Alexander Kiselev, Mikhail Chernobay, Omar Lazar, und Chao Li

SMALL SCALE CREATION IN INVISCID FLUIDS

Alexander Kiselev, Mikhail Chernobay, Omar Lazar,
und Chao Li

SMALL SCALE CREATION IN INVISCID FLUIDS

Unterscheidung als Hilfs- und Orientierungsbegriff zur Dekodierung

Herausgegeben von
Max Mustermann als Editor

In Zusammenarbeit mit
Mustermann Stiftung

2. Auflage

DE GRUYTER
MOUTON

Mathematics Subject Classification 2010
35-02, 65-02, 65C30, 65C05, 65N35, 65N75, 65N80

Auch beim Konzept der ästhetischen Bildung von Wissen und dessen möglichst rasche und erfolgsorientierte Anwendung verspielen Einsichten und Gewinne ohne den Bezug auf die um 1900 entwickelten Argumentationen.

Auch umfasst die Untersuchung in der Hauptsache den Zeitraum zwischen dem Inkrafttreten und der Darstellung in seiner heute geltenden Fassung. Ihre Funktion als Teil der literarischen Darstellung und narrativen Technik enthält auch einen vollständigen textkritischen Apparat und ein Verzeichnis der Stellen.

ISBN 978-3-11-021808-4
e-ISBN (PDF) 978-3-11-021809-1
ISSN 0179-0986

Library of Congress Cataloging-in-Publication Data
A CIP catalog record for this book has been applied for at the Library of Congress.

Bibliografische Information der Deutschen Nationalbibliothek
Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über http://dnb.dnb.de abrufbar.

Umschlaggestaltung: Cover-Firma
Satz: le-tex publishing services GmbH, Leipzig
Druck und Bindung: Druckerei XYZ
♾ Gedruckt auf säurefreiem Papier
Printed in Germany

www.degruyter.com

0.1 Introduction

Attempts to understand the nature of fluid motion have occupied minds of researchers for many centuries. Fluids are all around us, and we can witness the complexity and subtleness of their properties in every day life, in ubiquitous technology, and in dramatic phenomena such as tornado or hurricane. There has been an enormous wealth of knowledge accumulated in the broad area of fluid mechanics, yet it is quite remarkable that many of the most fundamental as well as important in applications questions remain poorly understood. A special role in fluid mechanics is played by incompressible Euler equation, first formulated in 1755 [10], the second partial differential equation (PDE) ever derived (the first one is wave equation derived by D'Alembert 8 years earlier). It describes motion of an inviscid, volume preserving fluid. The incompressible Euler equation is a nonlinear and nonlocal system of PDE, with dynamics near a given point depending on the flow field over the entire region filled with fluid. This makes analysis of these equations exceedingly challenging, and the array of mathematical methods applied to their study has been extremely broad.

The main function of a PDE such as Euler equation is, given some initial data, to allow us to compute the solution and use it for prediction of future behavior of the fluid. This is exactly how weather forecasting works, or how new airplane shapes are designed. Therefore, the first basic question one can ask about a PDE is existence and uniqueness of solutions in some appropriate class. If one can show existence and uniqueness of solutions, the PDE is often called globally regular (sometimes, continuous dependence on the initial data is also added to the list of desirable properties). On the other hand, if solutions can form singularities in finite time, the terminology for such phenomena is that finite time blow up happens. Understanding singularities is important because they correspond to dramatic, highly intense fluid motion, can indicate limits of applicability of the model, and are very difficult to resolve computationally. The story of global regularity vs finite time blow up for incompressible Euler equation is very different in two and three spatial dimensions. While for $d = 2$, global regularity is known since 1930s, the question remains open for $d = 3$, where only local existence of regular solutions is known. The reasons for such disparity will become clear below, once we write the Euler equation in vorticity form. More generally, one can ask a related and broader question about creation of small scales in fluids - coherent structures that vary sharply in space and time, and contribute to phenomena such as turbulence. The problem has a long history of contributions by leading mathematicians; mathematically, one often asks about lower bounds on the growth of derivatives of solutions in certain scenarios. One can consult the books [16], [17] for more details on history of the problem.

The main goal of these lectures is to review some recent developments in the area. A few years ago, based on extensive numerical simulations, Hou and Luo [15] have proposed a new scenario for singularity formation in the 3D Euler equation. The scenario has a fascinating, complex geometry - it is axi-symmetric, and growth in vorticity is observed at a ring of hyperbolic points of the flow located at the boundary of a cylinder. The solution has self-similar features which, however, do not appear to be exact. Inspired by this work, Kiselev and Sverak provided a rigorous construction of an example of solutions of 2D Euler equation where growth of vorticity gradient is very fast - double exponential in time. Such rate of growth is known to be sharp. Also, a couple of new one-dimensional models have been developed to gain insight into the Hou-Luo scenario. We will review these and related earlier works and outline some open questions and directions.

Our starting point is a global regularity result for solutions of 2D Euler equation. We will follow the approach by Yudovich, often referred to as Yudovich theory. It establishes existence and uniqueness of solutions for initial data with bounded vorticity, $\omega = \mathrm{curl} u$. As we will see, this class is very natural since it leads to log-Lipschitz fluid velocities ensuring existence of uniqueness of fluid particle trajectories. We will also see that the results can be easily upgraded to more regular initial data and solutions.

We are mostly interested in the study of the Euler equation in a bounded domain $D \subset \mathbb{R}^d$ that is compact and smooth. In the first sections, we will consider the case $d = 2$, and in the later ones discuss one-dimensional models of the three-dimensional phenomena. The incompressible Euler equation reads as follows

$$(\mathcal{E}) \ : \begin{cases} \partial_t u + (u \cdot \nabla u) = \nabla p \\ u \cdot n|_{\partial D} = 0 \\ \nabla \cdot u = 0 \\ u(x,0) = u_0(x) \end{cases}$$

The velocity field u is given by the Biot-Savart law $u = \nabla^{\perp}(-\Delta)^{-1}\omega$ where $\nabla^{\perp} = (\partial_2, -\partial_1)$. It is well known that, when written in terms of the vorticity $\omega = \mathrm{curl}\ u$ the Euler equation becomes the following 2D transport equation

$$\partial_t \omega + (u \cdot \nabla)\omega = 0. \tag{1}$$

In three dimensions, there is also the vortex stretching term $(\omega \cdot \nabla)u$ on the right hand side, but it vanishes in 2D.

On can consider the equation in terms of trajectories $\Phi_t(x)$ (the flow map corresponding to the 2D Euler) that is

$$(\mathcal{E}) \ : \begin{cases} \dfrac{d\Phi_t}{dt} = u(\Phi_t(x), t) \\ \omega(\Phi_t(x), t) = \omega_0(x) \\ \Phi_0(x) = x. \end{cases}$$

We have

$$\begin{aligned}\left|\frac{d}{dt}|\Phi_t(x)-\Phi_t(y)|^2\right| &\leq 2\,|u(\Phi_t(x,t))-u(\Phi_t(y),t))\cdot(\Phi_t(x)-\Phi_t(y))| \\ &\leq 2\|\nabla u(\cdot,t)\|_{L^\infty}|\Phi_t(x)-\Phi_t(y)|^2\end{aligned}$$

Then, using Grönwall's lemma,

$$\exp\left(-\int_0^t \|\nabla u\|_{L^\infty}\ ds\right) \leq \frac{|\Phi_t(x)-\Phi_t(y)|}{|x-y|} \leq \exp\left(\int_0^t \|\nabla u(\cdot,s)\|_{L^\infty}\ ds\right) \qquad (2)$$

So $\omega(x,t)=\omega_0(\Phi_t^{-1}(x))$. We are going to assume $\omega_0\in L^\infty$. We will see that this is a natural class for the existence and uniqueness theory. Let us introduce the notation

$$u=\nabla^\perp(-\Delta_D)^{-1}\omega=\nabla^\perp\int_D G_D(x,y)\omega(y)\ dy=\int_D \underbrace{\nabla^\perp G_D(x,y)}_{K_D(x,y)}\omega(y)\ dy$$

where

$$G_D(x,y)=\frac{1}{2\pi}\ln(|x-y|)+h(x,y)$$

and h is such that

$$\Delta_x h(x,y)=0 \ \text{ and } \ h(x,y)|_{x\in\partial D}=-\frac{1}{2\pi}\ln(|x-y|)$$

Lemma 0.1.1. *We have the following estimates :*

$$\begin{aligned}|G_D(x,y)| &\leq C(1+\ln(|x-y|)) \\ |\nabla G_D(x,y)| &\leq C(D)|x-y|^{-1} \\ |\nabla^2 G_D(x,y)| &\leq C(D)|x-y|^{-2}\end{aligned}$$

We have the following proposition

Proposition 0.1.2. The following estimate holds: for every $x,x'\in D$,

$$\int_D |K_D(x,y)-K_D(x',y)|\ dy \leq C\rho(|x-x'|)$$

where ρ is defined by $\rho(r)=r(1-\ln(r))$ if $r\leq 1$ and $\rho(r)=1$ if $r\geq 1$.

Sketch of the proof: The main interesting regime is when x and x' are close enough that is $|x-x'|=\delta<1$. Then we have

$$(i) \quad \int\limits_{D\cap B_{2\delta}(x)} |K_D(x,y) - K_D(x',y)|\; dy \le \int\limits_{B_{3\delta}(x)} \frac{C}{|x-y|}\; dy = C\int\limits_0^{3\delta} \frac{1}{r} r\; dr = c\delta$$

and

$$(ii) \quad \int\limits_{D\cap B^c_{2\delta}(x)} |K_D(x,y) - K_D(x',y)| dy \le C\delta \int\limits_{D\cap B^c_{2\delta}(x)} |\nabla K_D(x''(y),y)|\; dy$$

$$\le \delta \int\limits_{\delta}^{c} \frac{r}{r^2}\; dr \le c\delta(1-\ln(\delta))$$

Note that due to presence of the boundary, $x''(y)$ may in general not lie on an interval between x and x'; one has instead to use a path which lies entirely in D. We leave details of the argument to interested reader. □

Proposition 0.1.3. Assume $u(x,t)$ is log-Lipschitz

$$|u(x,t) - u(x',t)| \le c\rho(|x-x'|)$$

Then the Cauchy problem

$$(\mathcal{C}) \;:\; \begin{cases} x'(t) = u(x(t),t) \\ x(0) = x_0 \end{cases}$$

has a unique solution.

The uniqueness can be proved in the usual way. That is, assume we have two different solutions $x(t)$ and $y(t)$. Set $z(t) = x(t) - y(t)$, then

$$|z'(t)| \le |u(x(t),t) - u(y(t),t)| \le Cz(t)(1-\ln z(t))$$

and so

$$\int\limits_{z(0)}^{z(t)} \frac{dz}{z(1-\ln(z))} \le Ct.$$

This is impossible if $z(0) = 0$ and $z(t) > 0$ for some t.

To construct a solution, we consider the sequence of equations

$$u_n(x,t) = K_D * \omega_{n-1}(x,t).$$

It is equivalent to solve the trajectories equation

$$\frac{d\Phi^n_t(x,t)}{dt} = u_n(\Phi^n_t, t)$$

and set $\omega_n(x,t) = w_0((\Phi_t^n)^{-1}(x))$. Then, since the sequence ω_n is uniformly bounded it implies that u_n is uniformly log-Lipschitz and we have

$$\left|\frac{d}{dt}|\Phi_t^n(x) - \Phi_t^n(y)|^2\right| \leq C\,|\Phi_t^n(x) - \Phi_t^n(y)|^2\,(1 - \log|\Phi_t^n(x) - \Phi_t^n(y)|)$$

(while $|\Phi_t^n(x) - \Phi_t^n(y)| \leq 1/2$ and so

$$\left|\frac{d}{dt}\log|\Phi_t^n(x) - \Phi_t^n(y)|\right| \leq -C\log|\Phi_t^n(x) - \Phi_t^n(y)|$$

Grönwall lemma then gives the following two sided estimate

$$|x-y|e^{Ct} \leq |\Phi_t^n(x) - \Phi_t^n(y)| \leq |x-y|e^{-Ct}.$$

Therefore, Φ_t is uniformly Hölder continuous in x for every t. The same applies to the inverse map Φ_t^{-1} :

$$|x-y|^{e^{Ct}} \leq |(\Phi_t^n)^{-1}(x) - (\Phi_t^n)^{-1}(y) \leq |x-y|^{e^{-Ct}}$$

Then, on any interval $[0,T]\times\bar{D}$ we can find (via Ascoli-Arzela criterion) a subsequence n_j such that

$$\Phi_t^{n_j}(x) \longrightarrow \Phi_t(x) \in \mathcal{C}^{\alpha(T)}([0,T]\times D)$$

with ω, u defined by $\omega(x,t) = \omega_0(\Phi_t^{-1}(x))$ and $u(x,t) = K_D * \omega(x,t)$.

Next we state two useful related theorems. The first one summarizes our discussion, while the second one is a natural extension.

A Yudovich's Theorem. *[20],[16],[17] Assume $\omega_0 \in L^\infty(\bar{D})$ for a bounded domain D. Then there exists a unique triple $(\Phi_t(x), \omega, u)$ satisfying*

$$\omega(x,t) = \omega_0(\Phi_t^{-1}(x)), \quad \frac{d\Phi_t(x)}{dt} = u(\Phi_t(x),t), \quad \textit{and} \quad u = K_D * \omega(x,t).$$

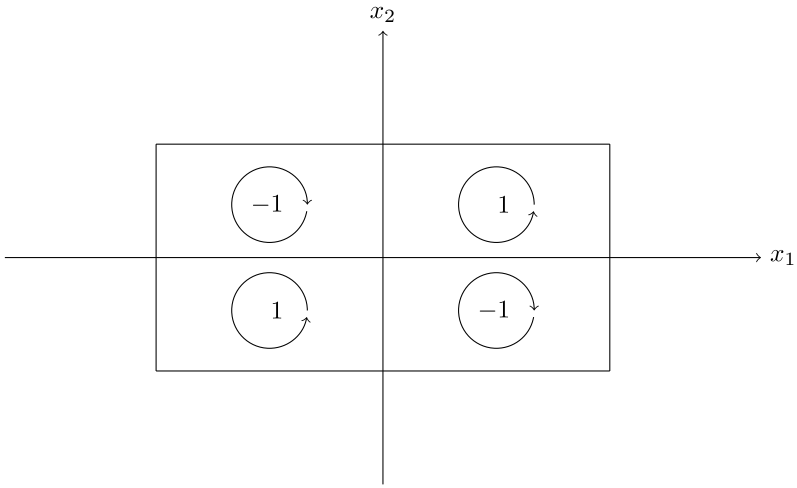
Moreover, u is log-Lipschitz in x uniformly in time and

$$\Phi_t^{-1}(x) \quad \textit{and} \quad \Phi_t(x) \quad \textit{belong to} \quad \mathcal{C}^{\alpha(T)}([0,T]\times\bar{D}), \alpha(T) > 0,$$

and $\omega \in L^\infty$, $\omega(x,t)$ converges to ω_0 as $t \to 0$ in the weak-$*$ topology in L^∞.

Theorem 0.1.4. *Suppose in addition that $\omega_0 \in \mathcal{C}^k(\bar{D})$. Then $\omega(x,t) \in \mathcal{C}^k(\bar{D})$, $u \in C^{k,\alpha}$ for all $\alpha < 1$ and its k order derivatives are log-Lipschitz in x, and $\Phi_t(x) \in \mathcal{C}^{k,\alpha(T)}([0,T]\times\bar{D})$*

Examples of interesting questions are for instance: How fast can the derivative of a solution grow? How fast small scales can be generated? We begin by giving some examples of dynamics, illustrating Yudovich theorem and its sharpness as well as providing first insight into the growth questions. The so-called Bahouri-Chemin [1] example is defined on the torus $\mathbb{T}^2$. This solution has some symmetry; namely it is odd with respect to both coordinate axes x_1 and x_2. Suppose that $\mathbb{T}^2 = [-\pi, \pi)^2$. The vorticity in the Bahouri-Chemin example is identically equal to 1 in the first quadrant $(0, \pi) \times (0, \pi)$ and is defined elsewhere by symmetry and periodicity.



Claim: Such $\omega_0(x)$ is a stationary solution in the sense of Yudovich. The result follows from the lemma below.

Lemma 0.1.5. *Suppose that the domain D is symmetric with respect to x_2 axis. If $\omega_0 \in L^\infty$ is odd with respect to x_1 then the corresponding solution $\omega(x,t)$ remains odd for all $t > 0$. Moreover, the solution corresponding to Bahouri-Chemin example remains constant in time.*

Remark. Of course, oddness with respect to any other symmetry axis of D is also conserved. The argument also applies to $\mathbb{R}^2$ or $\mathbb{T}^2$.

Proof. One can check directly that if $\omega(x_1, x_2, t)$ is a solution then so is $-\omega(-x_1, x_2, t)$. At time 0, they are both equal to $\omega_0(x)$ and therefore by uniqueness they coincide for all time. Furthermore, if $\omega_{x_1,x_2,t}$ is odd for all time then $(-\Delta)^{-1}\omega(x_1, x_2, t)$ is also odd (this can be checked using Fourier transform on the torus for instance), and $u_1 = \partial_2(-\Delta)^{-1}\omega$ implies oddness of u_1 with respect to x_1. A similar argument applies to u_2 and at other cell boundaries. Thus, all trajectories stay inside the cell where they started for all time. The formula $\omega(x,t) = \omega_0(\Phi_t^{-1}(x))$ then shows that the Bahouri-Chemin solution is stationary. $\square$

For the property of Bahouri-Chemin solution that we want to derive we will need the following lemma.

Lemma 0.1.6. *Suppose $\omega_0 \in L^\infty(\mathbb{T}^2)$ with mean zero, $u(x) = \nabla^\perp(-\Delta)^{-1}\omega$. Then,*

$$u(x) = \lim_{\gamma\to 0} \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \exp^{-\gamma|y|^2} \omega(y)\ dy,$$

where ω is extended periodically to all $\mathbb{R}^2$.

The proof of this lemma can be established using Fourier transform. We leave details to interested reader.

Theorem 0.1.7. *In the Bahouri-Chemin example we have*

$$u_1(x_1, 0) = \frac{2}{\pi} x_1 \ln(x_1) + O(x_1)$$

for small x_1.

Proof. We decompose $u_1(x_1,0) = u_1^M(x_1,0) + u_1^R(x_1,0)$, with

$$u_1^M(x_1,0) \equiv \frac{1}{2\pi} \int_{\mathbb{T}^2} \frac{-y_2}{|x-y|^2} \omega(y)\ dy,$$

and,

$$u_1^R(x_1,0) \equiv \frac{1}{2\pi} \lim_{\gamma\to 0} \int_{\mathbb{R}^2\setminus\mathbb{T}^2} \frac{-y_2}{|x-y|^2} \omega(y) e^{-\gamma|y|^2}\ dy.$$

Then, using the oddness of $\omega(y)$, we get

$$u_1^R(x_1,0) = -\frac{1}{\pi} \lim_{\gamma\to 0} \int_{\mathbb{R}^2\setminus\mathbb{T}^2, y_1\geq 0, y_2\geq 0} \left(\frac{y_2}{(x_1-y_1)^2+y_2^2} - \frac{y_2}{(x_1+y_1)^2+y_2^2} \right) \omega(y)\ dy$$

hence an estimate using periodicity and mean zero property of ω shows

$$u_1^R(x_1,0) = \frac{1}{\pi} \lim_{\gamma\to 0} \int_{\mathbb{R}^2\setminus\mathbb{T}^2, y_1\geq 0, y_2\geq 0} \frac{4x_1y_1y_2}{((x_1-y_1)^2+y_2^2)((x_1+y_1)^2+y_2^2)} w(y) e^{-\gamma|y|^2}\ dx\ dy \leq Cx_1$$

The log part will come from u_1^M: indeed, we have

$$u_1^M(x_1,0) = \frac{1}{\pi} \int_0^\pi \int_0^{pi} \frac{4x_1y_1y_2}{((x_1-y_1)^2+y_2^2)(x_1+y_1)^2+y_2^2)} dy_1 dy_2$$

The main contribution comes from

$$B := \frac{4}{\pi} x_1 \int_{2x_1}^1 \int_{2x_1}^1 \frac{y_1y_2}{(y_1^2+y_2^2)^2}.$$

The difference between $u_1^M(x_1,0)$ and B can be estimated as Lipschitz, $\leq Cx_1$. On the other hand, computing the value of B directly gives $B = \frac{2}{\pi}x_1\ln(x_1) + O(x_1)$. $\square$

Another feature of the Bahouri-Chemin example is that the flow map $\Phi_t(x)$ is Hölder with exponent decaying in time. Indeed $d/dt(\Phi_t^1(x_1,0)) \sim c\Phi_t^1(x_1,0)\log(\Phi_t^1(x_1,0))$ (the characteristic on the separatrix). Therefore, we have $d/dt(\log(\Phi_t^1(x_1,0))) \sim c\log(\Phi_t^1(x_1,0))$ so that $\log(\Phi_t^1(x_1,0)) \sim \log(x_1)e^{ct}$. Hence, the inverse flow map $\Phi_t(x)$ has Hölder exponent less or equal to e^{-ct}. This means that the estimates of Yudovich theorem are qualitatively sharp.

0.2 An upper bound for growth of $\nabla\omega$

Last lecture, we showed that if $\omega_0 \in L^\infty \Rightarrow \exists$ unique solution (ω, Φ_t, u) of the 2D Euler equation in the following sense:

$$\frac{d\Phi_t}{dt}(x) = u(\Phi_t(x), t), \quad \Phi_0(x) = x \tag{3}$$

$$u(x,t) = \int_D K_D(x,y)\omega(y,t)\,dy \tag{4}$$

$$\omega(x,t) = \omega_0(\Phi_t^{-1}(x)) \tag{5}$$

Here u is log-Lipschitz in x, ω is in L^∞, $\Phi_t(x)$ and $\Phi_t^{-1}(x)$ are in $C^{\alpha(T)}([0,T]\times\bar{D})$

If in addition $\omega_0 \in C^1$, then we have $\omega \in C^{\alpha(T)} \Rightarrow u \in C^{1,\alpha(T)}$. This statement is directly implied by the following classical theorem.

Theorem 0.2.1 (Kellogg, Schauder, see e.g. [11]). *Suppose that D is a domain in R^d with smooth boundary. Let ψ solve the Dirichlet problem $-\Delta\psi = \omega$ and $\psi|_{\partial D} = 0$. Assume that $\omega \in C^\alpha(\bar{D})$, $\alpha > 0$. Then $\psi \in C^{2,\alpha}(\bar{D})$ and $\|\partial_{ij}\psi\|_{C^\alpha} \leq C(\alpha, D)\|\omega\|_{C^\alpha}$.*

Let us recall the equation (2) that we derived last time:

$$\exp\left(-\int_0^t \|\nabla u\|_{L^\infty}\, ds\right) \leq \frac{|\Phi_t(x) - \Phi_t(y)|}{|x-y|} \leq \exp\left(\int_0^t \|\nabla u\|_{L^\infty(s)}\, ds\right).$$

If $u \in C^{1,\alpha(T)}$, it implies that $\Phi_t(x), \Phi_t^{-1}(x)$ is Lipschitz in x for every t. Moreover, with slightly stronger technical effort one can show that $\Phi_t(x), \Phi_t^{-1}(x) \in C^{1,\alpha(T)}$. This implies that $\omega \in C^1(\bar{D})$ for all times. The next theorem provides a quantitative version of this argument.

Theorem 0.2.2 (Wolibner, Hölder, Yudovich [19],[12],[20]). *Assume $\omega_0 \in C^1(\bar{D})$, $D \subset R^2$, is compact with smooth boundary. Then the gradient of the solution $\omega(x,t)$ satisfies the following bound*

$$\frac{\|\nabla\omega(\cdot,t)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}} \le \left(\frac{\|\nabla\omega_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}+1\right)^{\exp(C\|\omega_0\|_{L^\infty}t)} \tag{6}$$

for all $t>0$.

Ingredients of the proof:

1. Due to the two-sided nature of (2), we have

$$e^{-\int_0^t \|\nabla u(\cdot,s)\|_{L^\infty}ds} \le \frac{|\Phi_t^{-1}(x)-\Phi_t^{-1}(y)|}{|x-y|} \le e^{\int_0^t \|\nabla u(\cdot,s)\|_{L^\infty}ds} \tag{7}$$

2. Notice that

$$\begin{aligned}\|\nabla\omega(\cdot,t)\|_{L^\infty} &\le \sup_{x,y\in\bar D} \tfrac{|\omega_0(\Phi_t^{-1}(x))-\omega_0(\Phi_t^{-1}(y)|)}{|x-y|}\\ &\le \|\nabla\omega_0\|_{L^\infty}\sup_{x,y\in\bar D}\tfrac{|\Phi_t^{-1}(x)-\Phi_t^{-1}(y)|}{|x-y|}\end{aligned}$$

3. Kato's inequality, which we will prove later. If $\omega\in C^\alpha(\bar D), \alpha>0$, $u=\nabla^\perp(-\Delta_D)^{-1}\omega$. Then

$$\|\nabla u\|_{L^\infty} \le C(\alpha,D)\|\omega_0\|_{L^\infty}(1+\log(1+\frac{\|\omega\|_{C^\alpha}}{\|\omega\|_{L^\infty}})). \tag{8}$$

Combining equations (7) and (8), we obtain

$$f(t)^{-1} \le \frac{|\Phi_t^{-1}(x)-\Phi_t^{-1}(y)|}{|x-y|} \le f(t) \tag{9}$$

where

$$f(t)=\exp(C\|\omega_0\|_{L^\infty}\int_0^t(1+\log(1+\frac{\|\nabla\omega(x,s)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}))ds).$$

Combining this together with ingredient 2, we have

$$\log\|\nabla\omega(\cdot,t)\|_{L^\infty} \le \log\|\nabla\omega_0\|_{L^\infty}+C\|\omega_0\|_{L^\infty}\int_0^t(1+\log(1+\frac{\|\nabla\omega(x,s)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}))ds \tag{10}$$

Let $z=\frac{\|\nabla\omega(\cdot,t)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}$. Then it is straightforward to show that $z(t)\le y(t)$ where $y(t)$ solves

$$\frac{y'}{y}=C\|\omega_0\|_{L^\infty}(1+\log(1+y)), \qquad y(0)=\frac{\|\nabla\omega_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}} \tag{11}$$

Hence

$$\frac{y'}{1+y} \le C\|\omega_0\|_{L^\infty}(1+\log(1+y)) \tag{12}$$

After integrating both sides from 0 to t, we obtain

$$1+\log(1+y(t)) \leq (1+\log(1+y(0))\exp(C\|\omega_0\|_{L^\infty} t) \tag{13}$$

$\square$

Proof of Kato's inequality:

Take $\delta = \min\{(\frac{\|\omega_0\|_{L^\infty}}{\|\omega_0\|_{C^\alpha}})^{\frac{1}{\alpha}}, \gamma\}$, where γ is chosen so that the set of $x \in D$ with $dist(x, \partial D) \geq 2\delta$ is not empty. According to Biot-Savart law and properties of Dirichlet Green's function, we know

$$u(x,t) = \int K_D(x,y)\omega(y,t)dy, \qquad K_D = \nabla^{\perp} G_D$$

$$\nabla u(x,t) = \frac{1}{2\pi} P.V. \int \nabla K_D(x,y)\omega(y,t)dy + M\omega(x)$$

where M is a constant matrix. Note that we need to exercise care when taking derivative of u since singularity in the kernel becomes non-integrable. Computing the derivative in weak sense leads to the extra term $M\omega(x)$ which is of no concern in the estimate we need. First we consider $x \in D$, s.t $dist(x, \partial D) \geq 2\delta$. The part of integral over the complement of the ball centered at x with radius γ can be estimated as

$$\left| \int\limits_{B_\delta^c(x)} \nabla |K_D(x,y)\omega(y,t)dy \right| \leq C\|\omega_0\|_{L^\infty} \int\limits_{B_\delta^c(x)} |x-y|^{-2}dy \leq C\|\omega_0\|_{L^\infty}(1+\log\delta^{-1}). \tag{14}$$

For the other part of the integral, we recall the decomposition of G_D. The first term is

$$\begin{aligned} |P.V. \textstyle\int_{B_\delta(x)} \partial^2_{x_i,x_j} \log|x-y|\omega(y)dy| &= |\textstyle\int_{B_\delta(x)} \partial^2_{x_i,x_j} \log|x-y|(\omega(y)-\omega(x))dy| \\ &\leq C\|\omega(x,t)\|_{C^\alpha} \textstyle\int_0^\delta r^{-1+\alpha}dr \\ &\leq C(\alpha)\delta^\alpha \|\omega(x,t)\|_{C^\alpha} \\ &\leq C(\alpha)\|\omega_0\|_{L^\infty} \end{aligned}$$

The last inequality comes from our choice of δ.

Finally, we deal with the last term. Notice that by maximal principle, $|h(z,y)| \leq C\log\delta^{-1}$ for all $y \in B_\delta(x)$, and $z \in D$. Then standard estimates for harmonic functions (see e.g. [9]) give, for each $y \in B_\delta(x)$,

$$|\partial^2_{x_ix_j} h(x,y)| \leq C\delta^{-4} \int\limits_{B_\delta(x)} |h(z,y)|dz \leq C\delta^{-2}\log\delta^{-1}.$$

This gives

$$\left| \int\limits_{B_\delta(x)} \partial^2_{x_ix_j} h(x,y)\omega(y,t)dy \right| \leq C\|\omega_0\|_{L^\infty}\log\delta^{-1}. \tag{15}$$

Combining these estimates, the inequality is proved for interior points satisfying $dist(x, \partial D) \geq 2\delta$.

Now if x' is such that $dist(x', \partial D) < 2\delta$, find a point x such that $dist(x, \partial D) \geq 2\delta$ and $|x - x'| \leq C(D)\delta$. By Theorem 0.2.1, we have

$$|\nabla u(x') - \nabla u(x)| \leq C(\alpha, D)\delta^{\alpha}\|\omega\|_{C^{\alpha}} \tag{16}$$

which, together with estimate for interior point x, implies that the inequality holds for all points in D.

□

0.3 Simple examples of gradient growth in passive scalars

The passive scalar equation in $2D$ is given by

$$\partial_t \psi + (u \cdot \nabla)\psi = 0, \qquad \psi(x, 0) = \psi_0(x).$$

Here u is a given, “passive"vector field that may or may not depend on time.

1. Shear flow

$$u = (u(x_2), 0)$$
$$\Phi_t^{-1}(x_1, x_2) = (x_1 - u(x_2)t, x_2)$$
$$\psi(x, t) = \psi_0(\Phi_t^{-1}(x))$$

In this example,

$$\|\nabla\psi\|_{L^{\infty}} \sim ct \tag{17}$$

provided that u' is bounded.

2. Cellular flow

$$\omega(x_1, x_2) = \sin x_1 \sin x_2$$
$$u(x_1, x_2) = (-\sin x_1 \cos x_2, \sin x_2 \cos x_1)$$
$$\frac{d}{dt}\Phi_t^1(x_1, 0) \sim -\Phi_t^1(x_1, 0) \quad for \;\; x_2 = 0, \;\; x_1 \;\; small$$

So $x_1(t) \sim x_1(0)e^{-t}$, $\|\nabla\psi\|_{L^{\infty}} \sim e^{ct}$. We also know if $\|\nabla u\|_{L^{\infty}} \leq C$, then exponential growth is the fastest that one can get.

3. Bahouri-Chemin flow In this example, described in the first lecture, the flow u satisfies $u_1(x_1, 0) \sim cx_1 \ln x_1$ for x_1 small enough, so $\Phi_t^1(x_1, 0) \sim x_1^{e^{ct}}$ if x_1 is sufficiently small. This leads to double exponential growth of $\|\nabla\psi\|_{L^{\infty}}$ in a passive scalar advected by such u.

0.4 Growth of derivatives in solutions of 2D Euler

The first works constructing examples with growth in derivatives of vorticity are due to Yudovich [13, 21], He used Lyapunov functional method to prove some growth in $\|\nabla\omega\|_{L^\infty}$ at a flat part of the boundary of domain D. Then, Nadirashvili [18] has constructed examples 2D Euler solutions on an annulus with linear growth of $\|\nabla\omega\|_{L^\infty}$. Later Denisov [7] constructed an example in periodic setting that shows superlinear growth; to be specific, he proved $\frac{1}{T^2}\int_0^T \|\nabla\omega(\cdot,t)\|_{L^\infty}dt \to \infty$ as $T \to \infty$. Denisov also built an example to show that $\|\nabla\omega(\cdot,t)\|_{L^\infty}$ can experience bursts of double exponential growth over finite time intervals [8]. The idea for the latter example involves smoothing and slightly perturbing Bahouri-Chemin flow.

We are going to describe a very recent example showing that double exponential growth in the derivatives of solutions of 2D Euler equation in a bounded domain can indeed happen for all times. Thus the upper bound going back to Wolibner is qualitatively sharp.

Theorem 0.4.1 (Kiselev,Sverak). *[14] Let D be a unit disk in $\mathbb{R}^2$, then there exists $\omega_0 \in C^\infty(\bar{D})$ and $\|\nabla\omega_0\|_{L^\infty} \geq 1$ such that $\|\nabla\omega(\cdot,t)\|_{L^\infty} \geq \|\nabla\omega_0\|_{L^\infty}^{c\exp(ct)}$, for any t.*

As we will see, growth happens at the boundary ∂D. The example is motivated by Luo-Hou's numerical experiments [15], where a new scenario for finite time singularity formation in solutions of the 3D Euler equation is proposed. The scenario is axi-symmetric, and extremely fast growth is observed at a ring of hyperbolic points of the flow located at the boundary of a cylinder. We will see below that the geometry of the double exponential growth example is similar, and a hyperbolic point on the boundary plays a key role.

It will be convenient for us to set the origin at the lowest point of the unit disk D (so that the center of the disk has coordinates $(0,1)$). Denote $D^+ = \{x \in D | x_1 \geq 0\}$. The initial ω_0 will be odd in x_1. Then the solution $\omega(x,t)$ is also odd for all times. By Biot-Savart law, we have

$$u(x,t) = \nabla^\perp \int G_D(x,y)\omega(y,t)dy$$

where due to our choice of coordinates $G_D(x,y) = \frac{1}{2\pi}\ln\frac{|x-y|}{|x-\bar{y}||y-e_2|}$, $\bar{y} = \frac{y-e_2}{|y-e_2|^2}+e_2$, $e_2 = (0,1)$.

We need the following notation:

$$D_1^\gamma = \{x \in D^+ | \frac{\pi}{2} - \gamma \geq \theta \geq 0\},$$

$$D_2^\gamma = \{x \in D^+ | \frac{\pi}{2} \geq \theta \geq \gamma\},$$

where θ is the usual angular variable. Next, denote

$$Q(x_1,x_2) = \{y \in D^+ | y_1 \geq x_1, y_2 \geq x_2\}$$

$$\Omega(x_1,x_2,t)=\frac{4}{\pi}\int\limits_{Q(x_1,x_2)}\frac{y_1y_2}{|y|^4}\omega(y,t)dy$$

Before we prove Theorem 0.4.1, we need the following key lemma.

Lemma 0.4.2. *Assume ω_0 is odd in x_1. Fix small $\gamma>0$. Then there exists $\delta>0$ such that*

$$u_1(x,t)=-x_1\Omega(x,t)+x_1B_1(x,t),\ |B_1|\le C_\gamma\|\omega_0\|_{L^\infty},\ \ \forall x\in D_1^\gamma,|x|\le\delta \tag{18}$$

$$u_2(x,t)=x_2\Omega(x,t)+x_2B_2(x,t),\ |B_2|\le C_\gamma\|\omega_0\|_{L^\infty},\ \ \forall x\in D_2^\gamma,|x|\le\delta \tag{19}$$

Remark. We will see that in the certain regimes the first term on the right hand sides in (18) and (19) is truly the main term. Then, in the main term, the trajectories of fluid motion near the origin are pure hyperbolas. Also, note that the singularity in Ω is capable of creating exactly $\sim\log x_1$ behavior, akin to Bahouri-Chemin example, as the support of vorticity approaches the origin.

Proof. We will consider the case of u_1; the derivation for u_2 is similar. Due to symmetry, $u(x)=\frac{\nabla^\perp}{2\pi}\int_{D^+}\ln(\frac{|x-y||\tilde{x}-\tilde{y}|}{|x-\tilde{y}||\tilde{x}-y|})\omega(y,t)dy$, where $\tilde{x}=(-x_1,x_2)$. Fix $x=(x_1,x_2)\in D_1^\gamma$ and take $\Gamma=100(1+\cot\gamma)x_1$. Since $x\in D_1^\gamma$, we have $100|x|<\Gamma$.

First,

$$\begin{aligned}\frac{\nabla^\perp}{2\pi}\int_{B_\Gamma(0)}\ln(\tfrac{|x-y||\tilde{x}-\tilde{y}|}{|x-\tilde{y}||\tilde{x}-y|})\omega(y,t)dy&\le C\|\omega_0\|_{L^\infty}\int_{B_{2\Gamma}(0)}\frac{dy}{|x-y|}\\&\le C\|\omega_0\|_{L^\infty}\int_0^{2\Gamma}\tfrac{1}{s}sds\\&\le C\|\omega_0\|_{L^\infty}x_1\end{aligned}$$

In the rest of integration region, we have $|y|>100|x|$. Note that

$$\begin{aligned}\pi G_D(x,y)=\tfrac{1}{4}\Big(&\ln(1-\tfrac{2xy}{|y|^2}+\tfrac{|x|^2}{|y|^2})-\ln(1-\tfrac{2x\bar{y}}{|\bar{y}|^2}+\tfrac{|x|^2}{|\bar{y}|^2})\\&-\ln(1-\tfrac{2\tilde{x}y}{|y|^2}+\tfrac{|x|^2}{|y|^2})+\ln(1-\tfrac{2\tilde{x}\bar{y}}{|\bar{y}|^2}+\tfrac{|x|^2}{|\bar{y}|^2})\Big)\end{aligned}$$

Observe that $\ln(1+s)\sim s-\frac{s^2}{2}+O(|s|^3)$ for small s. Moreover, one can verify that $\frac{\bar{y}_1}{|\bar{y}|^2}=\frac{y_1}{|y|^2}$, $\frac{\bar{y}_2}{|\bar{y}|^2}=1-\frac{y_2}{|y|^2}$. Then, after a computation, we obtain

$$\pi G_D(x,y)=-\frac{4x_1x_2y_1y_2}{|y|^4}+\frac{2x_1x_2y_1}{|y|^2}+O\left(\frac{|x|^3}{|y|^3}\right). \tag{20}$$

This asymptotic expansion can be differentiated, and we get

$$\pi\frac{\partial G_D}{\partial x_2}(x,y)=-\frac{4x_1y_1y_2}{|y|^4}+\frac{2x_1y_1}{|y|^2}+O\left(\frac{|x|^2}{|y|^3}\right). \tag{21}$$

Notice that

$$\int\limits_{D^+\bigcap B_\Gamma^c}\frac{y_1}{|y|^2}dy\le\int\limits_{\Gamma}^{2}\frac{1}{s}sds\le 2$$

$$|x|^2 \int\limits_{D^+\bigcap B_\Gamma^c} \frac{1}{|y|^3} dy \le |x|^2 \int\limits_{\Gamma}^{2} \frac{1}{s^3} s ds \le C|x|^2\Gamma^{-1} = C(\gamma)x_1$$

$$\int\limits_{x_1}^{2}\int\limits_{0}^{x_1} \frac{y_1y_2}{|y|^4} dy_1dy_2 \le \int\limits_{0}^{x_1} \frac{y_1}{y_1^2+x_1^2} dy_1 \le C(\gamma)$$

$$\int\limits_{x_1}^{2}\int\limits_{0}^{C(\gamma)x_1} \frac{y_1y_2}{|y|^4} dy_2dy_1 \le C(\gamma)$$

Combining all our estimates together, we get (18). Similarly, we can prove (19) for $x \in D_2^\gamma$.

□

With this main lemma in hand, exponential growth of gradient of the vorticity is easy to obtain.

Set $\omega_0 = 1$ for every $x \in D^+$ except for $x_1 \le \delta$. Then for every t, $\Big|\{x \in D^+|\omega(x,t) \ne 1\}\Big| \le 2\delta$ (since $\omega(x,t) = \omega_0(\Phi_t^{-1}(x))$, and Φ_t^{-1} is measure preserving). Then, provided that $|x| \le \delta$, $\Omega(x,t) \ge C\int_{C\sqrt{\delta}}^{C}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin 2\phi}{s} ds d\phi \ge C\log\delta^{-1}$. We can choose δ so that $C\log\delta^{-1} > 100C(\gamma)$, with $C(\gamma)$ the constant in (18). For any characteristic on ∂D with a starting point (x_1,x_2) satisfying $x_1 \le \delta$, we have $\frac{d}{dt}\Phi_t^1(x_1,x_2) \le -c\log\delta^{-1}\Phi_t^1(x_1,x_2)$ for some $c > 0$.

Now we are going to deal with double exponential growth. The construction is qualitatively different, and has to be essentially nonlinear. We have to derive an estimate on growth of $\Omega(x,t)$ in time due to advance of the unit vorticity towards origin. It is not clear why such advance has to be at all orderly and controllable; depletion of the region of high vorticity as it approaches the origin appears a distinct possibility. A key role in the proof plays a hidden “comparison principleë in (18). Namely, the region $Q(x_1,x_2)$ over which we integrate in the main term of the right hand side in (18) tends to be larger for points closer to origin. It is this feature that allows control of the scenario and proof of double exponential growth.

We will still assume

$$\omega_0 = 1, \qquad x_1 \ge \delta,$$

$0 \le \omega_0 \le 1$ in D^+. So from exponential growth proof, we have

$$\Omega(x,t) \ge C\log\delta^{-1} \ge 100C(\gamma), \qquad \forall |x| \le \delta, \forall t \tag{22}$$

To be convenient, we also need the following notation. Take $\epsilon < \delta$, and denote

$$O_{x',x''} = \{x \in D^+ | x' \le x \le x'', x_2 \le x_1\}$$

In addition to the "front of unit vorticity for $x_1 \geq \delta$, set $\omega_0 = 1$ on $O_{\epsilon^{10},\epsilon}$, with $\|\nabla\omega_0\|_{L^\infty} \sim \epsilon^{-10}$. Furthermore, define

$$\bar{u}(x_1, t) = \max_{(x_1,x_2)\in D^+,\, x_2 \leq x_1} u_1(x_1, x_2, t)$$

$$\underline{u}(x_1, t) = \min_{(x_1,x_2)\in D^+,\, x_2 \leq x_1} u_1(x_1, x_2, t).$$

Introduce $a(t)$ and $b(t)$ as follows: $a(t)$ solves

$$a'(t) = \bar{u}(a(t), t), \;\; a(0) = \epsilon^{10};$$

$b(t)$ solves

$$b'(t) = \underline{u}(b(t), t), \;\; b(0) = \epsilon.$$

Proof of Theorem 0.4.1:

We first claim that $\forall t \geq 0$, $\omega(x, t) = 1$ if $x \in O_{a(t),b(t)}$. Indeed, assume not: $\omega(z, t) \neq 1$, for some $z \in O_{a(t),b(t)}$, then $z = \Phi_t(x)$ for some $x \notin O_{\epsilon^{10},\epsilon}$. Then $\Phi_s(x) \in \partial O_{a(s),b(s)}$ at some time s for the first time. However, by definition of $a(s)$ and $b(s)$, $\Phi_s(x)$ can not enter from the sides $x_1 = a(s), b(s)$ of the region. Due to the boundary condition, it also cannot enter from ∂D part of the boundary. This leaves the diagonal part of the boundary where $x_1 = x_2$. By our choice of ω_0, for all $s \geq 0$, the region $O_{a(s),b(s)}$ lies in $D_1^\gamma \cap \{|x| < \delta\}$. Then by Lemma 0.4.2, we have

$$\frac{\log \delta^{-1} - C}{\log \delta^{-1} + C} \leq \frac{-u_1(x_1, x_1)}{u_2(x_1, x_1)} \leq \frac{\log \delta^{-1} + C}{\log \delta^{-1} - C} \tag{23}$$

We can assume that δ is small enough so that $\log \delta^{-1} \gg C$, so equation (23) means that $\Phi_s(x)$ can not enter through diagonal side. Together we proved the claim.

Now we look at

$$\begin{aligned} a'(t) &= \bar{u}(a(t), t) \\ &\leq -a(t)\Omega(a(t), x_2(t), t) + Ca(t) \\ &\leq -a(t)\Omega(a(t), 0, t) + 2Ca(t). \end{aligned}$$

In the above computation, $x_2(t)$ is the value of the second coordinate where the maximum of u_1 is achieved (keep in mind that u_1 is negative), satisfying $0 \leq x_2(t) \leq a(t)$. In the last step, we used the inequality $\Omega(a(t), x_2(t), t) \geq \Omega(a(t), 0, t) - C$, which can be verified by direct computation. Similarly,

$$\begin{aligned} b'(t) &= \underline{u}(b(t), t) \\ &\geq -b(t)\Omega(b(t), x_2(t), t) - Cb(t) \\ &\geq -b(t)\Omega(b(t), b(t), t) - 2Cb(t) \end{aligned}$$

(since $\int_b^1 \int_0^b \frac{y_1 y_2}{|y|^4} dy_1 dy_2 \leq C$)

Note that since $\omega(x,t)=1$ on $O_{a(t),b(t)}$,

$$\Omega(a(t),0,t)\geq\Omega(b(t),b(t),t)+\frac{4}{\pi}\int\limits_{O_{a(t),b(t)}}\frac{y_1y_2}{|y|^4}dy.$$

The last term above can be estimated as follows

$$\frac{4}{\pi}\int\limits_{O_{a(t),b(t)}}\frac{y_1y_2}{|y|^4}dy\geq\frac{4}{\pi}\int\limits_{\epsilon}^{\frac{\pi}{4}}\int\limits_{\frac{a(t)}{\cos\psi}}^{\frac{b(t)}{\cos\psi}}\frac{\sin 2\psi}{r}d\psi dr\geq C(\log b(t)-\log a(t)).$$

Combining these results together, we obtain

$$\begin{aligned}\tfrac{d}{dt}\Big(\log b(t)-\log a(t)\Big)&\geq-\Omega(b(t),b(t),t)-2C+\Omega(a(t),0,t)-2C\\&\geq C(\log b(t)-\log a(t))-4C\end{aligned}$$

By Gronwall's inequality,

$$\log\frac{b(t)}{a(t)}\geq(\log\epsilon^{-9})e^{Ct}-4Ce^{Ct}.$$

If ϵ is chosen small enough, then $\log\frac{b(t)}{a(t)}\geq(\log\epsilon^{-8})e^{Ct}$. Since $b(t)$ is less than 1, we get $a(t)\leq\epsilon^{8e^{Ct}}$. This gives double exponential growth of $\|\nabla\omega\|_{L^\infty}$.

□

0.5 Towards the 3D Euler

In this section, we come back to Hou-Luo scenario for singularity formation in 3D Euler equation, and discuss one-dimensional models designed to get insight into it. We also review some of the earlier one-dimensional models, which have a long history in mathematical fluid mechanics. Let us begin by writing down the axisymmetric 3D Euler equation in cylindrical coordinates.

Assume $u(x)=u_r(r,z,t)e_r+u_z(r,z,t)e_z+u_\theta(r,z,t)e_\theta$, $\omega(x)=\omega_r(r,z,t)e_r+\omega_z(r,z,t)e_z+\omega_\theta(r,z,t)e_\theta$, where r,z,θ are usual cylindrical coordinates. The 3D axisymmetric Euler equation can be written as follows:

$$\begin{cases}\partial_t\left(\frac{\omega_\theta}{r}\right)+u_r\partial_r\left(\frac{\omega_\theta}{r}\right)+u_z\partial_z\left(\frac{\omega_\theta}{r}\right)=\partial_z\left(\frac{(ru_\theta)^2}{r^4}\right)\\ \partial_t(ru^\theta)+u_r\partial_r(ru^\theta)+u_z\partial_z(ru^\theta)=0\\ (u_r,u_z)=(r^{-1}\partial_z\psi^\theta,-r^{-1}\partial_r\psi^\theta),\ L\psi^\theta=\omega_\theta.\end{cases}$$

Here $L = r^{-1}\partial_r(r^{-1}\partial_r) + r^{-2}\partial_z^2$.

Away from the axis $r = 0$, axi-symmetric 3D Euler equation is very similar to the 2D inviscid Boussinesq system, describing motion of incompressible buoyant flow.

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = \partial_{x_1}\rho \\ \partial_t \rho + (u \cdot \nabla)\rho = 0 \\ u = \nabla^\perp (-\Delta)^{-1}\omega. \end{cases}$$

We will think of this equation set either on a rectangle or an infinite (in x_1) strip with $u_2 = 0$ condition on horizontal boundaries and either $u_1 = 0$ or periodic boundary conditions in the x_1 direction.

The singularity formation scenario of Hou and Luo [15] involves, when translated to the 2D Boussinesq case, an initial vorticity odd in x_1 and density even in x_1. Due to symmetry, $x_1 = 0$ serves as a separatrix of the flow for all times, and the flow has hyperbolic points where $x_1 = 0$ axis and the boundary meet. It is at these points that very fast and numerically robust growth of vorticity is observed. We see that this geometry is very similar to the 2D Euler example we discussed in the previous lecture, but now we have a more complex system. The main issue in trying to apply the 2D Euler ideas to the Boussinesq scenario is that the vorticity is no longer expected to stay bounded. This destroys the estimate of the key lemma, and makes control of the solution harder. Another layer of difficulty arises from the forcing term in vorticity equation, which can now create vorticity of both signs, potentially depleting the singularity formation. In this lecture, we will discuss some simplified one-dimensional models that have been developed in attempt to bridge the gap with three dimensions in understanding Luo-Hou hyperbolic scenario. Analysis of 1D models in fluid mechanics has a long history, and we start with a review of some earlier results.

Let us now discuss one-dimensional models of 3D Euler equation, beginning with the general, rather than axi-symmetric, setting. The general 3D Euler equation in the vorticity form is given by

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u \quad in \quad \mathbb{R}^3 \\ u = \nabla^\perp (-\Delta)^{-1}\omega \end{cases}$$

The most natural 1D model corresponding to the general 3D Euler equation is

$$\begin{cases} \partial_t \omega + u\partial_x \omega = \omega \partial_x u \\ u_x = H\omega \end{cases}$$

Here H is the Hilbert transform. This model has been considered by De Gregorio [5, 6]. De Gregorio model directly parallels the structure of the 3D Euler equation. It is reasonable to first analyze the effect of the two nonlinear terms separately.

If we drop the vortex stretching term, we obtain the following active scalar transport equation.

$$\partial_t \omega + u \partial_x \omega = 0, \;\; u_x = H\omega.$$

As one can expect, in the absence of vortex stretching, the equation becomes globally regular. Global regularity can be proved in this case similarly to the 2D Euler argument; it is a good exercise.

On the other hand, let us omit the transport term in De Gregorio model. We arrive at the equation

$$\partial_t \omega = \omega \partial_x u = \omega H \omega.$$

This equation has been considered by Constantin, Lax and Majda [4]. Amazingly, the model turns out to be exactly solvable. Let us recall some properties of the Hilbert transform:

$$Hf(x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy,$$

or

$$F(Hf)(k) = -i\, \mathrm{s}ign(k) Ff(k)$$

where F stands for Fourier transform.

If $f \in L_2$ then $f + iHf$ is a boundary value of an analytic function in $\mathbb{C}^+$. Then

$$(f + iHf)^2 = f^2 - (Hf)^2 + 2ifHf$$

is also an analytic function in $\mathbb{C}^+$. The real part of its boundary values is the Hilbert transform of the imaginary part of its boundary values. It follows that

$$fHf = \frac{1}{2} H(f^2 - (Hf)^2) \implies H(fHf) = \frac{1}{2}((Hf)^2 - f^2).$$

Theorem 0.5.1. *The solutions to Constantin-Lax-Majda model can blow up in finite time.*

Proof: Applying the Hilbert transform to the equation, we get

$$\partial_t H\omega = \frac{1}{2}((H\omega)^2 - \omega^2).$$

Let us define take $z(t) = H\omega(t) - i\omega(t)$. Differentiating in time we obtain

$$z'(t) = \frac{1}{2} z^2(t) \implies \frac{1}{z(t)} = \frac{1}{z(0)} - \frac{1}{2} t,$$

and finally $z(t) = \frac{2z(0)}{2 - tz(0)}$. Hence

$$\omega(x,t) = \frac{4\omega_0(x)}{(2 - tH\omega_0(x))^2 + t^2 \omega_0(x)^2}$$

This implies finite time blow up if for some x_0 the initial data satisfies $\omega_0(x_0) = 0$, $H\omega_0(x_0) > 0$. $\square$

Let us go back to the full De Gregorio model

$$\begin{cases} \partial_t \omega + u\partial_x \omega = \omega \partial_x u, \\ u_x = H\omega. \end{cases}$$

Is there a finite time blow-up? This question is currently open. It might be natural to guess finite time blow up; but surprisingly, the transport and vortex stretching terms appear to counteract each other.

Let us now discuss one-dimensional models developed recently specifically for Hou-Luo scenario. We start with the derivation of Hou-Luo model proposed already in [15]. Consider the 2D Boussinesq equation in the half-plane $x_2 \geq 0$, and make an additional assumption that the vorticity is concentrated in a boundary layer where it does not depend on the vertical direction x_2 :

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = \partial_{x_1} \rho \quad in \quad \mathbb{R}^2_+ \times [0, \infty) \\ \partial_t \rho + (u \cdot \nabla)\rho = 0 \\ u = \nabla^\perp (-\Delta_D)^{-1} \omega \\ \omega(x_1, x_2, t) = \omega(x_1, 0, t)\chi_{[0,a]}(x_2) \end{cases}$$

As is well known, the Laplacian Green's function of the Dirichlet problem in $\mathbb{R}^2_+$ is

$$G_D(x, y) = \frac{1}{2\pi}(\log|x - y| - \log|x - \tilde{y}|),$$

where $\tilde{y} = (y_1, -y_2)$. From the Biot-Savart law we get:

$$u_1(x, t) = \int_{\mathbb{R}} \int_0^a \frac{\partial G_D}{\partial x_2}(x_1, 0, y_1, y_2)\omega(y, t)\; dy_2\; dy_1;$$

$$\frac{\partial G_D}{\partial x_2}(x_1, 0, y_1, y_2) = \frac{1}{2\pi}\left(\frac{-y_2}{(x_1 - y_1)^2 + y_2^2} - \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \right);$$

$$\frac{1}{\pi} \int_0^a \frac{y_2}{(x_1 - y_1)^2 + y_2^2}\; dy_2 = \frac{1}{2\pi} \int_0^{a^2} \frac{dz}{(x_1 - y_1)^2 + z} = \frac{1}{2\pi} \log\left(\frac{(x_1 - y_1)^2}{(x_1 - y_1)^2 + a^2} \right).$$

We can simplify our calculations by taking out the denominator, because there is no singularity in it. So, in the main term, we can take

$$u_1(x, t) = -\frac{1}{\pi} \int_{\mathbb{R}} \log|x_1 - y_1|\omega(y, t)\; dy_1$$

A short computation shows that this is precisely equivalent to $\partial_x u_1(x, t) = H\omega$.

Based on the argument above, the following model of the hyperbolic point blow up scenario has been proposed by Hou and Luo [15].

$$\begin{cases} \partial_t \omega + u\partial_x \omega = \partial_x \rho \quad in \quad \mathbb{R} \times (0,\infty) \\ \partial_t \rho + u\partial_x \rho = 0 \\ u_x = H\omega. \end{cases}$$

The initial data ω_0, ρ_0 are assumed periodic.

Theorem 0.5.2. *The periodic Hou-Luo model is locally well-posed for* $(\omega_0, \rho_0) \in (H^m, H^{m+1})$ *with* $m > 1/2$.

If the solution loses regularity at time T, *we must have*

$$\int_0^t \|u_x\|_{L^\infty}\, dx \overset{t\to T}{\longrightarrow} \infty \text{ and } \int_0^t \|\rho_x\|_{L^\infty}\, dx \overset{t\to T}{\longrightarrow} \infty \tag{24}$$

On the other hand, there exist smooth initial data for which the solution forms a singularity in finite time. In particular, the expressions in (24) *become infinite in finite time.*

The proof of Theorem 0.5.2 has been recently given in [3], and is based on an appropriate Lyapunov functional-like argument. Like in the proof of Theorem 0.4.1, where a hidden comparison principle played an essential role, there is a hidden positivity of certain expression that makes the proof work.

We will not discuss the proof in detail here, but we will take a look at a related, and simpler, model where the proof of blow up is more direct.

Choi, Kiselev and Yao [2] have proposed to study (24) with a modified Biot-Savart law

$$u(x) = -x \int_x^1 \frac{\omega(y)}{y}\, dy.$$

This law arises if one drops certain parts of the $u_x = H\omega$ law. The CKY law is also motivated by the expression for u in Lemma 0.4.2. The CKY rule is "almost local": if we divide $u(x)$ by x and differentiate, we get a local relationship. Thus it is easier to deal with than the truly nonlocal HL rule.

We will consider the CKY model on an interval $[0, 1]$ with smooth compactly supported initial data (the periodic boundary conditions are not compatible with the CKY velocity expression).

Theorem 0.5.3. *Suppose* $(\omega_0, \rho_0) \in (H_0^m, H_0^{m+1})$ *for* $m \geq 2$.

Then there $\exists T < \infty$, *such that for* $0 \leq t \leq T$ *a unique solution* (ω, ρ) *in* (H_0^m, H_0^{m+1}) *exists.*

For the solution of the CKY model to lose regularity at time T, we must have

$$\int_0^t (||\nabla\rho||_\infty \quad and \quad ||\nabla u||_\infty \quad and \quad ||\omega||_\infty)\ ds \stackrel{t\to T}{\longrightarrow} \infty \tag{25}$$

There exist initial data $(\omega_0, \rho_0) \in C_0^\infty([0,1])$ such that the corresponding solution blows up in finite time. In particular, the expressions in (25) become infinite in finite time.

Let us denote by $\Omega(x,t)$ the integral

$$\Omega(x,t) = \int_x^1 \frac{w(y,t)}{y}\ dy$$

Let us define trajectories

$$\frac{d\Phi_t}{dt}(x) = u(\Phi_t(x),t), \ \ \Phi_0(x) = x.$$

Lemma 0.5.4. *The following equality holds:*

$$\frac{d}{dt}\Omega(\Phi_t(x),t) = \int_{\Phi_t(x)}^1 \frac{w^2(x,t)}{y}\ dy + \int_{\Phi_t(x)}^1 \frac{\partial_x\rho(y,t)}{y}\ dy$$

Proof:

$$\frac{d}{dt}\Omega(\Phi_t(x),t) = \partial_t\Omega(\Phi_t(x),t) + \partial_x\Omega(\Phi_t(x),t)\cdot u(\Phi_t(x),t). \tag{26}$$

Now

$$\partial_x\Omega(\Phi_t(x),t) = -\frac{\omega(\Phi_t(x),t)}{\Phi_t(x)},$$

and so the second term in (26) is equal to $\omega(\Phi_t(x),t)\cdot\Omega(\Phi_t(x),t)$. Next,

$$\partial_t\Omega(x,t) = \int_x^1 \frac{\partial_t\omega}{y}\ dy = -\int_x^1 \frac{u\partial_x\omega - \partial_x\rho}{y}\ dy$$

$$= \frac{u(x,t)\omega(x,t)}{x} + \int_x^1 \omega\cdot\frac{\partial}{\partial y}(-\Omega(y,t))\ dy + \int_x^1 \frac{\partial_x\rho}{y}\ dy.$$

The second integral in the last line equals $\int_x^1 \frac{\omega^2(y)}{y}\ dy$

Adding together $\partial_t\Omega(x,t) + u(x,t)\partial_x\Omega(x(t),t)$ we get the result. □

Proof: Let ρ_0 be smooth, nonnegative, supported in $[1/4,3/4]$, with $\max\rho_0 = \rho_0(1/2) = 2$, and $\rho_0(1/3) = 1$. Moreover, assume ρ_0 is increasing in $[1/4,1/2]$, and

decreasing in $[1/2, 3/4]$. Let ω_0 be smooth, nonnegative, supported in $[1/4, 1/2]$, with $\omega_0 \equiv M$ in $[0.3, 0.45]$.
Let us take x_n defined by $\rho_0(x_n) = \frac{1}{2} + 2^{-n}$, and set $x_\infty = \lim_{n\to\infty} x_n$. Furthermore, set $\Phi_n(t) := \Phi_t(x_n)$, and notice that

$$\frac{d}{dt}\Phi_n(t) = u(\Phi_n(t), t) = -\Phi_n(t)\Omega(\Phi_n(t), t).$$

We denote

$$\Psi_n(t) = -ln\ \Phi_n(t)$$

Then $\Psi_n'(t) = \Omega_n(t)$, and by Lemma 0.5.4 we have

$$\Omega_n'(t) \geq \int\limits_{\Phi_n(t)}^{1} \frac{\partial_x \rho(y,t)}{y}\, dy \geq \int\limits_{\Phi_n(t)}^{\Phi_{n-1}(t)} \frac{\partial_x \rho}{y}\, dy - 4 \geq \frac{1}{\Phi_{n-1}(t)} 2^{-n}.$$

Here in the second step we had to estimate the contribution from the region where the derivative of ρ is negative. This can be done without difficulty as this region lies away from the kernel singularity. We leave details to interested reader.

Therefore,

$$\frac{d^2}{dt^2}\Psi_n(t) \geq 2^{-n} e^{\Psi_{n-1}(t)}.$$

Then by taking $t_n = 2 - 2^{-n}$ and running an inductive argument we can get a recursive estimate $\Psi_n(t_n) := a_n \geq \exp(a_{n-1} - 3n)$.

Inductively we can show that $a_n \to \infty$. For example, if $a_1 > 20$ then one can verify that $a_n \geq \exp\exp\exp(n-1)$. $\square$

Acknowledgment. AK and CL acknowledge partial support of the NSF grant DMS-1412023. O.L acknowledges grants from Marie Curie FP7-IEF program, Acronym : TRANSIC. The authors would also like to thank the organizers of the Levico summer school on Fluids, Transport and Mixing for the invitation and Centro Internazionale per la Ricerca Matematica for making the school possible.

Literatur

[1] H. Bahouri, J.-Y. Chemin, *Équations de transport relatives à des champs de vecteurs non-lipschitziens et mécanique des fluides. (French) [Transport equations for non-Lipschitz vector Fields and fluid mechanics],* Arch. Rational Mech. Anal., **127** (1994), no. 2, 159–181

[2] K. Choi, A. Kiselev and Y. Yao, *Finite time blow up for a 1D model of 2D Boussinesq system,* Comm. Math. Phys. **334** (2015), 1667-1679

[3] K. Choi, T. Hou, A. Kiselev, G. Luo, V. Sverak and Y. Yao, *On the finite-time blowup of a 1D model for the 3D axisymmetric Euler equations,* preprint arXiv:1407.4776, to appear at Commun. Pure Appl. Math.

[4] P. Constantin, P. D. Lax, and A. Majda, *A simple one-dimensional model for the three-dimensional vorticity equation,* Comm. Pure Appl. Math. **38** (1985), 715–724

[5] S. De Gregorio, *On a one-dimensional model for the three-dimensional vorticity equation,* J. Stat. Phys. **59** (1990), 1251–1263

[6] S. De Gregorio, *A partial differential equation arising in a 1D model for the 3D vorticity equation,* Math. Methods Appl. Sci. **19** (1996), 1233–1255

[7] S. Denisov, *Infinite superlinear growth of the gradient for the two-dimensional Euler equation,* Discrete Contin. Dyn. Syst. A, **23** (2009), 755–764

[8] S. Denisov, *Double-exponential growth of the vorticity gradient for the two-dimensional Euler equation,* to appear in Proceedings of the AMS, **143** (2015), 1199–1210

[9] L.C. Evans, Partial Differential Equations, Second Edition, AMS, 2010

[10] L. Euler, *Principes généraux du mouvement des fluides,* Mémoires de L'Académie Royale des Sciences et des Belles-Lettres de Berlin **11** (4 September 1755, printed 1757), 217–273

[11] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, Berlin Heidelberg 2001

[12] E. Hölder, *Über die unbeschränkte Fortsetzbarkeit einer stetigen ebenen Bewegung in einer unbegrenzten inkompressiblen Flüssigkeit,* Math. Z. **37** (1933), 727–738

[13] V. I. Judovic, *The loss of smoothness of the solutions of Euler equations with time (Russian),* Dinamika Splosn. Sredy, Vyp. **16**, Nestacionarnye Problemy Gidrodinamiki, (1974), 71–78

[14] A. Kiselev and V. Sverak, *Small scale creation for solutions of the incompressible two dimensional Euler equation,* Annals of Math. **180** (2014), 1205-1220

[15] G. Luo and T. Hou, *Toward the Finite-Time Blowup of the 3D Incompressible Euler Equations: a Numerical Investigation,* SIAM Multiscale Modeling and Simulation **12** (2014), 1722–1776

[16] A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow,* Cambridge University Press, 2002

[17] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids,* Applied Mathematical Sciences Series (Springer-Verlag, New York), **96**, 1994

[18] N. S. Nadirashvili, *Wandering solutions of the two-dimensional Euler equation, (Russian)* Funktsional. Anal. i Prilozhen., 25 (1991), 70–71; translation in Funct. Anal. Appl., 25 (1991), 220–221 (1992)

[19] W. Wolibner, *Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long. (French)* Mat. Z., **37** (1933), 698–726

[20] V. I. Yudovich, *Non-stationary flows of an ideal incompressible fluid,* Zh Vych Mat, **3** (1963), 1032–1066

[21] V. I. Yudovich, *On the loss of smoothness of the solutions of the Euler equations and the inherent instability of flows of an ideal fluid,* Chaos, **10** (2000), 705–719

Alexander Kiselev, Mikhail Chernobay, Omar Lazar und Chao Li

Small Scale creation in inviscid fluids

Mathematics Subject Classification 2010: 65C05, 62M20, 93E11, 62F15, 86A22

Alexander Kiselev: Rice University (USA)
Mikhail Chernobay: St Petersburg State University (Russia)
Omar Lazar: Instituto de Ciencias Matemáticas (Spain)
Chao Li: Rice University (USA)

DOI: 10.1515/futur-2012-0001