AN EXPLICIT SELF-DUALITY
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Abstract. We provide an exposition of the canonical self-duality associated to a presenta-
tion of a finite, flat, complete intersection over a Noetherian ring, following work of Scheja
and Storch.

1. Introduction
Consider a finite, flat ring map \( f : A \to B \) and assume that \( A \) is Noetherian. Coherent
duality for proper morphisms provides a functor \( f^! : \text{D}(\text{Spec } A) \to \text{D}(\text{Spec } B) \)
on derived categories. The assumptions on \( f \) imply that \( f^! A \) is isomorphic to the sheaf on B associated to \( \text{Hom}_A(B, A) \). See for example [Sta18, 0AA2]. If we assume moreover that \( f : \text{Spec } B \to \text{Spec } A \) is a local complete intersection morphism, then \( f^! A \) is locally free [Sta18, 0B6V, 0FNT]. Thus there exists
an isomorphism
\[
(1.0.1) \quad \text{Hom}_A(B, A) \cong B
\]
of \( B \)-modules under additional hypotheses, for example if we assume that \( B \) is local. \(^1\)

There are many choices for the isomorphism \( (1.0.1) \). (The set of these isomorphisms form a
\( B^* \)-torsor.) An explicit presentation of \( B \) as
\[
(1.0.2) \quad B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)
\]
singles out a particular choice, which satisfies certain nice properties such as compatibility
with base change and the trace. In addition to the advantages of having a canonical choice
(e.g. gluing such isomorphisms together), this choice is closely related to the degree map in
\( A^1 \)-homotopy theory due to F. Morel. See Remark 1.1.

In this expository paper, we follow the approach of [SS75] to construct this canonical isomor-
phism for \( B \) a finite, flat \( A \)-algebra equipped with a presentation \( (1.0.2) \).

The approach is as follows: Consider the ideals
\[
(f_1 \otimes 1 - 1 \otimes f_1, \ldots, f_n \otimes 1 - 1 \otimes f_n) \subset (x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)
\]
of \( A[x_1, \ldots, x_n] \otimes A[x_1, \ldots, x_n] \). One writes
\[
f_j \otimes 1 - 1 \otimes f_j = \sum a_{ij}(x_i \otimes 1 - 1 \otimes x_i).
\]
and defines the element \( \Delta \in B \otimes_A B \) as the image of \( \text{det}(a_{ij}) \) under the morphism \( A[x_1, \ldots, x_n] \otimes A[x_1, \ldots, x_n] \to B \otimes_A B \). This is shown to be independent of the choice of \( a_{ij} \). There is a
canonical \( A \)-module morphism
\[
\chi : B \otimes_A B \to \text{Hom}_A(\text{Hom}_A(B, A), B).
\]
Let I denote the kernel of multiplication \( B \otimes_A B \to B \), or in other words the image of \( (x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n) \). One checks that \( \chi \) restricts to an isomorphism
\[
\chi : \text{Ann}_{B \otimes_A B} I \to \text{Hom}_B(\text{Hom}_A(B, A), B)
\]

\(^1\) An alternate point of view on the equivalence \( f^! A \cong B \) is that a factorization \( A \xrightarrow{i} A[x_1, \ldots, x_n] \xrightarrow{p} B \) of \( f \)
into a regular immersion and structure map for \( A \) allows one to compute \( f^! A \) as \( i^! p^! A \cong i^! (A[x_1, \ldots, x_n][n]) \cong \text{det} N_\ast^*[n] \cong B \), where \( N_\ast^* \) denotes the conormal bundle of the regular immersion \( \text{Spec } B \hookrightarrow A \). See for
example [Har66, Ideal Theorem p. 6, III, particularly Corollary 7.3].
of B-modules and identifies the annihilator as $\text{Ann}_{B \otimes_A B} I \cong \Delta$. Finally, one shows that

$$\chi(\Delta) := \Theta \in \text{Hom}_B(\text{Hom}_A(B, A), B)$$

provides the desired isomorphism of B-modules $\Theta: \text{Hom}_A(B, A) \to B$ guaranteed by the general theory of coherent duality. This is Theorem 3.4 (or [SS75, Satz 3.3]) and the main result. For the compatibility of $\Theta$ with base change and the trace see [SS75, p. 183-184 and Section 4] respectively.

Our arguments largely follow the outline of [SS75], although we make more use of Koszul homology in some proofs than the original did, and provide a self-contained proof of Lemma 2.4; the goal in large part is to provide an English reference for this material. See also [Kun05, Appendices H and I].

**Remark 1.1.** One motivation for providing an explicit description of this isomorphism is to describe the resulting $A$-valued bilinear form on $B$. This form is defined via

$$\langle b, c \rangle \mapsto \Theta^{-1}(b)(c) = \eta(bc) \in A,$$

where $\eta = \Theta^{-1}(1)$. The form $(-,-)$ has been used to give a notion of degree [EL77] [Eis78, some remaining questions (3)]. For example, it computes the local $A^1$-Brouwer degree of Morel [KW19] [BBM+21], and is useful in quadratic enrichments of results in enumerative geometry [Lev20] [KW21] [McK21] [Pau20].

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## 2. Commutative Algebra Preliminaries

**Lemma 2.1.** [SS75, 1.2] Let $A$ be a noetherian ring and suppose that $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are sequences satisfying the following hypotheses:

(i) $b = (g_1, \ldots, g_n) \subset a = (f_1, \ldots, f_n)$

(ii) If $p$ is a prime such that $a \subset p$, then the sequence $f_1, \ldots, f_n$ is a regular sequence in $A_p$, as is $g_1, \ldots, g_n$.

Write $g_i = \sum_{j=1}^{n} a_{ij} f_j$, and let $(a_{ij})$ be the resulting matrix of coefficients.

$$\Delta := \det(a_{ij}).$$

Define $\overline{\Delta}$ to be the image of $\Delta$ under the map $A \to A/b$. Then:

(a) The element $\overline{\Delta}$ is independent of the choices of $a_{ij}$.

(b) We have an equality (of $A/b$-ideals):

$$\langle \overline{\Delta} \rangle = \text{Fit}_{A/b}(a/b),$$

where $\text{Fit}$ denotes the 0-th Fitting ideal.

(c) We have an equality of ideals:

$$\langle \overline{\Delta} \rangle = \text{Ann}_{A/b}(a/b),$$

and

$$a/b = \text{Ann}_{A/b}(\overline{\Delta}).$$

**Remark 2.2.** We comment on condition (ii). If $(A, p)$ is a local ring and $a \subset p$, then condition (ii) is equivalent to asking that $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are regular sequences. In general, condition (ii) asks only that they are regular sequences after localizing at primes containing $a$ (e.g., they may not be regular sequences on $A$).
Proof. First, we may assume that \( A \) is a local ring and each of the \( f_i \)'s and \( g_i \)'s are in the maximal ideal \( \mathfrak{m} \).

(a): Write \( g_i = \sum_{j=1}^{n} b_{ij} f_j \). We want to show that \( \det(a_{ij}) - \det(b_{ij}) \) is in \( \mathfrak{b} \). It suffices to consider the case where \( a_{ij} = b_{ij} \) for all \( j \) and for \( i = 1, \ldots, n - 1 \), as this allows us to change the presentation of one \( g_i \) at a time, and thus all of them. Define
\[
c_{ij} = \begin{cases} a_{ij} = b_{ij} & i = 1, \ldots, n - 1 \\ a_{ij} - b_{ij} & i = n, \end{cases}
\]
By cofactor expansion along the \( j \)-th row, we have that
\[
\det(a_{ij}) - \det(b_{ij}) = \det(c_{ij}).
\]
But now
\[
(c_{ij}) \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n-1} \\ 0 \end{pmatrix}
\]
By Cramer's rule, for all \( k = 1, \ldots, n \) we have that
\[
\det(c_{ij}) \cdot f_k \in (g_1, \ldots, g_{n-1}),
\]
which means
\[
\det(c_{ij}) \cdot a \in (g_1, \ldots, g_{n-1}).
\]
But \( g_n \in \mathfrak{a} \) and hence
\[
\det(c_{ij}) \cdot g_n \in (g_1, \ldots, g_{n-1}),
\]
which means that \( \det(c_{ij}) \in (g_1, \ldots, g_n) = \mathfrak{b} \) since \( g_1, \ldots, g_n \) is a regular sequence.

(b): First observe that
\[
\text{Fit}_A(\mathfrak{a}/\mathfrak{b}) \ mod \ \mathfrak{b} = \text{Fit}_{A/\mathfrak{b}}(\mathfrak{a}/\mathfrak{b}).
\]
Therefore, to prove the claim, it suffices to prove that
\[
\text{Fit}_A(\mathfrak{a}/\mathfrak{b}) = \Delta + I,
\]
where \( I \subset \mathfrak{b} \).

To prove this claim, note that the Fitting ideal of the \( A \)-module \( \mathfrak{a}/\mathfrak{b} \) is computed by a presentation:
\[
\Lambda^{\oplus n} \oplus \Lambda^{\oplus \binom{n}{2}} \xrightarrow{T} \Lambda^{\oplus n} \to \mathfrak{a}/\mathfrak{b} \to 0,
\]
where \( T \) is given by:
\[
(a_{ij}) \times d_{\mathfrak{Kosz}}^{\binom{n}{2}}.
\]
In other words, the matrix of \( T \) has the first \( n \)-columns are just given by \( a_{ij} \) and, the last \( \binom{n}{2} \) columns are composed of the usual Koszul relations among the \( f_i \). (Note that the sequence \( f_1, \ldots, f_n \) is regular in our local ring, so the corresponding Koszul complex produces a resolution of \( \mathfrak{a} \) [Sta18, 062F].)

Now, the Fitting ideal is given by the \( n \times n \)-minors of the matrix of \( T \). The first minor is \( \Delta \). If \( \Delta' \) is another \( n \times n \) minor, then it is the determinant of a matrix \( T' \), which is composed of some \( r \) columns of \( (a_{ij}) \) and \( n - r \) columns of \( d_{\mathfrak{Kosz}}^{n-1} \); without loss of generality we may assume \( T' \) contains the first \( r \) columns of \( (a_{ij}) \) (if not, simply reorder the \( g_i \), using that the ring \( A \) is local and thus regularity of the sequence of \( g_i \) preserved). Applying \( T' \) to \( (f_k) \) we get
\[
(T') \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\]
We again conclude that \( \Delta' f_i = \det(T') f_i \in b \) for each \( i = 1, \ldots, n \). Thus,
\[
\Delta' \cdot a \in (g_1, \ldots, g_{n-1}),
\]
and in particular
\[
\Delta' \cdot g_n \in (g_1, \ldots, g_{n-1}),
\]
which by regularity of the \( g_i \) means that \( \Delta' \in b \) and thus \( \text{Fit}_A(a/b) = \Delta + I \) with \( I \subset b \).

(c): First, we claim that we have an isomorphism:
\[
\text{Ann}_A(a/b) \cong \text{Tor}_n^A(A/b, A/a).
\]
We will abbreviate \( \text{Tor}_j^A \) by \( \text{Tor}_j \) and \( \otimes_A \) by \( \otimes \) in what follows. To prove this, we deploy the Koszul complex. (As noted above, a regular sequence is Koszul-regular by \([\text{Sta18, 062F}]\).) We thus have a quasi-isomorphism:
\[
K_*(f_1, \ldots, f_n) \cong A/a
\]
Therefore the Tor group above is computed as the kernel of \( 1 \otimes d_n^{\text{Kosz}} \) in the complex \( A/b \otimes K_*(f_1, \ldots, f_n) \):
\[
0 \to A/b \xrightarrow{(f_1, \ldots, f_n)} (A/b)^{\oplus n}.
\]
Indeed, the cohomology of this small complex is the desired annihilator and thus we obtain the desired isomorphism.

On the other hand, we claim that \( \text{Tor}_n(A/a, A/b) \cong \Delta \cdot A/b \). To see this note that we have a short exact sequence of \( A \)-modules:
\[
0 \to a/b \to A/b \to A/a \to 0.
\]
We claim that the induced long exact sequence splits into short exact sequences for \( j \geq 1 \)
\[
0 \to \text{Tor}_j(A/b, a/b) \to \text{Tor}_j(A/b, A/b) \to \text{Tor}_j(A/b, A/a) \to 0
\]
Indeed, via the Koszul complex for \( A/b \), we see that for \( j \geq 1 \):
\[
(2.0.1) \quad \text{Tor}_j(A/b, a/b) \cong (a/b)^{(j)} \quad \text{Tor}_j(A/b, A/b) \cong (A/b)^{(j)},
\]
and the map \( \text{Tor}_j(A/b, a/b) \to \text{Tor}_j(A/b, A/b) \) is identified with the direct sum of copies of the injection \( a/b \hookrightarrow A/b \). To conclude, the functoriality of the Koszul complex \([\text{Sta18, 0624}]\) yields a morphism of complexes
\[
A/b \otimes K_*(g_1, \ldots, g_n) \to A/b \otimes K_*(f_1, \ldots, f_n);
\]
where the left end is as follows:
\[
\begin{array}{ccc}
A/b & \xrightarrow{0} & (A/b)^{\oplus n} \\
\downarrow & & \downarrow \\
A/b(f_1, \ldots, f_n) & \xrightarrow{a} & (A/b)^{\oplus n}.
\end{array}
\]
Since the map \( \text{Tor}_j(A/b, A/b) \to \text{Tor}_j(A/b, A/a) \) is a surjection, we conclude that
\[
\text{Tor}_n(A/b, A/a) \cong \text{Im}(\Delta) \cong \Delta \cdot A/b
\]
as desired.

For the second claim, note that the ideal \( \text{Ann}_A/\text{Ann}_A(\Delta) \) is obtained as the kernel of the left vertical map in (2.0.2), and is thus isomorphic to \( \text{Tor}_n(A/b, a/b) \), which we already know is isomorphic to \( a/b \) by (2.0.1).

A module \( M \) over a ring \( R \) is said to be reflexive if the natural map \( R \to \text{Hom}_R(\text{Hom}_R(M, R), R) \) is an isomorphism \([\text{Sta18, 0AVY}]\). A form of the following lemma is in the stacks project \([\text{Sta18, 0AVA}]\), but assumes that \( A \) is integral and that \( A = B \). The following is \([\text{SS75, 1.3}]\).

**Lemma 2.3.** Let \( A \) be a Noetherian ring and \( B \) a finite flat \( A \)-algebra. A finite \( B \)-module \( M \) is reflexive if and only if the following conditions hold:

\[\Box\]
(i) If \( \mathfrak{p} \subset A \) is a prime ideal with \( \text{depth} \, A_{\mathfrak{p}} \leq 1 \), then \( M_{\mathfrak{p}} \) is a reflexive \( B_{\mathfrak{p}} \)-module.
(ii) If \( \mathfrak{p} \subset A \) is a prime ideal with \( \text{depth} \, A_{\mathfrak{p}} \geq 2 \), then \( \text{depth} \, A_{\mathfrak{p}} \) \( (M_{\mathfrak{p}}) \geq 2 \).

**Proof.**

The property of being reflexive is preserved under any localization of \( B \) [Sta18, 0EB9], and can be checked locally on \( B \) [Sta18, 0AV1]. Therefore reflexivity of \( M \) implies (i). Reflexivity implies (ii): Any regular sequence in \( A \) \( \mathfrak{p} \) and can be checked locally on \( B \) [Sta18, 0AV1]. Therefore reflexivity of \( M \) implies (i). Reflexivity of \( M \) implies (i). Reflexivity of \( M \) implies (i). Reflexivity of \( M \) implies (i).

Conversely, suppose \( M \) is not reflexive. We assume for the sake of contradiction that properties (i) and (ii) hold. Since reflexivity can be checked locally, there is some minimal \( \mathfrak{p} \subset A \) among all prime ideals of \( A \) for which \( M_{\mathfrak{p}} \) is not a reflexive \( B_{\mathfrak{p}} \)-module. Without loss of generality, we may assume that \( A \) is local with maximal ideal \( \mathfrak{p} \). Since \( M_{\mathfrak{p}} \) is not reflexive, we must have that \( \text{depth} \, A_{\mathfrak{p}} \geq 2 \) and therefore \( \text{depth} \, A_{\mathfrak{p}} (M_{\mathfrak{p}}) \geq 2 \). We consider the exact sequence

\[
0 \to \text{Ker} \, \varphi \to M \to \text{Hom}_B(\text{Hom}_B(M, B), B) \to \text{Coker} \, \varphi \to 0,
\]

where \( \varphi \) is the canonical map to the double-dual. By assumption, \( \varphi \) becomes an isomorphism after localizing at any prime of \( A \) different from \( \mathfrak{p} \). It follows that \( \text{Ker} \, \varphi \) and \( \text{Coker} \, \varphi \) have finite length. Since \( \text{depth} \, A_{\mathfrak{p}} \geq 1 \), there exists some \( x \in A \) which is a nonzerodivisor on \( M \). But then \( x \) is a nonzerodivisor on the finite-length module \( \text{Ker} \, \varphi \), which therefore must vanish. Since \( \text{Hom}_B(\text{Hom}_B(M, B), B) \) is reflexive (as a \( B \)-module), it has \( A \)-depth \( \geq 2 \) by the forward implication of the lemma. The exact sequence

\[
0 \to M \to \text{Hom}_B(\text{Hom}_B(M, B), B) \to \text{Coker} \, \varphi \to 0,
\]

then shows that \( \text{depth} \, A_{\mathfrak{p}} \) \( \text{Coker} \, \varphi \geq 1 \) by the standard behavior of depth in short exact sequences [Sta18, 00LX]. Therefore the cokernel must vanish, which shows that \( M \) is reflexive. \( \square \)

**Lemma 2.4.** [SS75, 1.4] Let \( A \) be a Noetherian ring and let \( B \) be a finite flat \( A \)-algebra. Let \( M \) be a finite \( B \)-module, which is projective as an \( A \)-module. If \( \text{Hom}_B(M, B) \) is projective as a \( B \)-module, then \( M \) is projective as a \( B \)-module. In particular, if \( \text{Hom}_B(M, B) \) is free, then \( M \) is free.

**Proof.** It is enough to show that \( M \) is reflexive. We are therefore reduced to checking the conditions (i) and (ii) of Lemma 2.3. Clearly, (ii) holds, since \( M \) is projective over \( A \). It remains to check (i). We may therefore assume that \( A \) is a Noetherian local ring with \( \text{depth} \, A \leq 1 \), and we want to show that \( M \) is projective as a \( B \)-module. Since \( B \) is finite flat over \( A \), we have \( \text{depth} \, B_m = \text{depth} \, A \) for every maximal ideal \( m \) of \( B \) [Sta18, 0337].

Throughout, we will write \( N^* := \text{Hom}_B(N, B) \) for a \( B \)-module \( N \). Consider the map

\[
\varphi : M \to M^*.
\]

Let \( C := \text{Coker} \, \varphi \). Taking a presentation of \( M \), we obtain an exact sequence

\[
0 \to U \to F \to M \to 0
\]

with \( F \) free. Consider the dual sequence

\[
0 \to M^* \to F^* \to U^*,
\]

and let \( Q := \text{Im}(F^* \to U^*) \). Since \( M^* \) is projective by assumption, \( Q \) has projective dimension \( 0 \) or \( 1 \) as a \( B \)-module.

We have the commutative diagram

\[
\begin{array}{c}
F \\
\downarrow \\
F^{**} \\
\downarrow \\
M^{**} \\
\downarrow \\
\text{Ext}^1_B(Q, B) \\
\downarrow \\
0
\end{array}
\]
with exact lower row. Since $F \to M$ is a surjection, we see that $C = \text{Ext}_B^1(Q, B)$. Suppose
for each maximal ideal $m$. We find that $Q_m$ has projective dimension zero, i.e., is projective.
Therefore $C_m = 0$ and $C = 0$.

Now suppose that depth $A = 1$. Then depth $B_m \geq 1$ by [Sta18, 0AV5], whence
depth $B_m Q_m \geq 1$ by [Sta18, 00LX]. Again by Auslander–Buchsbaum, we find that $Q_m$ is projective,
and that $C = 0$.

We have shown that in any case $M \to M^*$ is surjective. Since $M^*$ is projective, this implies
that $M^* \cong M^{**} \oplus N$ for some $B$-module $N$. It follows that $N^* = 0$ and that $N$ is again free as an
$A$-module.

By assumption both $M$ and $M^{**}$ are free over the local ring $A$. A surjection of finite free
modules is an isomorphism if they have the same rank. To show two finite free modules
have the same rank, we may localize at a minimal prime ideal $q$ of $A$, so that also $B_q$ is a zero-
dimensional ring. Over the Artinian ring $B_q$, $\text{Hom}_B(N_q, B_q) = 0$ implies $N_q = 0$. (To see this,
note that we may assume that $B$ is local, with maximal ideal $m$. Then $N_q \to m N_q$ is nonzero
by Nakayama’s lemma. Since $B_q$ has finite length, there is a nonzero element annihilated by $m$, whence a $B$-homomorphism $B/m \to B_q$.) Thus $M_q$ and $M_q^{**}$ have the same rank, and therefore
$M \to M^{**}$ is an isomorphism. \hfill \Box

3. The explicit isomorphism

Recall that a ring map $A \to B$ is a relative global complete intersection if there exists a
presentation $A[x_1, \ldots, x_n]/(f_1, \ldots, f_c) \cong B$, and every nonempty fiber of $\text{Spec} B \to \text{Spec} A$ has
dimension $n - c$ [Sta18, 00SP]. Note that in this case the $f_i$ form a regular sequence [Sta18, 00SV].

We note that a global complete intersection is flat [Sta18, 00SW], and thus syntomic. We will be interested in the situation where $A \to B$ is furthermore assumed to be a finite flat global complete intersection.

**Construction 3.1.** Suppose that $A \to B$ is a finite flat global complete intersection. Choose
a presentation

$$A[x_1, \ldots, x_n] \cong B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_n).$$

Consider the commutative diagram

$$
\begin{array}{ccc}
A[x_1, \ldots, x_n] & \xrightarrow{m_1} & A[x_1, \ldots, x_n] \\
\downarrow{\pi} & & \downarrow{\pi} \\
B \otimes_A B & \xrightarrow{m} & B,
\end{array}
$$

with $m_1, m$ the obvious multiplication maps. We note that the elements

$$\{f_j \otimes 1 - 1 \otimes f_j\}_{j=1, \ldots, n}$$

are all in $\ker(m_1)$, which is generated by the $x_i \otimes 1 - 1 \otimes x_i$ for $i = 1, \ldots, n$, whence we have a relation

$$f_j \otimes 1 - 1 \otimes f_j = \sum_{i=1}^n a_{ij}(x_i \otimes 1 - 1 \otimes x_i).$$

Define $\Delta := (\pi \otimes \pi)(\det(a_{ij})) \in B \otimes_A B$. Define also $I := \ker m$.

**Proposition 3.2.** The following properties of $\Delta$ hold:

(a) The element $\Delta$ is independent of the choice of $a_{ij}$.

(b) We have an equality of $B \otimes_A B$-ideals:

$$\langle \Delta \rangle = \text{Fitt}_{B \otimes_A B} I,$$
(c) we have an equality of ideals
\[ (\Delta) = \Ann_{B \otimes A} I = \Ann_{B \otimes A}(\Delta) = 1. \]

Proof. Consider the ring map
\[ \pi \otimes 1 : A[x_1, \ldots, x_n] \otimes_A A[x_1, \ldots, x_n] \to B \otimes_A A[x_1, \ldots, x_n] \cong B[x_1, \ldots, x_n]. \]
Since
\[ f_i \otimes 1 \equiv 1 \otimes f_i = \sum_{i=1}^{n} a_{ij}(x_i \otimes 1 - 1 \otimes x_i) \]
in \( A[x_1, \ldots, x_n] \otimes_A A[x_1, \ldots, x_n], \) we have that
\[ -1 \otimes f_i = \sum_{i=1}^{n} a_{ij}(\pi(x_i) \otimes 1 - 1 \otimes x_i) \]
in \( B \otimes_A A[x_1, \ldots, x_n]. \)

Note that \( \Delta \) is the image of \( \det(a_{ij}) \) under the obvious morphism \( B \otimes_A A[x_1, \ldots, x_n] \to B \otimes A B, \) and that if \( a \) is the ideal generated by the \( \pi(x_i) \otimes 1 - 1 \otimes x_i \) and \( b \) the ideal generated by the \( (-1 \otimes f_i) \), then \( I \) is \( a/b. \) The desired properties will then follow immediately from applying Lemma 2.1 to \( b = (-1 \otimes f_i) \subset (\pi(x_i) \otimes 1 - 1 \otimes x_i) = a, \) once we show that the conditions of the Lemma are satisfied. It suffices to show that each is a regular sequence.

We claim that \( \{-1 \otimes f_j\} \subset B \otimes_A A[x_1, \ldots, x_n] \) is a regular sequence. Indeed, since relative global complete intersections are flat [Sta18, 00SW] and regular sequences are preserved under flat morphisms, this follows by regularity of the \( f_i \) in \( A[x_1, \ldots, x_n] \) and flatness of \( A \to B. \) It is immediate also that \( (\pi(x_i) - x_i) \) forms a regular sequence in \( B[x_1, \ldots, x_n] \) as well (the \( \pi(x_i) \) are just elements \( b_i \) of \( B \), and \( (x_i - b_i) \) is always a regular sequence in \( B[x_1, \ldots, x_n] \)).

Thus, the proposition follows by Lemma 2.1. \( \square \)

Now, retain our setup from Construction 3.1. There is a canonical map of \( A \)-modules
\[ \chi : B \otimes A B \to \text{Hom}_A(\text{Hom}_A(B, A), B) \quad \chi(b \otimes c) = (\varphi \mapsto \varphi(b)c). \]
Both \( B \otimes A B \) and \( \text{Hom}_A(\text{Hom}_A(B, A), B) \) each carry two natural \( B \)-module structures:

1. \( B \) acts on \( B \otimes A B \) as multiplication on either the left or right factor (i.e., either \( a(b \otimes c) = ab \otimes c \) or \( a(b \otimes c) = b \otimes ac \)).
2. \( B \) acts on \( \text{Hom}_A(\text{Hom}_A(B, A), B) \) as either pre- or post-composing a homomorphism by multiplication (i.e., either \( a\varphi : \psi \mapsto \varphi(a)\psi \) or \( a\varphi : \psi \mapsto a\varphi(\psi) \)).

Lemma 3.3. \( \chi \) induces a \( B \)-module isomorphism \( \Ann_{B \otimes A} B \cong \text{Hom}_B(\text{Hom}_A(B, A), B). \)

Proof. We note first that this map is an isomorphism of \( A \)-modules, for which it suffices to check that it’s bijective: Since \( B \) is a projective \( A \)-module we have that \( B \) is canonically isomorphic to \( B^{\vee \vee} \) (where we denote by \( ^\vee \) the \( A \)-module dual), so that we have isomorphisms of \( A \)-modules
\[ B \otimes_A B \cong (B^{\vee})^{\vee} \otimes_A B \cong \text{Hom}_A(B^{\vee}, B) \cong \text{Hom}_A(\text{Hom}_A(B, A), B); \]
one can check that \( \chi \) is simply the composition of these canonical isomorphisms.

It’s immediately checked that the morphism \( \chi \) is in fact a \( B \)-bimodule homomorphism for the \( B \)-module structures of \( B \otimes A B \) and \( \text{Hom}_A(\text{Hom}_A(B, A), B) \) given by right multiplication and post-composition.

Now, we note the following:

1. The largest submodule of \( B \otimes A B \) where the two \( B \)-module structures agree is \( \Ann_{B \otimes A} B \): this follows since an element \( r \in B \otimes A B \) is annihilated by all \( a \otimes 1 - 1 \otimes a \) exactly when \( (a \otimes 1)r = (1 \otimes a)r \) for all \( a \), which occurs exactly when the action of every \( a \) on \( r \) is the same under the two \( B \)-module structures.
(2) The largest submodule of $\text{Hom}_A(\text{Hom}_A(B, A), B)$ where the two $B$-module structures agree is

$$\text{Hom}_B(\text{Hom}_A(B, A), B) \subset \text{Hom}_A(\text{Hom}_A(B, A), B);$$

this is clear since the condition of pre- and post-multiplying by elements of $B$ being the same is exactly $B$-linearity.

Putting this together, we have that $\chi$ induces an isomorphism of $B$-modules

$$\chi : \text{Ann}_{B \otimes A} I \to \text{Hom}_B(\text{Hom}_A(B, A), B),$$

which was our desired claim. □

Theorem 3.4. The map $\chi(\Delta) : \text{Hom}_A(B, A) \to B$ is an isomorphism of $B$-modules.

Proof. Applying Lemma 3.2(c) we have that $\text{Ann}_{B \otimes A} I = \Delta(B \otimes A) B$, and further that $\text{Ann}_{B \otimes A} \Delta(B \otimes A) = I$. Thus, we have that

$$\text{Ann}_{B \otimes A} I = \Delta(B \otimes A) B \cong \Delta(B \otimes A) B / \text{Ann}_{B \otimes A} \Delta = \Delta(B \otimes A) B / I \cong m(\Delta) B.$$

Applying Lemma 3.3, we have then that $\text{Hom}_B(\text{Hom}_A(B, A), B)$ is a free $B$-module with basis $\chi(\Delta)$. Applying Lemma 2.4, this implies that $\text{Hom}_A(B, A)$ is a free $B$-module of rank 1. We must then have that the $B$-module homomorphism $\chi(\Delta) : \text{Hom}_A(B, A) \to B$ is an isomorphism, as desired. □

References


