# FINITE TIME BLOW UP FOR A 1D MODEL OF 2D BOUSSINESQ SYSTEM 

KYUDONG CHOI, ALEXANDER KISELEV, AND YAO YAO


#### Abstract

The 2D conservative Boussinesq system describes inviscid, incompressible, buoyant fluid flow in gravity field. The possibility of finite time blow up for solutions of this system is a classical problem of mathematical hydrodynamics. We consider a 1D model of 2D Boussinesq system motivated by a particular finite time blow up scenario. We prove that finite time blow up is possible for the solutions to the model system.


## 1. Introduction

The 2D Boussinesq system for vorticity of the fluid $\omega(x, t)$ and density (or temperature) $\rho(x, t)$ is given by

$$
\begin{array}{r}
\partial_{t} \omega+(u \cdot \nabla) \omega=\partial_{x_{1}} \rho ; \quad \partial_{t} \rho+(u \cdot \nabla) \rho=0 ;  \tag{1}\\
u=\nabla^{\perp}(-\Delta)^{-1} \omega, \quad \omega(x, 0)=\omega_{0}(x), \quad \rho(x, 0)=\rho_{0}(x) .
\end{array}
$$

The 2D Boussinesq system models motion of buoyant incompressible fluid that takes place in atmosphere, ocean, inside Earth or stars, and in every kitchen. Global regularity of solutions is known when classical dissipation is present in at least one of the equations [3], [9], or under a variety of more general conditions on dissipation (see e.g. [2] for more information). The regularity vs finite time blow up question for the inviscid 2D Boussinesq system (1) is a well known open problem that has appeared, for example, on the "eleven great problems of mathematical hydrodynamics" list proposed by Yudovich [17]. There is also an interesting connection between (1) and axi-symmetric three dimensional Euler equation: the equations are closely related and virtually identical away from the rotation axis (see e.g. [15], page 186).

There has been much numerical work on trying to find possible singular scenario for solutions of axi-symmetric 3D Euler equation with swirl or 2D Boussinesq system. Often, situations where strong growth of solutions has been observed were later determined to be regular by further numerical or analytic research. For numerical studies,

[^0]see for example [16], [7], or a review [8]. Analytical tools for ruling out blow up scenario include nonlinearity depletion mechanisms discovered by Constantin, Fefferman and Majda [5], [6] and later extensions in [10], [11].

In a recent work [12], Tom Hou and Guo Luo suggested a new scenario for possible singularity formation in 3D Euler equation. In their scenario, the flow is axi-symmetric and confined in a rotating cylinder with no flow condition on the boundary. The numerically observed growth of vorticity happens at the boundary of the cylinder, away from rotation axis. So one can equivalently work with (1) set on a square $D$ (corresponding to a fixed angular variable in the 3D case). Motivated by [12], Kiselev and Sverak [14] considered a similar setting for the 2D Euler equation on a disk. The solutions of 2D Euler equation with smooth initial data are well known to be globally regular. However the work [14] constructs examples with double exponential growth of the vorticity gradient. This is known to be the fastest possible rate of growth, and [14] provides the first example where such growth happens. The growth in [14] also happens on the boundary, and their result confirms that the scenario of [12] is indeed an interesting candidate to consider for blow up in solutions of 3D Euler equation or 2D Boussinesq system.

Compared to the 2D Euler case, the 2D Boussinesq system presents significant new difficulties for analysis. There are nonlinear effects coming from the coupling in (1), and possible growth in vorticity makes solutions harder to control. A simplified onedimensional model has been suggested in [12] and analyzed in [13]. It is given by

$$
\begin{array}{r}
\partial_{t} \omega+u \partial_{x} \omega=\partial_{x} \rho ; \quad \partial_{t} \rho+u \partial_{x} \rho=0  \tag{2}\\
u_{x}=H \omega, \quad \omega(x, 0)=\omega_{0}(x), \quad \rho(x, 0)=\rho_{0}(x)
\end{array}
$$

where the the initial data is periodic with period two, the density function is even, the vorticity is odd with respect to $x=0$ and $x=1$, and $H \omega$ denotes the periodic Hilbert transform of vorticity. Local well-posedness and a number of useful estimates have been proved in [13] for the system (2), and both numerical simulations as well as formal arguments suggesting blow up have been carried out. However a fully rigorous proof of finite time blow up is currently not available for the system (2).

Our goal in this paper is to analyze a related but further simplified system that is inspired by [14]. The system is set on an interval [0, 1] with Dirichlet boundary conditions for $\omega$ and $\rho$.

$$
\left\{\begin{array}{l}
\partial_{t} \rho(t, x)+u(t, x) \partial_{x} \rho(t, x)=0  \tag{3}\\
\partial_{t} \omega(t, x)+u(t, x) \partial_{x} \omega(t, x)=\partial_{x} \rho(t, x) \\
u(t, x)=-x \Omega(t, x), \quad \Omega(t, x)=\int_{x}^{1} \frac{\omega(t, y)}{y} d y \\
\omega(0, x)=\omega_{0}(x), \quad \rho(0, x)=\rho_{0}(x), \quad \omega_{0}(0)=\omega_{0}(1)=\rho_{0}(0)=\rho_{0}(1)=0
\end{array}\right.
$$

We choose to work with Dirichlet boundary conditions for both $\omega$ and $\rho$, which is more natural than periodic setting for our version of the Biot-Savart law. The BiotSavart law linking fluid velocity to vorticity is the main difference between (2) and (3).

The law for the system (3) is simpler, even though closely related to the law for the system (2). This facilitates the analysis. Such simplified Biot-Savart law is motivated by the result proved in [14]. It is shown there that under certain conditions on the initial data $\omega_{0}$, the flow near the origin $O$ is hyperbolic for all times. Namely, apart from small exceptional sectors, the velocity $u$ near $O$ satisfies

$$
\begin{align*}
& u_{1}\left(x_{1}, x_{2}\right)=-\frac{4}{\pi} x_{1} \int_{Q\left(x_{1}, x_{2}\right)} \frac{y_{1} y_{2}}{|y|^{4}} \omega(y, t) d y_{1} d y_{2}+x_{1} B_{1}\left(x_{1}, x_{2}, t\right)  \tag{4}\\
& u_{2}\left(x_{1}, x_{2}, t\right)=\frac{4}{\pi} x_{2} \int_{Q\left(x_{1}, x_{2}\right)} \frac{y_{1} y_{2}}{|y|^{4}} \omega(y, t) d y_{1} d y_{2}+x_{2} B_{2}\left(x_{1}, x_{2}, t\right) \tag{5}
\end{align*}
$$

where $x_{1}, x_{2} \geq 0,\left|B_{1,2}\left(x_{1}, x_{2}, t\right)\right| \leq C(\gamma)\|\omega\|_{L^{\infty}}$ and $Q\left(x_{1}, x_{2}\right)=\left\{y \mid y \in D, x_{1} \leq\right.$ $\left.y_{1}, x_{2} \leq y_{2}\right\}$. The first term on the right hand side of (4), (5) is the main term, and it is this term that is modeled by $u(x, t)=-x \int_{x}^{1} \omega(y, t) / y d y$ in (3). Thus one can expect the system (3) to be a reasonable model of the true 2D Boussinesq dynamics as far as the hyperbolic flow formulas like (4), (5) remain valid, in particular all the time up to blow up if it happens. This is far from clear, even though the numerical simulations of Hou and Luo [12] seem to suggest that this might be the case.

In the first two sections below we will establish local well-posedness and conditional regularity results for the system (3), in particular proving an analog of the celebrated Beale-Kato-Majda criterion [1]. Then we will prove our main result

Theorem 1.1. There exist $\omega_{0}, \rho_{0} \in C_{0}^{\infty}([0,1])$ for which the solution of (3) blows up in finite time. In particular,

$$
\int_{0}^{T^{*}}\|\omega(t)\|_{L^{\infty}} d t \rightarrow \infty
$$

for some $T^{*}<\infty$.

Roughly speaking, the blow-up proof is done by tracking the evolution of $\Omega(x, t)$ along a family of characteristics originating from a sequence of points $x_{1} \geq x_{2} \geq \ldots$, where $x_{\infty}:=\lim _{n \rightarrow \infty} x_{n}>0$ satisfies $\rho_{0}\left(x_{\infty}\right)>0$. By obtaining lower bound on $\Omega$ on this family of characteristics, we conclude that the characteristic originating from $x_{\infty}$ must touch the origin before some finite time $T$, which implies that the classical solution has to break down at (or before) time $T$.

The main new effect reflected in Theorem 1.1 is a rigorous understanding of the mechanism how coupling in 2D Boussinesq can in principle lead to blow up. The main simplifications the result utilizes are lack of two-dimensional geometry which makes certain monotonicity properties easier to control as well as reliance on the stable hyperbolic form of fluid velocity akin to [14]. These simplifications are clearly significant, but one has to take the first step.

## 2. LOCAL WELL-POSEDNESS

It will be often useful for us to solve equations for $\omega$ and $\rho$ on characteristics. Denote $\phi_{t}(x)$ the solution of

$$
\left\{\begin{array}{l}
\frac{d}{d t} \phi_{t}(x)=u\left(t, \phi_{t}(x)\right) \\
\phi_{0}(x)=x
\end{array}\right.
$$

Then we have

$$
\begin{gathered}
\rho\left(t, \phi_{t}(x)\right)=\rho_{0}(x) \\
\omega\left(t, \phi_{t}(x)\right)=\omega_{0}(x)+\int_{0}^{t}\left(\partial_{x} \rho\right)\left(s, \phi_{s}(x)\right) d s
\end{gathered}
$$

First we consider the following lemma which says that $u$ has almost one more derivative than $\omega$ has.
Lemma 2.1. Let $\omega \in C_{0}^{\infty}((0,1))$ be a smooth function that is compactly supported in $(0,1)$.
Then we have $\left\{\begin{array}{l}\|u\|_{H^{m+1}} \leq C_{m} \cdot\|\omega\|_{H^{m}} \text { for } m \geq 0 \text { and } \\ \left\|u^{(m+1)}\right\|_{L^{\infty}} \leq C_{m} \cdot\left\|\omega^{(m)}\right\|_{L^{\infty}} \text { for } m \geq 1 .\end{array}\right.$
Proof. Observe that $\|u\|_{L^{\infty}} \leq\|\omega\|_{L^{1}}$ and $u^{\prime}=-\Omega+\omega$. For $p \in[1, \infty)$, we obtain $\|\Omega\|_{L^{p}} \leq p \cdot\|\omega\|_{L^{p}}$ by using the following Hardy's inequality with $f(x)=\omega(x) / x$ :

$$
\left(\int_{0}^{\infty}\left(\int_{x}^{\infty}|f(x)| d x\right)^{p} d x\right)^{1 / p} \leq p\left(\int_{0}^{\infty}|f(x)|^{p} x^{p} d x\right)^{1 / p}
$$

It shows that $\|u\|_{H^{1}} \leq\|\omega\|_{L^{2}}$.
For $u \in H^{m+1}$ estimate with $m \geq 1$, observe that $u^{(m+1)}(x)=\sum_{i=0}^{m} C_{m, i} \cdot \frac{\omega^{(m-i)}(x)}{x^{i}}$ for some constants $C_{m, i}$. We claim $\left\|\frac{\omega^{(m-i)}(x)}{x^{i}}\right\|_{L^{2}} \leq C\left\|\omega^{(m)}\right\|_{L^{2}}$. Indeed, observe that for $n \geq 1$ and for smooth $f$ which is compactly supported in $(0,1)$,

$$
\begin{aligned}
\int_{0}^{1}\left(\frac{f(x)}{x^{n}}\right)^{2} d x & =\left.\frac{f^{2}(x)}{(1-2 n) x^{2 n-1}}\right|_{x=0} ^{x=1}+\int_{0}^{1} \frac{2 f f^{\prime}}{(2 n-1) x^{2 n-1}} d x \\
& \leq \frac{2}{2 n-1}\left(\int_{0}^{1}\left(\frac{f(x)}{x^{n}}\right)^{2} d x\right)^{1 / 2}\left(\int_{0}^{1}\left(\frac{f^{\prime}(x)}{x^{n-1}}\right)^{2} d x\right)^{1 / 2}
\end{aligned}
$$

This gives us $\left\|f(x) / x^{n}\right\|_{L^{2}} \leq C\left\|f^{\prime}(x) / x^{n-1}\right\|_{L^{2}}$. We can iterate until we get $\|u\|_{H^{m+1}} \leq$ $C\|\omega\|_{H^{m}}$.

The $u^{(m+1)} \in L^{\infty}$ estimate follows from Taylor error estimates $\left|f(x) / x^{n}\right| \leq C\left\|f^{(n)}\right\|_{L^{\infty}}$.

Remark 2.1. The above $u^{(m+1)} \in L^{\infty}$ estimate does not hold for the case $m=0$. Instead, we have only pointwise estimate:

$$
\left|u^{\prime}(x)\right| \leq\|\omega\|_{L^{\infty}} \cdot(1-\ln (x)) \text { for } x \in(0,1)
$$

However, it will be proved for a solution $\omega$ on $[0, T)$ with finite $T$ that $\int_{0}^{T}\|\omega(t)\|_{L^{\infty}} d t<$ $\infty$ implies $\int_{0}^{T}\left\|\partial_{x} u(t)\right\|_{L^{\infty}} d t<\infty$ (see Proposition 3.1).

Remark 2.2. We can weaken the condition that $\omega$ is compactly supported in $(0,1)$. For example, in order to get $u \in H^{m+1}$ estimate assuming $\omega \in H_{0}^{m}((0,1))$ is enough (where $H_{0}^{m}((0,1))$ is the completion of $\left(C_{0}^{\infty} \cap H^{m}\right)((0,1))$ by using the topology of $\left.H^{m}((0,1))\right)$. Recall that we used the fact that $\omega$ is compactly supported in $(0,1)$ only to say the boundary term $\left.\left(\frac{f(x)}{x^{n-(1 / 2)}}\right)^{2}\right|_{x=0} ^{1}$ from integration by parts vanishes. From Sobolev embedding, $\omega \in H_{0}^{m}$ implies $\omega \in C^{m-1}$ and $\omega^{(i)}(0)=\omega^{(i)}(1)=0$ for $i=0,1, \ldots,(m-1)$. Moreover, the embedding gives us $\omega^{(m-1)} \in C^{1 / 2}$-Holder space, which implies $\frac{\omega^{(m-1)}(x)}{\sqrt{x}} \leq C\|\omega\|_{H^{m}}$. Taking $\omega \in H_{0}^{m}$ suffices to carry out the same computation in the same manner as for compactly supported function. Similarly, it is enough for $u^{(m+1)} \in L^{\infty}$ estimate to assume $\omega^{(m)} \in L^{\infty}$ and $\omega^{(i)}(0)=\omega^{(i)}(1)=0$ for $i=0,1, \ldots,(m-1)$ instead of assuming that $\omega$ is compactly supported in $(0,1)$.

Proposition 2.2. Given any initial data $\left(\omega_{0}, \rho_{0}\right) \in H_{0}^{m}((0,1)) \times H_{0}^{m+1}((0,1))$ with $m \geq 2$, there exists $T=T\left(\left\|\omega_{0}\right\|_{H^{m}}+\left\|\rho_{0}\right\|_{H^{m+1}}\right)>0$ such that the system has a unique classical solution $(\omega, \rho) \in C\left([0, T] ; H_{0}^{m} \times H_{0}^{m+1}\right)$.

Proof. Consider a function $\psi \in C^{\infty}(\mathbb{R})$ such that $\int \psi=1, \psi \geq 0$ and $\operatorname{supp}(\psi) \subset$ $[-1,1]$, and set $\psi_{\epsilon}(x):=\psi(x / \epsilon) / \epsilon$ for $\epsilon>0$. First we replace the initial data $\left(\omega_{0}, \rho_{0}\right)$ with approximations compactly supported in $(0,1)$, given by $\left(\tilde{\omega_{0}}, \tilde{\rho_{0}}\right)(x):=$ $\left(\omega_{0}, \rho_{0}\right)\left(\frac{x-2 \epsilon}{1-4 \epsilon}\right)$. Then we mollify the initial data $\left(\tilde{\omega_{0}}, \tilde{\rho_{0}}\right)$ by convolution: $\omega_{0}^{\epsilon}:=\tilde{\omega_{0}} * \psi_{\epsilon}$ and $\rho_{0}^{\epsilon}:=\tilde{\rho_{0}} * \psi_{\epsilon}$. Note that $\omega_{0}^{\epsilon}$ and $\rho_{0}^{\epsilon}$ lie in $C^{\infty}$ and they are compactly supported in $[\epsilon, 1-\epsilon] \subset(0,1)$.
Define $u_{0}^{\epsilon}(t, x):=-x \int_{x}^{1} \frac{\omega_{0}^{\epsilon}(y)}{y} d y$. Then consider the following iteration scheme for $n \geq 1$ :

$$
\left\{\begin{array}{l}
\partial_{t} \rho_{n}^{\epsilon}+u_{n-1}^{\epsilon} \partial_{x} \rho_{n}^{\epsilon}=0 \text { with } \rho^{\epsilon}(0)=\rho_{0}^{\epsilon}  \tag{6}\\
\partial_{t} \omega_{n}^{\epsilon}+u_{n-1}^{\epsilon} \partial_{x} \omega_{n}^{\epsilon}=\partial_{x} \rho_{n}^{\epsilon} \text { with } \omega_{n}^{\epsilon}(0)=\omega_{0}^{\epsilon} \\
u_{n}^{\epsilon}(t, x)=-x \int_{x}^{1} \frac{\omega_{n}^{\epsilon}(t, y)}{y} d y
\end{array}\right.
$$

Namely, for each $n \geq 1$, we can solve the characteristic equations

$$
\left\{\begin{array}{l}
\frac{d}{d t} \phi_{n}^{\epsilon}(t, x)=u_{n-1}^{\epsilon}\left(t, \phi_{n}(t, x)\right) \\
\phi_{n}^{\epsilon}(0, x)=x
\end{array}\right.
$$

for $t \in[0, \infty)$ since $u_{n-1}^{\epsilon} \in C_{t, x}^{\infty}$. Then define $\rho_{n}^{\epsilon}, \omega_{n}^{\epsilon}$ for $t \in[0, \infty)$ via the characteristics so that $\rho_{n}^{\epsilon}\left(t, \phi_{n}^{\epsilon}(t, x)\right)=\rho_{0}^{\epsilon}(x)$ and $\omega_{n}^{\epsilon}\left(t, \phi_{n}^{\epsilon}(t, x)\right)=\omega_{0}^{\epsilon}(x)+\int_{0}^{t}\left(\partial_{x} \rho_{n}^{\epsilon}\right)\left(s, \phi_{n}^{\epsilon}(s, x)\right) d s$. Note that this process can be repeated and we get $\rho_{n}^{\epsilon}, \omega_{n}^{\epsilon} \in C_{t, x}^{\infty}$ which are are compactly supported in $(0,1)$ for each $t>0$ since $x=0$ and 1 are stationary points under the flow.

Let $m \geq 2$. Simple energy estimates give us that, for any $n \geq 1$,

$$
\frac{d}{d t}\left(\left\|\omega_{n}^{\epsilon}(t)\right\|_{H^{m}}^{2}+\left\|\rho_{n}^{\epsilon}(t)\right\|_{H^{m+1}}^{2}\right) \leq C\left(\left\|u_{n-1}^{\epsilon}(t)\right\|_{H^{m+1}}+1\right)\left(\left\|\omega_{n}^{\epsilon}(t)\right\|_{H^{m}}^{2}+\left\|\rho_{n}^{\epsilon}(t)\right\|_{H^{m+1}}^{2}\right) .
$$

Since $\rho_{n}^{\epsilon}(t), \omega_{n}^{\epsilon}(t)$ are compactly supported in $(0,1)$, we have $\left\|u_{n-1}^{\epsilon}(t)\right\|_{H^{m+1}} \leq C\left\|\omega_{n-1}^{\epsilon}(t)\right\|_{H^{m}}$ by the previous lemma. As a result, we obtain $\left\{\begin{array}{l}\frac{d}{d t} f_{n}^{\epsilon}(t) \leq C \sqrt{f_{n-1}^{\epsilon}(t)} f_{n}^{\epsilon}(t), \\ f_{n}^{\epsilon}(0)=f_{0}^{\epsilon}\end{array}\right.$ where $f_{n}^{\epsilon}(t):=\left\|\omega_{n}^{\epsilon}(t)\right\|_{H^{m}}^{2}+\left\|\rho_{n}^{\epsilon}(t)\right\|_{H^{m+1}}^{2}+1$ and $f_{0}^{\epsilon}:=\left\|\omega_{0}^{\epsilon}\right\|_{H^{m}}^{2}+\left\|\rho_{0}^{\epsilon}\right\|_{H^{m+1}}^{2}+1$. After a straightforward monotonicity argument, this implies

$$
\begin{equation*}
\left(f_{n}^{\epsilon}(t)\right) \leq 1 /\left(\left(f_{0}^{\epsilon}\right)^{-1 / 2}-C t\right)^{2}, \quad \text { for } n \geq 1 \text { and for } 0 \leq t<C / \sqrt{f_{0}^{\epsilon}} \tag{7}
\end{equation*}
$$

Denote $f_{0}:=\left\|\omega_{0}\right\|_{H^{m}}^{2}+\left\|\rho_{0}\right\|_{H^{m+1}}^{2}+1$. Take $T$ between 0 and $C / \sqrt{f_{0}}$. Thanks to the fact that $f_{0}^{\epsilon}$ converges to $f_{0}$ as $\epsilon \rightarrow 0$, we know $T<C / \sqrt{f_{0}^{\epsilon}}$ for sufficiently small $\epsilon>0$. Then, for small $\epsilon>0$, we get

$$
\begin{equation*}
\sup _{t \in[0, T]}\left(\left\|\omega_{n}^{\epsilon}(t)\right\|_{H^{m}}+\left\|\rho_{n}^{\epsilon}(t)\right\|_{H^{m+1}}\right)<\infty \tag{8}
\end{equation*}
$$

and, by using the structure of (6),

$$
\begin{equation*}
\sup _{t \in[0, T]}\left(\left\|\partial_{t} \omega_{n}^{\epsilon}(t)\right\|_{H^{m-1}}+\left\|\partial_{t} \rho_{n}^{\epsilon}(t)\right\|_{H^{m}}\right)<\infty \tag{9}
\end{equation*}
$$

Note that the above estimates are uniform in $n \geq 1$. Then the existence of a solution $\left(\omega^{\epsilon}, \rho^{\epsilon}\right) \in C\left([0, T] ; H_{0}^{m} \times H_{0}^{m+1}\right)$ to (3) corresponding the mollified initial data $\left(\omega_{0}^{\epsilon}, \rho_{0}^{\epsilon}\right)$ follows the standard argument (e.g. see [15]).

We briefly sketch this argument here. First there exists a weak-* limit $\left(\omega^{\epsilon}, \rho^{\epsilon}\right) \in$ $L^{\infty}\left(0, T ; H_{0}^{m} \times H_{0}^{m+1}\right)$, which follows from (8) by Banach-Alaoglu theorem. Then, by using (8) and (9), we can show strong convergence $\left(\omega_{n}^{\epsilon}, \rho_{n}^{\epsilon}\right) \rightarrow\left(\omega^{\epsilon}, \rho^{\epsilon}\right)$ in $C\left([0, T] ; H^{m-\delta} \times\right.$ $\left.H^{m+1-\delta}\right)$ for all real $\delta>0$. Recall that we assumed $m \geq 2$. Thus, from Sobolev's inequality, all terms in (3) become continuous (pointwise). Moreover (6) converges pointwise to (3). It shows that $\left(\omega^{\epsilon}, \rho^{\epsilon}\right)$ is a classical solution to (3). Since $H^{-(m-\delta)} \times$ $H^{-(m+1-\delta)}$ is dense in $H^{-m} \times H^{-(m+1)}$, our solution $\left(\omega^{\epsilon}, \rho^{\epsilon}\right)$ is weakly continuous in time variable as a $H_{0}^{m} \times H_{0}^{m+1}$ valued function. Lastly, thanks to weak continuity in time and the estimate (7), we can show $\left(\omega^{\epsilon}, \rho^{\epsilon}\right) \in C\left([0, T] ; H_{0}^{m} \times H_{0}^{m+1}\right)$ by showing that both $\left\|\omega_{n}^{\epsilon}(t)\right\|_{H^{m}}$ and $\left\|\rho_{n}^{\epsilon}(t)\right\|_{H^{m+1}}$ are continuous in time variable $t \in[0, T]$. In addition, we have

$$
\begin{equation*}
f^{\epsilon}(t) \leq 1 /\left(\left(f_{0}^{\epsilon}\right)^{-1 / 2}-C t\right)^{2}, \quad \text { for } 0 \leq t \leq T . \tag{10}
\end{equation*}
$$

To find a solution for the original initial data $\left(\omega_{0}, \rho_{0}\right)$, recall that $T$ does not depend on $\epsilon$, the estimate (10) is uniform in $\epsilon>0$, and $f_{0}^{\epsilon}$ converges to $f_{0}$ as $\epsilon \rightarrow 0$. Then we repeat the above procedure as $\epsilon \rightarrow 0$ in order to get a solution $(\omega, \rho) \in C\left([0, T] ; H_{0}^{m} \times\right.$
$H_{0}^{m+1}$ ) to (3) corresponding to $\left(\omega_{0}, \rho_{0}\right)$ with the same estimate

$$
\left(\|\omega(t)\|_{H^{m}}^{2}+\|\rho(t)\|_{H^{m+1}}^{2}+1\right) \leq 1 /\left(\left(f_{0}\right)^{-1 / 2}-C t\right)^{2} \quad \text { for } 0 \leq t \leq T
$$

Its uniqueness in the space $C\left([0, T] ; H_{0}^{m} \times H_{0}^{m+1}\right)$ is easy to show (e.g. see [4]).

## 3. Beale-Kato-Majda type criteria

Proposition 3.1. Let $(\omega, \rho) \in C\left([0, T) ; H_{0}^{m} \times H_{0}^{m+1}\right)$ be the unique solution provided by Proposition 2.2 for initial data $\left(\omega_{0}, \rho_{0}\right) \in H_{0}^{m} \times H_{0}^{m+1}$ with $m \geq 2$. Then for any finite $T^{*} \leq T$, the followings are equivalent:
(1). $\sup _{t \in\left[0, T^{*}\right]}\left(\|\omega(t)\|_{H^{m}}+\|\rho(t)\|_{H^{m+1}}\right)<\infty$.
(2). $\int_{0}^{T^{*}}\left\|\partial_{x} u(t)\right\|_{L^{\infty}} d t<\infty$.
(3). $\int_{0}^{T^{*}}\|\omega(t)\|_{L^{\infty}} d t<\infty$.
(4). $\int_{0}^{T^{*}}\left\|\partial_{x} \rho(t)\right\|_{L^{\infty}} d t<\infty$.

Remark 3.1. It is well known that for a full 2D inviscid Boussinesq system, either $\int_{0}^{T^{*}}\|\nabla u(t)\|_{L^{\infty}} d t<\infty$ or $\int_{0}^{T^{*}}\|\nabla \rho(t)\|_{L^{\infty}} d t<\infty$ implies (1) (see e.g. [4], [2]). Whether (3) implies (1) for 2D inviscid Boussinesq system is an interesting open question.

Proof. The implication $(1) \Rightarrow(2),(3)$ and (4) is obvious from Sobolev's inequality.
The direction $(2) \Rightarrow(1)$ follows from a standard energy estimate. Indeed, if we denote $M:=\int_{0}^{T^{*}}\left\|\partial_{x} u(t)\right\|_{L^{\infty}} d t<\infty$, then we get for any $t \in\left[0, T^{*}\right]$,

$$
\begin{aligned}
& \left\|\partial_{x} \rho(t)\right\|_{L^{2}}^{2} \leq e^{C M}\left\|\partial_{x} \rho_{0}\right\|_{L^{2}}^{2}, \\
& \|\omega(t)\|_{L^{2}}^{2} \leq e^{C M}\left(1+T^{*}\right)\left(\left\|\omega_{0}\right\|_{L^{2}}^{2}+\left\|\partial_{x} \rho_{0}\right\|_{L^{2}}^{2}\right), \\
& \left\|\partial_{x} \rho(t)\right\|_{L^{\infty}} \leq e^{M}\left\|\partial_{x} \rho_{0}\right\|_{L^{\infty}}, \text { and } \\
& \|\omega(t)\|_{L^{\infty}} \leq\left\|\omega_{0}\right\|_{L^{\infty}}+e^{M} T^{*}\left\|\partial_{x} \rho_{0}\right\|_{L^{\infty}} .
\end{aligned}
$$

Then straightforward estimates lead us to

$$
\begin{aligned}
& \left\|\omega^{\prime}(t)\right\|_{L^{2}}+\left\|\rho^{\prime \prime}(t)\right\|_{L^{2}} \leq C_{M, T^{*},\left\|\omega_{0}\right\|_{H^{1}},\left\|\rho_{0}\right\|_{H^{2}} \quad \text { and }}\left\|\omega^{\prime}(t)\right\|_{L^{\infty}}+\left\|\rho^{\prime \prime}(t)\right\|_{L^{\infty}} \leq C_{M, T^{*},\left\|\omega_{0}\right\|_{W^{1}, \infty},\left\|\rho_{0}\right\|_{W^{2}, \infty}}
\end{aligned}
$$

where $W^{n, p}$ is the usual Sobolev space. We repeat this procedure until we get

$$
\left\|\omega^{(m)}(t)\right\|_{L^{2}}+\left\|\rho^{(m+1)}(t)\right\|_{L^{2}} \leq C_{M, T^{*},\left\|\omega_{0}\right\|_{H^{m}},\left\|\rho_{0}\right\|_{H^{m+1}}}
$$

For $(4) \Rightarrow(3)$, we use the characteristic representation for $\omega$ :

$$
\omega\left(t, \phi_{t}(x)\right)=\omega_{0}(x)+\int_{0}^{t}\left(\partial_{x} \rho\right)\left(s, \phi_{s}(x)\right) d s
$$

For the direction (3) $\Rightarrow(2)$, we denote $M:=\int_{0}^{T^{*}}\|\omega(t)\|_{L^{\infty}} d t<\infty$. Then we make an $L^{\infty}$-estimate for $\partial_{x} u$ in the following way.

1. From $|u(t, x)| \leq\|\omega(t)\|_{L^{\infty}} \cdot x \cdot(-\ln (x))$, we get $\phi_{t}(x) \geq x^{\exp \left(\int_{0}^{t}\|\omega(s)\|_{L^{\infty} d s}\right.} \geq x^{\exp (M)}$ for $t \leq T^{*}$. We also get $\phi_{-t}(x) \leq x^{\exp (-M)}$.
2. From $\partial_{x} u=-\Omega+\omega$, we get

$$
\begin{aligned}
\left|\left(\partial_{x} u\right)\left(t, \phi_{t}(x)\right)\right| & \leq\left|\omega\left(t, \phi_{t}(x)\right)\right|+\left|\Omega\left(t, \phi_{t}(x)\right)\right| \leq\|\omega(t)\|_{L^{\infty}}\left(1+\left(-\ln \left(\phi_{t}(x)\right)\right)\right. \\
& \leq\|\omega(t)\|_{L^{\infty}}\left(1+e^{M}(-\ln (x))\right) .
\end{aligned}
$$

3. From $\partial_{t}\left(\partial_{x} \rho\right)+u \partial_{x}\left(\partial_{x} \rho\right)=-\left(\partial_{x} u\right)\left(\partial_{x} \rho\right)$, we obtain

$$
\left|\left(\partial_{x} \rho\right)\left(t, \phi_{t}(x)\right)\right| \leq\left|\left(\partial_{x} \rho_{0}\right)(x)\right|+\int_{0}^{t}\left|\left(\partial_{x} u\right)\left(s, \phi_{s}(x)\right)\right| \cdot\left|\left(\partial_{x} \rho\right)\left(s, \phi_{s}(x)\right)\right| d s
$$

This implies

$$
\begin{aligned}
\left|\left(\partial_{x} \rho\right)\left(t, \phi_{t}(x)\right)\right| & \leq\left|\left(\partial_{x} \rho_{0}\right)(x)\right| \exp \left(\int_{0}^{t}\left|\left(\partial_{x} u\right)\left(s, \phi_{s}(x)\right)\right| d s\right) \\
& \leq\left|\left(\partial_{x} \rho_{0}\right)(x)\right| \exp \left(\int_{0}^{t}\|\omega(s)\|_{L^{\infty}}\left(1+e^{M}(-\ln (x)) d s\right)\right. \\
& \leq\left|\left(\partial_{x} \rho_{0}\right)(x)\right| e^{M}\left(\frac{1}{x}\right)^{e^{M} \cdot M} .
\end{aligned}
$$

4. For a moment, assume that $M$ is so small that $M \cdot e^{M} \leq \frac{1}{2}$. Thanks to $\omega_{0}(0)=$ $\partial_{x} \rho_{0}(0)=0$, we can estimate

$$
\begin{aligned}
\left|\omega\left(t, \phi_{t}(x)\right)\right| & \leq\left|\omega_{0}(x)\right|+\int_{0}^{t}\left|\left(\partial_{x} \rho\right)\left(s, \phi_{s}(x)\right)\right| d s \\
& \leq\left|\omega_{0}(x)\right|+\left|\left(\partial_{x} \rho_{0}\right)(x)\right| \cdot e^{M} \cdot\left(\frac{1}{x}\right)^{e^{M} \cdot M} \cdot T^{*} \\
& \leq\left\|\omega_{0}^{\prime}\right\|_{L^{\infty}} \cdot x+\left\|\rho_{0}^{\prime \prime}\right\|_{L^{\infty}} \cdot x \cdot e^{M} \cdot\left(\frac{1}{x}\right)^{1 / 2} \cdot T^{*} \\
& \leq C_{0} \sqrt{x} e^{M}\left(T^{*}+1\right)
\end{aligned}
$$

where $C_{0}:=\left\|\omega_{0}^{\prime}\right\|_{L^{\infty}}+\left\|\rho_{0}^{\prime \prime}\right\|_{L^{\infty}}$. So we get a decay estimate of $\omega(t, x)$ near $x=0$ :

$$
|\omega(t, x)| \leq C_{0} \sqrt{\phi_{-t}(x)} e^{M}\left(T^{*}+1\right) \leq C_{0} x^{\frac{1}{2} \exp (-M)} e^{M}\left(T^{*}+1\right)
$$

This implies $L^{\infty}$ estimate of $\Omega$ :

$$
|\Omega(t, x)| \leq \int_{0}^{1} \frac{|\omega(t, y)|}{y} d y \leq C_{0} e^{M}\left(T^{*}+1\right) \int_{0}^{1} y^{\frac{1}{2} \exp (-M)-1} d y \leq 2 C_{0} e^{2 M}\left(T^{*}+1\right)
$$

Then we use $\partial_{x} u=-\Omega+\omega$ to get

$$
\left\|\partial_{x} u(t)\right\|_{L^{\infty}} \leq\|\omega(t)\|_{L^{\infty}}+2 C_{0} e^{2 M}\left(T^{*}+1\right) \quad \text { for } t \in\left[0, T^{*}\right]
$$

5. For general large $M$, we find $\sigma \in\left(0, T^{*}\right)$ such that $M_{\sigma}:=\int_{\sigma}^{T^{*}}|\omega(s)|_{L^{\infty}} d s$ is so small that $M_{\sigma} \cdot e^{M_{\sigma}} \leq \frac{1}{2}$. We do the same process not from $t=0$ but from $t=\sigma$ to get

$$
\left\|\partial_{x} u(t)\right\|_{L^{\infty}} \leq\|\omega(t)\|_{L^{\infty}}+2 C_{\sigma} e^{2 M}\left(\left(T^{*}-\sigma\right)+1\right) \quad \text { for } t \in\left[\sigma, T^{*}\right]
$$

where $C_{\sigma}:=\sup _{t \in[0, \sigma]}\left(\|\omega(t)\|_{H_{0}^{2}}+\|\rho(t)\|_{H_{0}^{3}}\right)$. Note that $C_{\sigma}$ is finite because $(\omega, \rho)$ lies in $C\left([0, T) ; H_{0}^{2} \times H_{0}^{3}\right)$ and $\sigma<T^{*} \leq T$.

Since $\left\|\partial_{x} u(t)\right\|_{L^{\infty}} \leq C\|u(t)\|_{H^{2}} \leq C\|\omega(t)\|_{H^{1}} \leq C C_{\sigma}$ for any $t \in[0, \sigma]$, we conclude

$$
\left\|\partial_{x} u(t)\right\|_{L^{\infty}} \leq\|\omega(t)\|_{L^{\infty}}+2 C_{\sigma} e^{2 M}\left(T^{*}+1\right)+C C_{\sigma} \quad \text { for } t \in\left[0, T^{*}\right] .
$$

## 4. Finite-time blow up examples

Before we construct a finite-time blow up example, let us first state a lemma concerning the growth of $\Omega$ along the characteristics $\phi_{t}(x)$.

Lemma 4.1. Along the characteristic $\phi_{t}(x)$, we have

$$
\begin{equation*}
\frac{d}{d t} \Omega\left(t, \phi_{t}(x)\right)=\int_{\phi_{t}(x)}^{1} \frac{\omega(t, y)^{2}}{y} d y+\int_{\phi_{t}(x)}^{1} \frac{\partial_{x} \rho(t, y)}{y} d y \tag{11}
\end{equation*}
$$

Proof. Note that

$$
\begin{equation*}
\frac{d}{d t} \Omega\left(t, \phi_{t}(x)\right)=\partial_{t} \Omega\left(t, \phi_{t}(x)\right)+u\left(t, \phi_{t}(x)\right) \partial_{x} \Omega\left(t, \phi_{t}(x)\right) \tag{12}
\end{equation*}
$$

Let us compute $\partial_{x} \Omega$ and $\partial_{t} \Omega$ respectively. The definition of $\Omega$ directly gives that

$$
\begin{equation*}
\partial_{x} \Omega(t, x)=-\frac{\omega(t, x)}{x} \tag{13}
\end{equation*}
$$

whereas $\partial_{t} \Omega(t, x)$ can be computed as follows:

$$
\begin{align*}
\partial_{t} \Omega(t, x) & =\int_{x}^{1} \frac{\partial_{t} \omega(t, y)}{y} d y=-\int_{x}^{1} \frac{u(t, y) \partial_{x} \omega(t, y)}{y} d y+\int_{x}^{1} \frac{\partial_{x} \rho(t, y)}{y} d y \\
& =\int_{x}^{1} \Omega(t, y) \partial_{x} \omega(t, y) d y+\int_{x}^{1} \frac{\partial_{x} \rho(t, y)}{y} d y  \tag{14}\\
& =-\Omega(t, x) \omega(t, x)+\int_{x}^{1} \frac{\omega(t, y)^{2}}{y} d y+\int_{x}^{1} \frac{\partial_{x} \rho(t, y)}{y} d y
\end{align*}
$$

In order to obtain (11), it suffices to replace $x$ by $\phi_{t}(x)$ in (13) and (14), and plug them into (12).

We now prove the following Proposition from which, given Proposition 3.1, Theorem 1.1 follows.

Proposition 4.2. There exist a pair of smooth functions $\rho_{0}$ and $\omega_{0}$ supported in $\left[\frac{1}{4}, \frac{3}{4}\right]$, such that there is no global classical solution to (3) with initial data $\left(\rho_{0}, \omega_{0}\right)$.

Proof. Step 1. We construct a pair of initial data $\left(\rho_{0}, \omega_{0}\right)$ as follows. Let $\rho_{0}$ be smooth, nonnegative, supported in $\left[\frac{1}{4}, \frac{3}{4}\right]$, with $\max \rho_{0}=\rho_{0}\left(\frac{1}{2}\right)=2$, and $\rho_{0}\left(\frac{1}{3}\right)=1$. Moreover, assume $\rho_{0}$ is increasing in $\left[\frac{1}{4}, \frac{1}{2}\right]$, and decreasing in $\left[\frac{1}{2}, \frac{3}{4}\right]$. Let $\omega_{0}$ be smooth, nonnegative, supported in $\left[\frac{1}{4}, \frac{1}{2}\right]$, with $\omega_{0} \equiv M$ in $[0.3,0.45]$, where $M$ is a large constant to be determined later. Figure 1 gives a sketch of the initial data.


Figure 1. A sketch of the initial data $\left(\rho_{0}, \omega_{0}\right)$.

Towards a contradiction we assume that there is a global classical solution. Let us first make a few observations. Note that for all $x \in(0,1)$, the characteristic $\phi_{t}(x)$ must be well-defined for all time, and $\rho$ is conserved along $\phi_{t}(x)$, i.e. $\rho\left(t, \phi_{t}(x)\right)=\rho_{0}(x)$. Moreover, for all $t \geq 0$, we have

$$
\begin{equation*}
\omega(x, t) \leq 0 \text { for } x \in\left[\phi_{t}(1 / 2), 1\right] . \tag{15}
\end{equation*}
$$

To see this, recall that by definition, $\rho_{0}$ is decreasing in $\left[\frac{1}{2}, 1\right]$. If there is a global classical solution, then the characteristics do not cross, hence for all $t \geq 0$, we have $\rho_{x}(x, t) \leq 0$ in $\left[\phi_{t}(1 / 2), 1\right]$. We then obtain (15) as a direct consequence, since the time derivative of $\omega$ along the characteristics $\phi_{t}(x)$ is equal to $\rho_{x}$.

Moreover, we have that $\phi_{t}(1 / 2)$ is increasing for all $t$. Note that

$$
\frac{d}{d t} \phi_{t}(1 / 2)=-\phi_{t}(1 / 2) \Omega\left(t, \phi_{t}(1 / 2)\right)=-\phi_{t}(1 / 2) \int_{\phi_{t}(1 / 2)}^{1} \frac{\omega(y, t)}{y} d y
$$

which is always non-negative due to (15).
Step 2. Our goal is to find a point $x_{\infty}\left(\right.$ with $\left.\rho_{0}\left(x_{\infty}\right)>0\right)$ and a finite time $T$, such that $\phi_{T}\left(x_{\infty}\right)=0$. This would imply that the classical solution has to break down at (or before) time $T$. To show this, the main idea is to consider a family of characteristics originating from a sequence of points $\left\{x_{n}\right\}$. Let $x_{1}=1 / 3$ (recall that we let $\rho_{0}\left(\frac{1}{3}\right)=1$ ).

For $n>1$, find $x_{n} \in\left[0, \frac{1}{2}\right]$, such that $\rho_{0}\left(x_{n}\right)=\frac{1}{2}+2^{-n}$. Observe that we have $x_{1}>x_{2}>x_{3}>\cdots$ since $\rho_{0}$ is increasing in $\left[0, \frac{1}{2}\right]$. Denote $x_{\infty}:=\lim _{n \rightarrow \infty} x_{n}$, and it follows that $x_{\infty}>0$ and $\rho\left(x_{\infty}\right)=1 / 2$. The choice of $\left\{x_{n}\right\}$ is illustrated in Figure 2.

Also, we choose $M$ large enough such that $C_{0}:=\Omega\left(0, x_{1}\right)=\int_{1 / 3}^{1} \frac{\omega_{0}(y)}{y} d y>20$ (e.g. $M=200$ should work). Note that at $t=0, \Omega(0, x)$ is decreasing in $x$ due to the non-negativity of $\omega_{0}$. This implies that at $t=0$, we have $\Omega\left(0, x_{n}\right)>20$ for all $n \geq 1$.


Figure 2. A sketch of the choice of $\left\{x_{n}\right\}$.

Let us denote $\rho_{n}:=\rho_{0}\left(x_{n}\right), \Phi_{n}(t):=\phi_{t}\left(x_{n}\right), \Omega_{n}(t):=\Omega\left(t, \Phi_{n}(t)\right)$. Observe that $\frac{d}{d t} \Phi_{n}(t)=u\left(t, \Phi_{n}(t)\right)=-\Phi_{n}(t) \Omega_{n}(t)$. Denoting $\psi_{n}(t):=-\ln \Phi_{n}(t)$ for $n \geq 1$, we get

$$
\frac{d}{d t} \psi_{n}(t)=\Omega_{n}(t)
$$

To see how $\Omega_{n}(t)$ grows in time, we apply Lemma 4.1 to $x_{n}$, and use the fact that $\phi_{t}(1 / 2) \geq 1 / 2$ for all $t \geq 0$. This gives

$$
\begin{align*}
\frac{d}{d t} \Omega_{n}(t) & \geq \int_{\Phi_{n}(t)}^{\phi_{t}(1 / 2)} \frac{\partial_{x} \rho(t, y)}{y} d y+\int_{\phi_{t}(1 / 2)}^{1} \frac{\partial_{x} \rho(t, y)}{y} d y \\
& \geq \int_{\Phi_{n}(t)}^{\phi_{t}(1 / 2)} \underbrace{\frac{\partial_{x} \rho(t, y)}{y}}_{\geq 0} d y+\underbrace{\frac{1}{\phi_{t}(1 / 2)}}_{\leq 2} \underbrace{\left(\rho(t, 1)-\rho\left(t, \phi_{t}(1 / 2)\right)\right.}_{=-2}  \tag{16}\\
& \geq \int_{\Phi_{n}(t)}^{\phi_{t}(1 / 2)} \frac{\partial_{x} \rho(t, y)}{y} d y-4 \quad \text { for all } n \geq 1 .
\end{align*}
$$

Recall that $\omega_{0}$ is chosen such that $\Omega_{n}(0) \geq 20$, hence (16) immediately implies $\Omega_{n}(t) \geq 0$ for all $n$ and all $t \in[0,5)$. Since $\frac{d}{d t} \psi_{n}(t)=\Omega_{n}(t)$, we have that $\psi_{n}(t)$ is increasing for $t \in[0,5)$.

For $n \geq 2$, using (16), we have

$$
\begin{align*}
\frac{d}{d t} \Omega_{n}(t) & \geq \int_{\Phi_{n}(t)}^{\Phi_{n-1}(t)} \frac{\partial_{x} \rho(t, y)}{y} d y-4 \\
& \geq\left(\rho_{n-1}-\rho_{n}\right) \frac{1}{\Phi_{n-1}(t)}-4  \tag{17}\\
& =2^{-n} e^{\psi_{n-1}(t)}-4
\end{align*}
$$

Collecting everything together, we arrive at the following system of inequalities:

$$
\left\{\begin{array}{l}
\psi_{n}^{\prime \prime}(t) \geq 2^{-n} e^{\psi_{n-1}(t)}-4  \tag{18}\\
\psi_{n-1}^{\prime}(t)=\Omega_{n-1}(t) \geq 0 \quad \text { for } n \geq 2,0 \leq t<5 \\
\psi_{n}(t) \geq \psi_{n-1}(t) \geq 0
\end{array}\right.
$$

Step 3. Take $t_{1}=1$, and let $t_{n+1}=t_{n}+2^{-n}$ and $\tilde{t}_{n}=t_{n}+2^{-(n+1)}$ for $n \geq 1$. Let $T:=\lim _{n \rightarrow \infty} t_{n}=2$ (Note that (18) holds until $t=5$, hence it holds for all $t \leq T$ ). We will show that $a_{n}:=\psi_{n}\left(t_{n}\right) \rightarrow \infty$ as $n \rightarrow \infty$.

Take $n \geq 2$. Since $\psi_{n-1}(t)$ is increasing in $t$ for all $t<5$, we have

$$
\begin{equation*}
\psi_{n}^{\prime \prime}(t) \geq 2^{-n} e^{\psi_{n-1}\left(t_{n-1}\right)}-4 \quad \text { for } t_{n-1} \leq t<5 . \tag{19}
\end{equation*}
$$

This implies that for $\tilde{t}_{n-1} \leq t \leq t_{n}$ (note that all $t_{n}$ 's are less than 5),

$$
\begin{aligned}
\psi_{n}^{\prime}(t) & \geq \psi_{n}^{\prime}\left(\tilde{t}_{n-1}\right)-4\left(t_{n}-\tilde{t}_{n-1}\right) \quad\left(\text { since } \psi_{n}^{\prime \prime}(t) \geq-4\right) \\
& \geq\left(2^{-n} e^{\psi_{n-1}\left(t_{n-1}\right)}-4\right)\left(\tilde{t}_{n-1}-t_{n-1}\right)+\psi_{n}^{\prime}\left(t_{n-1}\right)-4\left(t_{n}-\tilde{t}_{n-1}\right) \quad(\text { using }(19)) \\
& \geq\left(2^{-n} e^{\psi_{n-1}\left(t_{n-1}\right)}-4\right) 2^{-n}-4 \cdot 2^{-n} \\
& =\left(2^{-n} e^{\psi_{n-1}\left(t_{n-1}\right)}-8\right) 2^{-n} .
\end{aligned}
$$

Once we have the lower bound for $\psi_{n}^{\prime}(t)$ for $\tilde{t}_{n-1} \leq t \leq t_{n}$, we can use it get a lower bound for $\psi_{n}\left(t_{n}\right)$ as follows:

$$
\begin{align*}
\psi_{n}\left(t_{n}\right) & \geq\left(2^{-n} e^{\psi_{n-1}\left(t_{n-1}\right)}-8\right) 2^{-n} \cdot\left(t_{n}-\tilde{t}_{n-1}\right)+\psi_{n}\left(\tilde{t}_{n-1}\right) \\
& \geq\left(2^{-n} e^{\psi_{n-1}\left(t_{n-1}\right)}-8\right) 2^{-2 n}+\psi_{n-1}\left(\tilde{t}_{n-1}\right)  \tag{20}\\
& \geq\left(2^{-n} e^{\psi_{n-1}\left(t_{n-1}\right)}-8\right) 2^{-2 n}+\psi_{n-1}\left(t_{n-1}\right),
\end{align*}
$$

where in the second inequality we used the fact that $\psi_{n} \geq \psi_{n-1}$, and in the last inequality we used that $\phi_{n-1}$ is increasing for $t \leq 5$, hence $\phi_{n-1}\left(\tilde{t}_{n-1}\right) \geq \phi_{n-1}\left(t_{n-1}\right)$.

Step 4. Denoting $a_{n}:=\psi_{n}\left(t_{n}\right)$, we obtain the following recursive relation from (20):

$$
\begin{aligned}
a_{n} & \geq 2^{-2 n}\left(2^{-n} e^{a_{n-1}}-8\right)+a_{n-1} \\
& \geq e^{a_{n-1}-3 n}-1+a_{n-1} \quad \text { for } n \geq 2
\end{aligned}
$$

One can then use induction to show that if $a_{1} \geq 9$, then $a_{n} \geq 3 n+6$ for all $n \geq 1$, hence $a_{n} \rightarrow \infty$ as $n \rightarrow \infty$.

Finally it remains to check that whether $a_{1} \geq 9$ is satisfied, i.e. whether $\psi_{1}(1) \geq 9$. Recall that $\psi_{1}(0) \geq 0$, and $\psi_{1}^{\prime}(t)=\Omega_{1}(t)$, with $\Omega_{1}(t) \geq 20$ and $\Omega_{1}^{\prime}(t) \geq-4$. Hence we have $\Omega_{1}(t) \geq 16$ for $0 \leq t \leq 1$, which gives $\psi_{1}(1) \geq 16$, and this concludes the proof.

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[^0]:    Date: December 18, 2013.
    Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA; email: kchoi@math.wisc.edu.

    Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA; email: kiselev@math.wisc.edu.

    Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA; email: yaoyao@math.wisc.edu.

