocity field u can be recovered as

$$u(x,t) = \Lambda^\alpha\partial_x^{-1}(\rho(x,t) - \bar\rho) + \partial_x^{-1}\big(G(x,t) - \psi_L\star\rho(x,t)\big) + I_0(t), \tag{3.8}$$

where $I_0(t)$ can be determined by conservation of momentum (2.6). The second term on the right-hand side is well-defined since

$$\int_{\mathbb{T}}\big(G(x,t) - \psi_L\star\rho(x,t)\big)dx = \int_{\mathbb{T}}\big(\partial_x u(x,t) - \Lambda^\alpha\rho(x,t)\big)dx = 0 \quad \forall\ t\geq 0.$$

We can decompose u into two parts $u = u_S + u_L$, where u_S is the singular part

$$\begin{aligned}
u_S(x,t) =&\Lambda^\alpha\partial_x^{-1}(\rho(x,t)-\bar\rho) + \partial_x^{-1}\left(G(x,t) - \int_{\mathbb{T}} G(x,0)dx\right), \\
\partial_x u_S =&\Lambda^\alpha\rho + G - \int_{\mathbb{T}} G(x,0)dx,
\end{aligned} \tag{3.9}$$

and u_L is the Lipschitz part

$$
\begin{aligned}
u_L(x,t) &= -\,\partial_x^{-1}\left(\psi_L \star \rho(x,t) - \int_{\mathbb{T}} G(x,0)dx\right) + I_0(t),\\
\partial_x u_L &= -\,\psi_L \star \rho + \int_{\mathbb{T}} G(x,0)dx.
\end{aligned}
\tag{3.10}
$$

Now, we follow the same procedure as the fractional EPA system to show global regularity of system (3.6), (3.8). We first take the Newtonian potential (1.15). General interaction potentials will be discussed in the next section. The arguments below follow the same outline, so we focus on indicating changes.

Step 1: A priori lower bound on ρ. The statement and proof are identical to Theorem 2.1, except that estimate (2.14) is replaced by

$$
\begin{aligned}
-c\Lambda^\alpha \rho(\underline{x},t) &+ \psi_L \star \rho(\underline{x},t)\\
&= \int_{\mathbb{T}} \big(c\psi_\alpha(y) + \psi_L(y)\big)\big(\rho(\underline{x}-y,t) - \rho(\underline{x},t)\big)dy + \rho(\underline{x},t)\int_{\mathbb{T}} \psi_L(y)dy\\
&\geq \psi_m \int_{\mathbb{T}} \big(\rho(\underline{x}-y,t) - \rho(\underline{x},t)\big)dy - \rho(\underline{x},t)\|\psi_L\|_{L^\infty}\\
&= \psi_m\bar{\rho} - (\psi_m + \|\psi_L\|_{L^\infty})\rho(\underline{x},t).
\end{aligned}
\tag{3.11}
$$

Hence, estimate (2.15) becomes

$$
\partial_t \rho(\underline{x},t) \geq \big(\psi_m\bar{\rho}\big)\rho(\underline{x},t) - \left[\psi_m + \|\psi_L\|_{L^\infty} + \|F_0\|_{L^\infty} + |k|t + |k|\bar{\rho}\int_0^t \frac{1}{\rho_m(s)}ds\right]\rho(\underline{x},t)^2,
\tag{3.12}
$$

where the only extra term $\|\psi_L\|_{L^\infty}\rho(\underline{x},t)^2$ is quadratic in ρ and can be controlled by the linear term $\big(\psi_m\bar{\rho}\big)\rho(\underline{x},t)$ if ρ_m is small enough.

Following the same proof, we obtain the lower bound (2.13) with coefficient A_m, C_m satisfying (2.17) for any $\epsilon \in (0,\epsilon_*)$, where

$$
\epsilon_* = \frac{1}{2}\left(\sqrt{1 + \frac{4\psi_m}{e(\psi_m + \|\psi_L\|_{L^\infty} + \|F_0\|_{L^\infty})}} - 1\right).
$$

Step 2: A priori upper bound on ρ. We follow the proof of Theorem 2.3. The estimate (2.21) becomes

$$
\begin{aligned}
\frac{d}{dt}\rho(\bar{x},t) &\leq -\,C_1\rho(\bar{x},t)^{2+\alpha} + F_M(t)\rho(\bar{x},t)^2 + \rho(\bar{x},t)\cdot\psi_L \star \rho(\bar{x},t)\\
&\leq -\,C_1\rho(\bar{x},t)^{2+\alpha} + F_M(t)\rho(\bar{x},t)^2 + \|\psi_L\|_{L^\infty}\bar{\rho}\rho(\bar{x},t).
\end{aligned}
$$

Both second and third terms are dominated by the first term if $\rho(\bar{x},t)$ is big enough. In particular $\partial_t\rho(\bar{x},t) < 0$ if $\rho(\bar{x},t) > \max\{(2F_M/C_1)^{1/\alpha}, (2\|\psi_L\|_{L^\infty}\bar{\rho}/C_1)^{1/(1+\alpha)}\}$. Therefore,

$$
\rho(x,t) \leq \rho(\bar{x},t) \leq \max\left\{\|\rho_0\|_{L^\infty}, 3\bar{\rho}, \left(\frac{2F_M(t)}{C_1}\right)^{1/\alpha}, \left(\frac{2\|\psi_L\|_{L^\infty}\bar{\rho}}{C_1}\right)^{1/(1+\alpha)}\right\},
\tag{3.13}
$$

and (2.19) holds with $A_M = A_m/\alpha$ and

$$
C_M = \max\left\{\max_{x\in\mathbb{T}}\rho_0(x),\ 3\bar{\rho},\ \left[\frac{2}{C_1}\left(\|F_0\|_{L^\infty} + \frac{|k|}{eA_m} + \frac{|k|\bar{\rho}}{A_mC_m}\right)\right]^{\frac{1}{\alpha}}, \left(\frac{2\|\psi_L\|_{L^\infty}\bar{\rho}}{C_1}\right)^{\frac{1}{1+\alpha}}\right\}.
$$

Step 3: Local wellposedness. We write the system (3.6), (3.8) in terms of $\theta = \rho - \bar{\rho}$ and G as follows:

$$\partial_t \theta + \partial_x(\theta u_S) + \partial_x(\theta u_L) = -\bar{\rho}\partial_x u_S - \bar{\rho}\partial_x u_L, \tag{3.14}$$

$$\partial_t G + \partial_x(G u_S) + \partial_x(G u_L) = -k\theta, \tag{3.15}$$

where u_S and u_L are defined in (3.9) and (3.10), respectively.

We proceed with a Gronwall estimate on the quantity Y in (2.28). The estimates in Theorem 2.4 can be applied directly to the u_S part. We will focus on the Lipschitz part u_L. The procedure is similar to [1, Theorem A.1, Appendix A.1]. We will summarize it below.

For the term $\partial_x(\theta u_L)$, we have

$$\int_{\mathbb{T}} \Lambda^s\theta \cdot \Lambda^s \partial_x(\theta u_L) dx = \int_{\mathbb{T}} \Lambda^s\theta \cdot \Lambda^s\partial_x\theta \cdot u_L dx + \int_{\mathbb{T}} \Lambda^s\theta \cdot [\Lambda^s\partial_x, u_L]\theta \; dx =: L_1 + L_2.$$

We estimate the two terms one by one. For L_1,

$$|L_1| = \left|\int_{\mathbb{T}} \partial_x\left(\frac{(\Lambda^s\theta)^2}{2}\right) u_L dx\right| \leq \frac{1}{2}\int_{\mathbb{T}} (\Lambda^s\theta)^2 |\partial_x u_L| dx \leq \frac{1}{2}\|\psi_L\|_{L^\infty}\bar{\rho}\|\theta\|_{H^s}^2. \tag{3.16}$$

For L_2, we apply commutator estimate (e.g., [1, Lemma A.1, Appendix A.1]) and get

$$\begin{aligned}|L_2| &\leq \|\theta\|_{H^s}\left\|[\Lambda^s\partial_x, u_L]\theta\right\|_{L^2} \lesssim \|\theta\|_{H^s}\left(\|\partial_x u_L\|_{L^\infty}\|\theta\|_{H^s} + \|\partial_x u_L\|_{H^s}\|\theta\|_{L^\infty}\right)\\ &\leq \|\theta\|_{H^s}\left(\|\psi_L\|_{L^\infty}\bar{\rho}\|\theta\|_{H^s} + \|\psi_L\|_{L^\infty}\|\rho\|_{H^s}\|\theta\|_{L^\infty}\right)\\ &\leq \|\psi_L\|_{L^\infty}(2\bar{\rho} + \|\theta\|_{L^\infty})\|\theta\|_{H^s}^2 + \frac{1}{4}\|\psi_L\|_{L^\infty}\bar{\rho}. \end{aligned}\tag{3.17}$$

Note that for the last inequality, we have used $\|\rho\|_{H^s} \leq \|\theta\|_{H^s} + \|\bar{\rho}\|_{H^s} = \|\theta\|_{H^s} + \bar{\rho}$.

For the term $-\bar{\rho}\partial_x u_L$,

$$\left|-\bar{\rho}\int_{\mathbb{T}} \Lambda^s\theta \cdot \Lambda^s\partial_x u_L dx\right| \leq \bar{\rho}\|\theta\|_{H^s}\|(\partial_x\psi_L)\star(\Lambda^s\rho)\|_{L^2} \leq \bar{\rho}\|\partial_x\psi_L\|_{L^\infty}\|\theta\|_{H^s}(\|\theta\|_{H^s}+\bar{\rho}). \tag{3.18}$$

For the term $\partial_x(G u_L)$, the estimate is similar to the term $\partial_x(\theta u_L)$.

$$\begin{aligned}&\int_{\mathbb{T}} \Lambda^{s-\frac{\alpha}{2}}G \cdot \Lambda^{s-\frac{\alpha}{2}}\partial_x(G u_L) dx\\ &\qquad = \int_{\mathbb{T}} \Lambda^{s-\frac{\alpha}{2}}G \cdot \Lambda^{s-\frac{\alpha}{2}}\partial_x G \cdot u_L dx + \int_{\mathbb{T}} \Lambda^{s-\frac{\alpha}{2}}G \cdot [\Lambda^{s-\frac{\alpha}{2}}\partial_x, u_L]G \; dx =: L_4 + L_5,\end{aligned}$$

where

$$|L_4| = \left|\int_{\mathbb{T}} \partial_x\left(\frac{(\Lambda^{s-\frac{\alpha}{2}}G)^2}{2}\right) u_L dx\right| \leq \frac{1}{2}\int_{\mathbb{T}} (\Lambda^{s-\frac{\alpha}{2}}G)^2|\partial_x u_L| dx \leq \frac{1}{2}\|\psi_L\|_{L^\infty}\bar{\rho}\|G\|_{H^{s-\frac{\alpha}{2}}}^2, \tag{3.19}$$

and

$$\begin{aligned}|L_5| &\leq \|G\|_{H^{s-\frac{\alpha}{2}}}\left\|[\Lambda^{s-\frac{\alpha}{2}}\partial_x, u_L]G\right\|_{L^2}\\ &\lesssim \|G\|_{H^{s-\frac{\alpha}{2}}}\left(\|\partial_x u_L\|_{L^\infty}\|G\|_{H^{s-\frac{\alpha}{2}}} + \|\partial_x u_L\|_{H^s}\|G\|_{L^\infty}\right)\\ &\leq \|\psi_L\|_{L^\infty}\bar{\rho}\|G\|_{H^{s-\frac{\alpha}{2}}}^2 + \|\psi_L\|_{L^\infty}\bar{\rho}(\|\theta\|_{H^s}+\bar{\rho})\|G\|_{H^{s-\frac{\alpha}{2}}}\\ &\leq \|\psi_L\|_{L^\infty}\bar{\rho}\left[2\|G\|_{H^{s-\frac{\alpha}{2}}}^2 + \frac{1}{2}\|\theta\|_{H^{s-\frac{\alpha}{2}}}^2 + \frac{\bar{\rho}^2}{2}\right].\end{aligned}\tag{3.20}$$

Combining (2.33), (3.16), (3.17), (3.18), (3.19), (3.20), and the fact $\|G(\cdot,t)\|_{L^\infty}$ is controlled from above by a finite (growing in time) bound, we obtain that for all $t\in[0,T]$,

$$\frac{d}{dt}Y(t)\le C(T)(1+\|\partial_x\theta(\cdot,t)\|^2_{L^\infty})Y(t)-\frac{\rho_m(t)}{6}\|\theta\|^2_{H^{s+\frac{\alpha}{2}}}, \tag{3.21}$$

where the constant C depends on initial data and T. The same Gronwall's inequality yields local wellposedness as well as BKM-type blowup condition (2.25).

Step 4: Global wellposedness. To check the condition (2.25), we will use the procedure identical to that in section 2.3. Let us decompose u as in (3.8), $u=u_1+u_2$, where

$$u_1(x,t)=\Lambda^\alpha\partial_x^{-1}(\rho(x,t)-\bar\rho),\quad u_2(x,t)=\partial_x^{-1}\big(G(x,t)-\psi_L\star\rho(x,t)\big)+I_0(t). \tag{3.22}$$

The only difference between our system (3.6), (3.8) and the EPA system is that there is an extra term in u_2. Throughout the proof in section 2.3, the only property of u_2 we have used is that $\partial_x u_2$ is bounded, namely,

$$\|\partial_x u_2(\cdot,t)\|_{L^\infty}\le\rho_M(T)F_M(T)<\infty\quad\forall\, t\in[0,T].$$

For our u_2 defined in (3.22), we also have a bound on $\partial_x u_2$:

$$\|\partial_x u_2(\cdot,t)\|_{L^\infty}=\|G(\cdot,t)-\psi_L\star\rho(\cdot,t)\|_{L^\infty}\le\rho_M(T)F_M(T)+\|\psi_L\|_{L^\infty}\bar\rho<\infty$$

for any $t\in[0,T]$. Hence, global regularity follows from the same procedure by controlling the modulus of continuity.

3.2. General interaction potential. In this part, we consider system (3.1)–(3.2) with a general interaction potential $K\in W^{2,\infty}(\mathbb{T})$. This class of potentials is more regular than the Newtonian potential $\mathcal{N}$ defined in (1.15), as $\partial^2_{xx}\mathcal{N}=k(\delta_0-1)\notin L^\infty$, where δ_0 is the Dirac delta at $x=0$. We will show global wellposedness of the Euler-3Zone system with $W^{2,\infty}$ potentials. The result automatically extends to systems with potentials that can be decomposed into a sum of a Newtonian potential and a $W^{2,\infty}$ potential.

Now, let us assume $K\in W^{2,\infty}(\mathbb{T})$. After the transformation, the dynamics for (ρ,G) becomes (3.6), with velocity field u defined as (3.8). We shall run through the same procedure and point out the differences.

Step 1: A priori lower bound on ρ. Due to the change of the potential, the dynamics of F (2.8) becomes

$$(\partial_t+u\partial_x)F=-\frac{\partial^2_{xx}K\star\rho}{\rho}. \tag{3.23}$$

Therefore, we get

$$F(X(x,t),t)=F_0(x)-\int_0^t\frac{\partial^2_{xx}K\star\rho(X(x,s),s)}{\rho(X(x,s),s)}ds, \tag{3.24}$$

where $X(x,t)$ is the characteristic path defined in (2.9). Then, we obtain a bound

$$\|F(\cdot,t)\|_{L^\infty}\le\ \|F_0\|_{L^\infty}+\|\partial^2_{xx}K\|_{L^\infty}\bar\rho\int_0^t\frac{1}{\rho_m(s)}ds, \tag{3.25}$$

which is similar to (2.12). In fact, it is a simpler bound as the right-hand side does not contain a linear term on t.

The lower bound (2.13) follows then by the same argument, with

$$A_m = \frac{\|\partial_{xx}^2 K\|_{L^\infty}}{\psi_m}, \quad C_m = \min\left\{\rho_m(0), \frac{\psi_m \bar{\rho}}{\psi_m + \|\psi_L\|_{L^\infty} + \|F_0\|_{L^\infty}}\right\}.$$

Step 2: A priori upper bound on ρ. The upper bound estimate (3.13) can be obtained without any additional difficulties. Since we have

$$F_M(t) = \|F_0\|_{L^\infty} + \frac{\|\partial_{xx}^2 K\|_{L^\infty}\bar{\rho}}{A_m C_m} e^{A_m t}$$

by (3.25) and the lower bound estimate on ρ, the upper bound (2.19) holds with $A_M = A_m/\alpha$ and

$$C_M = \max\left\{\max_{x\in\mathbb{T}} \rho_0(x),\ 3\bar{\rho},\ \left[\frac{2}{C_1}\left(\|F_0\|_{L^\infty} + \frac{\|\partial_{xx}^2 K\|_{L^\infty}\bar{\rho}}{A_m C_m}\right)\right]^{\frac{1}{\alpha}}, \left(\frac{2\|\psi_L\|_{L^\infty}\bar{\rho}}{C_1}\right)^{\frac{1}{1+\alpha}}\right\}.$$

Step 3: Local wellposedness. Since the potential only enters the dynamics of the G equation, so the system in terms of (θ, G) is identical to (3.14)–(3.15), except the right-hand side of (3.15) is replaced by $-\partial_{xx}^2 K \star \rho$. Hence, we only need to estimate this extra term.

$$\begin{aligned}\left|\int_{\mathbb{T}} \Lambda^{s-\frac{\alpha}{2}} G \cdot \Lambda^{s-\frac{\alpha}{2}}(\partial_{xx}^2 K \star \rho)\, dx\right| &= \left|\int_{\mathbb{T}} \Lambda^{s-\frac{\alpha}{2}} G \cdot (\partial_{xx}^2 K \star \Lambda^{s-\frac{\alpha}{2}} \rho)\, dx\right| \\ &\lesssim \|\partial_{xx}^2 K\|_{L^\infty} \|G\|_{H^{s-\frac{\alpha}{2}}} \|\theta\|_{H^s} \leq \frac{1}{2}\|\partial_{xx}^2 K\|_{L^\infty} Y(t).\end{aligned}$$

The local wellposedness and BKM-type blowup condition (2.25) follow by applying the same Gronwall's inequality on Y.

Step 4: Global wellposedness. The argument for EPA system can be directly applied to the general system as the ρ equations in both cases are the same. The different potential requires different estimates on ρ_m, ρ_M, F_M, C_F, which are needed to construct the modulus ω_B. Since ρ_m, ρ_M, and F_M have been treated in the previous steps, we are left with estimating C_F, namely, proving Lemma 2.10 for the general system.

Proof of Lemma 2.10. Let $f = \partial_x F$. Differentiate (3.23) with respect to x and get

$$\partial_t f + \partial_x(uf) = \frac{-(\partial_{xxx}^3 K \star \rho)\rho + (\partial_{xx}^2 K \star \rho)\partial_x \rho}{\rho^2}. \tag{3.26}$$

Let $q = f/\rho$. Using (1.18) and (3.26), we obtain

$$\partial_t q + u\partial_x q = \frac{-(\partial_{xxx}^3 K \star \rho)\rho + (\partial_{xx}^2 K \star \rho)\partial_x \rho}{\rho^3}. \tag{3.27}$$

For $t \leq t_1$, $\rho(\cdot, t)$ obeys ω_B. Then $\|\partial_x \rho(\cdot, t)\|_{L^\infty} \leq \omega_B'(0) = B$. Therefore, we can bound the right-hand side of (3.27) as follows:

$$\begin{aligned}\left|\frac{-(\partial_{xxx}^3 K \star \rho)\rho + (\partial_{xx}^2 K \star \rho)\partial_x \rho}{\rho^3}\right| &\leq \frac{\|\partial_{xx}^2 K\|_{L^\infty}\|\partial_x \rho\|_{L^1}}{\rho_m(t)^2} + \frac{\|\partial_{xx}^2 K\|_{L^\infty}\bar{\rho}\|\partial_x \rho\|_{L^\infty}}{\rho_m(t)^3} \\ &\leq B\|\partial_{xx}^2 K\|_{L^\infty}\left(\frac{1}{\rho_m(t)^2} + \frac{\bar{\rho}}{\rho_m(t)^3}\right).\end{aligned}$$

Then, we obtain the bound on q for all $0 \le t \le t_1 < T$,

$$\|q(\cdot,t)\|_{L^\infty} \le \|q_0\|_{L^\infty} + B\|\partial^2_{xx}K\|_{L^\infty} \int_0^t \left(\frac{1}{\rho_m(t)^2} + \frac{\bar{\rho}}{\rho_m(t)^3}\right) ds \le C'(T)B, \tag{3.28}$$

where the finite constant C' depends on $\|\partial^2_{xx}K\|_{L^\infty}, T$ and initial data. This implies

$$|F(x) - F(y)| \le \|f\|_{L^\infty}|x-y| \le \rho_M(T)C'(T)B\xi =: C_F(T)B|x-y|. \qquad \square$$

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