A RELATIVE ORIENTATION FOR THE MODULI SPACE OF
STABLE MAPS TO A DEL PEZZO SURFACE

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Abstract. We prove orientation results for moduli stacks of rational curves on del Pezzo surfaces over a field, both in positive characteristic and in characteristic 0. These orientations and a theory of degree developed in [22] produce quadratically enriched Gromov–Witten invariants, counting rational curves over non-algebraically closed fields of characteristic not 2 or 3. The orientations come from the following geometric results, valid under appropriate hypotheses on the del Pezzo surface and degree of the curve.

(1) After removing a codimension 2 locus of the moduli stack \( \bar{M}_{0,n}(S, D) \) of stable degree \( D \) maps from a genus 0 curve with \( n \) marked points to a characteristic 0 del Pezzo surface \( S \), the evaluation map

\[
ev: \bar{M}_{0,n}(S, D) \to S^n
\]

is étale away from \( D_{\text{cusp}} \), defined as a closure of the locus of parametrized curves whose images have a simple cusp, and is simply ramified along \( D_{\text{cusp}} \).

(2) In characteristic 0, the discriminant of the double point locus over \( \bar{M}_{0,n}(S, D) \) is \( D_{\text{cusp}} + 2D_{\text{tac}} \), where \( D_{\text{tac}} \) denotes the closure of the locus of parametrized curves whose image contains a tac node.

(3) In positive characteristic, the image under the evaluation map of the complement of the locus of parametrized curves with only ordinary double points is codimension at least 1 in \( S^n \). More simply, we have many parametrized curves with only ordinary double points.

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1. Introduction

By a del Pezzo surface, we mean a smooth, projective surface $S$ over a field $k$ whose inverse canonical class $-K_S$ is ample. Examples of interest in this paper include blow-ups of $\mathbb{P}^2$ at fewer than 9 points, $\mathbb{P}^1 \times \mathbb{P}^1$, and cubic surfaces. Let $D$ be an effective Cartier divisor on $S$. A rational curve of degree $D$ on $S$ means here a map $f : \mathbb{P} \to S$ from a genus 0 curve $\mathbb{P}$ to $S$ such that $f_*[\mathbb{P}] = D$ in $CH^1(S)$. Let $d = D \cdot -K_S - 1 \in CH^2(S) \cong \mathbb{Z}$. An “invariance of number principle” and a well-known dimension count (Lemma 2.19) shows that one expects there to be a fixed finite number $N_d$ of rational curves on $S$ through $d-1$ generally chosen points on $S$ over the complex numbers $\mathbb{C}$, or more generally, over an algebraically closed field, at least of characteristic 0. The numbers $N_d$ are examples of Gromov–Witten invariants studied in [3][14][28][35][38][45].

Over non-algebraically closed fields, “invariance of number” fails, and the number of rational curves depends on the generally chosen points. This failure was overcome over $\mathbb{R}$ by assigning each curve a $\pm 1$ using the geometry of the curve [53][48][33][19][20]. To give meaningful counts over non-algebraically closed fields more generally, we will give an enrichment of $N_d$ in quadratic forms, analogous to the quadratic or $A^1$-enrichments of other enumerative invariants found in, e.g. [24][33][32][44][29]. Andrés Jaramillo Puentes and Sabrina Pauli have work in progress giving the count through rational points over toric surfaces via a tropical correspondence theorem, building on their previous work [21]. In such enriched counts, each curve is weighted by a class in the group completion $GW(k)$ of unimodular symmetric bilinear forms over $k$. The group $GW(k)$ arises from Morel’s theorem on $A^1$-homotopy groups of spheres [39][40][Theorem 1.23, Theorem 6.40, Theorem 3.37]. In concrete terms, the weight is connected to the field of definition of the curve and the fields of definitions of tangent directions at nodes.

Following ideas of the third-named author in [48], we define the quadratic enrichment of $N_d$ as an appropriate degree and inherit invariance of number under suitable notions of connectivity, e.g. $A^1$-connectivity of $S$. This notion of degree is developed in [22] and is connected with Morel and Voevodsky’s $A^1$-homotopy theory [11] and Morel’s degree of a map of spheres [39]. As one would expect, it is a generalization of the topological definition, which assigns an integer valued degree to a map $f : M \to N$ between oriented, compact, connected manifolds of the same dimension a degree. The topological degree may be computed as a weighted count of the points of $f^{-1}(y)$ for a general point $y$ of $N$. One may drop the hypothesis that $M$ and $N$ are themselves oriented by instead constructing an orientation of the map $f$, defined to be an orientation of the vector bundle $\text{Hom}(\text{det } TM, f^* \text{det } TN)$ [43][2.1]. Like its topological counterpart, the degree of [22] is only defined in the presence of appropriate orientations.
The subject of this paper is to construct such orientations. In the process, we give results on the geometry of the moduli stack \( \bar{M}_{0,n}(S,D) \) of stable maps from \( n \)-marked genus 0 curves to \( S \) in curve class \( D \) which may be of some independent interest.

Define the evaluation map \( ev: \bar{M}_{0,n}(S,D) \to S^n \) by taking the class of \( (f: \mathbb{P} \to S, (p_1, \ldots, p_n)) \), where \( \mathbb{P} \) is a genus 0 curve with marked points \( p_1, \ldots, p_n \) to \( (f(p_1), \ldots, f(p_n)) \in S^n \). Note that the fiber above a point of \( S^n \) consists of parametrized rational curves passing through the corresponding \( p_1, \ldots, p_n \). As in the case of the topological degree, the \( \mathbb{A}^1 \)-degree of \( [22] \) is a weighted count of the points of the fiber, when it is defined. It follows that a well-defined degree of \( ev \) along with degrees of twists and symmetrizations of \( ev \) which we will discuss shortly, give rise to a generalization of the Gromov–Witten invariants \( N_d \). The degree is moreover insensitive to removing subspaces of the domain or codomain of codimension \( \geq 2 \). We thus orient \( ev \) away from codimension 2.

To do this, we compute the ramification of \( ev \). Let \( M_{0,n}^{\text{bir}}(S,D) \subset \bar{M}_{0,n}(S,D) \) denote the locus of birational maps with irreducible domain curve. See Definition 2.4. Let \( D_{\text{cusp}} \) (respectively \( D_{\text{tac}} \)) denote the Cartier divisor on \( M_{0,n}^{\text{bir}}(S,D) \) defined as the closure of \( \langle f: \mathbb{P} \to S, (p_1, \ldots, p_n) \rangle \) such that \( f(\mathbb{P}) \) has one simple cusp (respectively tacnode) and nodes, but no other singularities. See Definition 2.10 and Lemma 2.12.

**Theorem 1.1.** Let \( S \) be a del Pezzo surface of degree \( d_S \) over a field \( k \) of characteristic 0. Let \( D \) be an effective Cartier divisor which is not an \( m \)-fold multiple of a \(-1\)-curve for \( m > 1 \). Suppose that \( d_S \geq 4 \), or \( d_S = 3 \) and \( d := -K_S \cdot D \neq 6 \), or \( d_S = 2 \) and \( d \geq 7 \).

Then, after pullback to the complement of a codimension \( \geq 2 \) subscheme of \( S^n \), the map \( ev \) is a map between smooth schemes. Moreover, \( ev \) is étale on the complement of \( D_{\text{cusp}} \) and \( dev \) vanishes to order 1 along \( D_{\text{cusp}} \).

Theorem 1.1 is proven in Theorem 4.5 and Lemma 4.8. As is consistent with Welschinger’s invariants and local contributions to open Gromov-Witten invariants, the nodes of \( f(\mathbb{P}) \) play a role in the orientation of \( ev \), and thus also in its degree and the resulting count of rational curves. To study the role of these nodes, define the locus of parametrized curves with only ordinary double points \( M_{0,n}^{\text{ordp}}(S,D) \) to parametrize \( \langle f: \mathbb{P}^1 \to S, p_1, \ldots, p_n \rangle \) in curve class \( D \) such that \( f(\mathbb{P}^1) \) has only ordinary nodes. See Definition 2.9. We show that the complement of \( M_{0,n}^{\text{ordp}}(S,D) \) in \( \bar{M}_{0,n}(S,D) \) maps under \( ev \) to codimension 1 in characteristic not 2 or 3 under appropriate hypotheses. See Corollary 3.15.

**Theorem 1.2.** Suppose that \( k \) is a field of characteristic \( \neq 2,3 \) and let \( S \) be a del Pezzo surface over \( k \) equipped with an effective Cartier divisor \( D \) which is not an \( m \)-fold multiple of a \(-1\)-curve for \( m > 1 \). Suppose furthermore that for every effective Cartier divisor \( D' \) on \( S \), there is a geometric point \( f \) in each irreducible component of \( M_{0}^{\text{bir}}(S,D') \) with \( f \) unramified. Then \( ev_k(\bar{M}_{0,n}(S,D) - M_{0,n}^{\text{ordp}}(S,D)) \) has positive codimension in \( S^n \).

We show the hypothesis of Theorem 1.2 holds for toric del Pezzo surfaces, such as \( \mathbb{P}^2 \), blow-ups of \( \mathbb{P}^2 \) at 3 or fewer points, and \( \mathbb{P}^1 \times \mathbb{P}^1 \) in Appendix A provided \( d_S \geq 2 \) and \( D \) is not an \( m \)-fold multiple of a \(-1\)-curve for \( m > 1 \). See Corollary A.15.

Above every node of \( f(\mathbb{P}) \), there are two points. The collection of all such points forms the double point locus \( D \), which is a locally closed subscheme in the product.
\[ \bar{\mathcal{M}}_{0,n+1}(S,D) \times \mathcal{M}_{0,n}(S,D) \] of universal curves. It contains the locus parameterizing stable maps \((f : \mathbb{P}^1 \to S, (p_1, \ldots, p_n)) \in M_{0,n}^{\text{odp}}(S,D)\) equipped with a pair of distinct points \(x_1 \neq x_2\) on \(\mathbb{P}^1\) such that \(f(x_1) = f(x_2)\). See Definitions 5.3 and 5.5. By construction \(D\) comes with a projection map \(\pi : D \to \bar{\mathcal{M}}_{0,n}(S,D)\). Over \(M_{0,n}^{\text{odp}}(S,D)\), \(\pi\) is finite étale. See Lemma 5.4 and the construction which implies properness. Since a curve in the linear system \([D]\) of \(D\) has arithmetic genus \(\delta = \frac{1}{2}D \cdot (K_S + D) + 1\), it follows that \(f(\mathbb{P}^1)\) will generically have \(\delta\) nodes for \((f : \mathbb{P}^1 \to S, (p_1, \ldots, p_n)) \in M_{0,n}^{\text{odp}}(S,D)\) and that the degree of the finite étale map \(\pi_{|\pi^{-1}(M_{0,n}^{\text{odp}}(S,D))}\) is \(2\delta\). We compute the ramification and discriminant of \(\pi\).

To state the computation, we introduce the following notation. Under the assumptions of Theorem 1.1 there is a smooth subscheme \(\bar{\mathcal{M}}_{0,n}(S,D)^{\text{good}}\) of the stack \(\bar{\mathcal{M}}_{0,n}(S,D)\) defined so \(\bar{\mathcal{M}}_{0,n}(S,D)^{\text{good}}\) is the inverse image \(\bar{\mathcal{M}}_{0,n}(S,D)^{\text{good}} := \bar{\mathcal{M}}_{0,n}(S,D) \setminus \text{ev}^{-1}(A)\) of the complement of \(A \subset S^n\) with \(\text{codim } A \geq 2\) and such that \(\bar{\mathcal{M}}_{0,n}(S,D)^{\text{good}}\) contains a dense open subset of \(M_{0,n}^{\text{odp}}(S,D)\). See Theorem 4.5.

Let \(\mathcal{D}_{\text{cusp}}\) denote the locus of points \((f, p)\) where \(f : \mathbb{P}^1 \to S\) is a map and \(p \in \mathbb{P}^1\) is a point where \(f(\mathbb{P}^1)\) has a simple cusp. The locus \(\mathcal{D}_{\text{cusp}}\) is in fact naturally a subscheme of the double point locus \(\mathcal{D}\). See Lemma 5.7. Let \(\mathcal{D}_{\text{tac}} \subset \mathcal{D}\) denote the locus \((f, p, q)\) where \(f : \mathbb{P}^1 \to S\) is a map and \(p, q \in \mathbb{P}^1\) is a pair of points where \(f(\mathbb{P}^1)\) has a simple tacnode.

**Theorem 1.3.** Under the assumptions of Theorem 1.1, the double point locus \(\mathcal{D}\) is smooth and the map \(\pi : \mathcal{D} \to \bar{\mathcal{M}}_{0,n}(S,D)^{\text{good}}\) is finite, flat and generically étale. The ramification of \(\pi\) is supported on \(\mathcal{D}_{\text{cusp}}\) and \(\mathcal{D}_{\text{tac}}\), where it is simply ramified, and the divisor of the discriminant is given

\[ \text{div } \text{disc}_\pi = 1 \cdot D_{\text{cusp}} + 2 \cdot D_{\text{tac}} \]

The definition of the discriminant is recalled in (6.1). Theorem 1.3 is proven as Corollary 5.13 and Proposition 6.1.

The definition of an orientation is as follows. Let \(f : X \to Y\) be a local complete intersection (lcI) morphism of schemes over \(k\) as in [49 Tag 068E]. For example, if \(X\) and \(Y\) are smooth \(k\)-schemes, any such \(f\) is lcI. Then there is a perfect cotangent complex \(L_f\) [49 Tag 08SH] defining a line bundle \(\text{det } L_f\), analogous to the topological line bundle \(\text{Hom}(f^* \text{det } T^* N, \text{det } T^* M)\) which appeared earlier. An orientation of \(f\) is defined to be the data of a line bundle \(L\) on \(X\) and an isomorphism \(L^{\otimes 2} \cong \text{det } L_f\). See [22 Definition 2.2] and [40 Definition 4.3]. In particular, a finite étale map (which has \(\text{det } L_f \cong \mathcal{O}_X\)) has a canonical orientation, and for \(X\) and \(Y\) smooth schemes, an orientation of \(f\) is an isomorphism \(L^{\otimes} \cong \text{Hom}(f^* T^* Y, T^* X)\).

Theorems 1.1 and 1.3 combine to construct an orientation of ev as follows. Discriminant bundles are canonically isomorphic to the square of a line bundle. Thus Theorem 1.3 identifies \(\mathcal{O}_{\bar{\mathcal{M}}_{0,n}^{\text{good}}(S,D)}(D_{\text{cusp}})\) canonically as the square of a line bundle, and Theorem 1.1 identifies \(\mathcal{O}_{\bar{\mathcal{M}}_{0,n}^{\text{good}}(S,D)}(D_{\text{cusp}})\) with \(\text{det } ev|_{\mathcal{M}_{0,n}^{\text{odp}}(S,D)}\). This orientation is given in Theorem 6.3.

We need to orient additional, but closely related, maps to obtain the Gromov–Witten invariants of \(\bar{\mathcal{M}}_d\). Given \(d - 1\) points \(p_1, \ldots, p_{d-1}\) of the plane, the rational curves through these points do not depend on the order in which the points are listed. We therefore define a symmetrization \(ev_{\bar{\mathcal{M}}_d}\) of \(ev\) in Section 7 which maps to the quotient \(\text{Sym}_n^0 S\) of the complement of the pairwise diagonals in \(S^n\) by the
symmetric group on \( n \)-letters. We orient \( ev_S \) in Theorem \([7.1]\). This orientation is defined in characteristic 0 under appropriate hypothesis. We will use additional tools to obtain the desired invariants in characteristic \( p \), which will be described shortly. The degree of \( ev_S \) corresponding to the constructed orientation assembles counts for different choices of points \( p_i \) into a section of a sheaf of quadratic forms on \( \text{Sym}^n S \). This degree is not pulled back from \( GW(k) \), even when \( S \) is \( \mathbb{A}^1 \)-connected, as is consistent with Welschinger’s invariants over \( \mathbb{R} \).

Invariance of number is equivalent to the corresponding degree being pulled back from \( GW(k) \). To obtain such invariance, we construct a twists of the evaluation map. Each twist corresponds to a choice of fields of definition of the points \( p_i \) counted with multiplicity. The orientation of \( ev \) twists along with the map. See Theorem \([8.1]\) For \( \mathbb{A}^1 \)-connected \( S \), the degree must be pulled back from \( GW(k) \). See \([22] \) Section 3.4 Proposition 3.23, Definition 3.25] This allows for a count in \( GW(k) \) of rational curves through generally chosen points with a given multi-set of field extensions of \( k \) that is independent of the chosen points \([22] \) Theorem 3]. It likewise proves that the degree of \( ev_S \) discussed in the previous paragraph has fibers which only depend on the fields of definition of the points, when \( S \) is \( \mathbb{A}^1 \)-connected, generalizing a result of the second named author \([34]\).

We furthermore construct Gromov–Witten invariants in positive characteristic. To define a degree in positive characteristic, it is enough to lift the map to characteristic 0 and orient the lift away from codimension 1 on the special fiber. For an exact statement, we refer the reader to \([22] \) Assumption 3.13, Definition 3.19]. We lift del Pezzo surfaces equipped with Cartier divisors from characteristic not 2, 3 to characteristic 0, and then use Theorem \([1.2]\) and our results in characteristic 0 to define orientations of \( ev \) its twists and symmetrizations in Section \([9]\).

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2. Rational curves on a del Pezzo surface

2.1. Definitions of moduli spaces of stable maps to del Pezzo surfaces. We will be using the moduli space of pointed, stable maps, and we set up notation and references for this now. Let \( \mathcal{O} \) be a Noetherian ring. Let \( S \rightarrow \text{Spec} \mathcal{O} \) be a smooth projective \( \mathcal{O} \)-scheme and let \( \mathcal{D} \) be an effective relative Cartier divisor on \( S \).

Let \( \overline{M}_{0,n}(S, \mathcal{D}) \) denote the scheme of morphisms \( f : \mathbb{P}^1 \rightarrow S \) with \( f_*([\mathbb{P}^1]) \in |\mathcal{D}| \), together with \( n \) disjoint points (i.e. disjoint sections from the base scheme) \( p_1, \ldots , p_n \) of \( \mathbb{P}^1 \) with its natural \( \text{PGL}_2 \)-action. Let
Definition 2.1. \( d := \deg(-D \cdot K_S) \)

in \( CH^0(O) \) denote the degree of \( D \) with respect to \(-K_S\) and suppose that \( d \) is everywhere greater than 0. (Intersection numbers are locally constant. See for example [25, B18].) This implies that each \( f : \mathbb{P}^1 \rightarrow S \) with \( f_*([\mathbb{P}^1]) \in |D| \) is a stable map (in the sense of having finite automorphisms – see [1, p. 91(6)]).

This stability gives a natural map \( \tilde{M}_{0,n}(S, D) \rightarrow \bar{M}_{0,n}(S, D) \) to a moduli stack \( \bar{M}_{0,n}(S, D) \) of stable maps given by the quotient by \( \text{PGL}_2 \). \( M_{0,n}(S, D) \) is an open substack of a compactified moduli stack \( \bar{M}_{0,n}(S, D) \) of \( n \)-pointed, stable maps of a genus zero curve to \( S \), in the curve class \( D \). See [1] Theorem 2.8 and comment p.90 for more information. We use the notation \( M_0(S, D), \bar{M}_0(S, D), M_0(S, D) \) to denote \( M_{0,n}(S, D), \bar{M}_{0,n}(S, D), \) and \( \tilde{M}_{0,n}(S, D) \), respectively, with \( n = 0 \).

The moduli stack \( \tilde{M}_{0,n}(S, D) \) is a proper (in particular, separated) algebraic stack over \( O \) with projective coarse moduli space. \( \tilde{M}_{0,n}(S, D) \) is constructed by representing the functor of morphisms of \( n \)-pointed curves as a quasi-projective subscheme \( \tilde{M}_{0,n}(S, D) \) of a suitable Hilbert scheme and then defining \( \tilde{M}_{0,n}(S, D) \) as the quotient stack of this scheme by the natural \( \text{PGL}_N \)-action (for suitable \( N \)).

In particular, over an open substack \( V \) with trivial groupoid structure, \( \tilde{M}_{0,n}(S, D) \) has a free and stable \( \text{PGL}_N \)-action, so \( V \) is isomorphic to its image in the coarse moduli space. In particular, \( V \) is a quasi-projective scheme over \( O \).

We will be interested in the case where \( S \) is a del Pezzo surface. The references [25, Chap III, §3] and [8, Chap. 8] contain pertinent information on del Pezzo surfaces, and we now give a definition and some description.

**Definition 2.1.** A \( O \)-scheme \( S \) is a **del Pezzo surface over \( O \)** if \( S \) is smooth of relative dimension 2 and projective over \( O \) and the anti-canonical sheaf \(-K_S\) is relatively ample. The **degree** \( d_S \) of a del Pezzo surface \( S \) is the self-intersection \( K_S^{(2)} \).

2.2. **Over a field.** Now let \( k \) be a perfect base field and let \( S \) be a del Pezzo surface over \( k \) of degree \( d_S \). Over the algebraic closure of \( k \), we may represent \( S \) as the blow-up of \( \mathbb{P}^2 \) at \( 9 - d_S \) points or as \( S = \mathbb{P}^1 \times \mathbb{P}^1 \) (in this case, \( d_S = 8 \)). In case \( d_S \geq 3 \), the anti-canonical divisor \(-K_S\) is very ample and in case \( d_S = 2 \), the linear system \( |-K_S| \) defines a finite, 2-1 cover of \( \mathbb{P}^2 \). We call \( S \) a **general del Pezzo** of degree \( d_S \) if \( d_S \geq 5 \) or if \( S \) is the blow-up of \( \mathbb{P}^2 \) at \( 9 - d_S \) “general” points (that is, an assertion about \( S \) is true for all sets of points outside a closed algebraic subset of \( S^{9-d_S} \)).

2.3. **Properties of rational curves.** We will need to examine the geometry of \( M_{0,n}(S, D) \) and \( \bar{M}_{0,n}(S, D) \) at some length. To do this, we need to recall quite a number of definitions of properties of rational curves on \( S \) which affect the geometry of their neighborhoods in these moduli spaces.

For later use in §2.1 in this section we use a separated Noetherian scheme \( B \) as base-scheme. We fix a del Pezzo surface \( S \rightarrow B \) over \( B \), endowed with relative effective Cartier divisor \( D \). Following §2.1, we have the moduli stack \( M_{0,n}(S, D) \rightarrow B \).

Let \( C \rightarrow B \) be a separated morphism of a noetherian scheme \( C \) to \( B \) and let \( f : \mathbb{P}^1_C \rightarrow S_C \) be a morphism. We say that \( f \) is non-constant, resp. separable, if for each geometric point \( x \rightarrow C \), the base-change \( f_x : \mathbb{P}^1_x \rightarrow S_x \) is non-constant,
resp. separable to the image curve \( f_x(\mathbb{P}^1) \subset S_x \). For \( f : \mathbb{P}^1_C \to S_C \) non-constant and separable, we have the normal sheaf \( \mathcal{N}_f \) defined by the exactness of the sheaf sequence

\[
0 \to T_{\mathbb{P}^1_C/T} \to f^* T_S \to \mathcal{N}_f \to 0
\]

When additionally, \( f_*([\mathbb{P}^1_C]) \in |D_C| \), we have that \( d := \deg(-D \cdot K_S) \) is the degree of the determinant of \( f^* T_{S_C} \) over \( C \). If \( C = \text{Spec} \ K \) for some field \( K \), we say that \( f : \mathbb{P}^1_K \to S_K \) is defined over \( K \); to save notation, we call this a morphism \( f : \mathbb{P}^1 \to S \), defined over \( K \). Similarly, we write \( f_*([\mathbb{P}^1]) \in |D| \) for \( f_*([\mathbb{P}^1_K]) \in |D_K| \).

**Remark 2.2.** Suppose \( f : \mathbb{P}^1 \to S \) is a nonconstant and separable morphism defined over a field \( F \) and \( f_*([\mathbb{P}^1]) \in |D| \). Then the sequence (2.1) yields \( \mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{N}^\text{tor}_f \) with \( m = d - 2 - \dim F H^0(\mathbb{P}^1, \mathcal{N}^\text{tor}_f) \) where \( \mathcal{N}^\text{tor}_f \) denotes the torsion subsheaf of \( \mathcal{N}_f \).

**Definition 2.3.** For \( f : \mathbb{P}^1 \to S \) defined over some field \( F \), we call \( f \) free if \( \mathcal{N}_f \) is generated by global sections and \( H^1(\mathbb{P}^1, \mathcal{N}_f) = 0 \); equivalently, \( H^1(\mathbb{P}^1, \mathcal{N}_f(-1)) = 0 \) (see [20] Chap. II, Definition 3.1). In general, we call \( f : \mathbb{P}^1_C \to S_C \) free if \( f_x \) is free for all geometric points \( x \to C \).

For \( f : \mathbb{P}^1 \to S \) defined over some field \( F \), we let \( \mathcal{N}^\text{tor}_f \subset \mathcal{N}_f \) be the torsion subsheaf of \( \mathcal{N}_f \). We have \( \mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{N}^\text{tor}_f \), where \( m = d - 2 - \ell \) and \( \ell \) the length of \( \mathcal{N}^\text{tor}_f \).

For a morphism \( f : \mathbb{P}^1 \to S \) defined over \( F \), we write \( H^i(\mathbb{P}^1, \mathcal{N}_f) \) for the \( F \)-vector space \( H^i(\mathbb{P}^1, \mathcal{N}_f) \) and drop the subscript \( F \) in other situations if the context makes the meaning clear.

Let \( (\mathbb{P}, p_+) \) be a semi-stable genus zero curve with \( n \) marked points and let \( f : \mathbb{P} \to S \) be a stable map with (reduced) image curve \( C = f(\mathbb{P}) \subset S \), defined over some field \( F \). We call \( f \) birational if \( f : \mathbb{P} \to C \) is a birational map of curves, that is, \( f^* \) is an isomorphism on total quotient rings; \( f \) is non-birational if \( f \) is not birational.

**Definition 2.4.** Let \( M^\text{bir}_{0,n}(S, D) \subset M_{0,n}(S, D) \) be the open subscheme with geometric points \( [(f, p_+)] \) such that \( f : \mathbb{P} \to f(\mathbb{P}) \) is birational. Let \( \overline{M}^\text{bir}_{0,n}(S, D) \subset M_{0,n}(S, D) \) denote the closure of \( M^\text{bir}_{0,n}(S, D) \).

Since a birational map has no automorphisms, \( M^\text{bir}_{0,n}(S, D) \) is in fact an open subscheme of the moduli stack of stable maps \( M_{0,n}(S, D) \).

**Definition 2.5.** We let \( M^\text{birf}_{0,n}(S, D) \subset M_{0,n}(S, D) \) be the open subscheme with geometric points \( [(f, p_+)] \) such that \( f : \mathbb{P} \to f(\mathbb{P}) \) is birational and free.

**Definition 2.6.** We say that a map \( f : \mathbb{P} \to S \) from a genus 0 curve \( \mathbb{P} \) is unramified if \( f : \mathbb{P} \to f(\mathbb{P}) \) is an unramified map of relative curves. For \( \mathbb{P} \) smooth, this is equivalent to the induced map on cotangent spaces \( df : f^* T^* S \to T^* \mathbb{P} \) being surjective. Let \( M^\text{unf}_{0,n}(S, D) \) represent those \( [(f, p)] \) in \( M_{0,n}(S, D) \) such that \( f : \mathbb{P} \to f(\mathbb{P}) \) is unramified.

**Remark 2.7.** Suppose \( f : \mathbb{P}^1 \to S \) is unramified. Then we have the exact sheaf sequence

\[
\mathcal{N}_f^\text{tor} \cong 0, \quad \text{and} \quad \mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(d - 2).
\]

In addition, letting \( \hat{M}^\text{birf}_{0,n}(S, D) \), \( \hat{M}^\text{unf}_{0,n}(S, D) \) be the corresponding subschemes of \( M_{0,n}(S, D) \), the map \( \hat{M}^\text{birf}_{0,n}(S, D) \to \hat{M}^\text{unf}_{0,n}(S, D) \) is a PGL2-bundle; for \( F \) an
algebraically closed field, we will often identify an \([f] \in \mathcal{M}^{\text{bir}}_{0,n}(S,D)(F)\) with a choice of lifting \(f \in \mathcal{M}^{\text{bir}}_{0,n}(S,D)(F)\), leaving the context to make the meaning clear.

**Lemma 2.8.** \(\mathcal{M}^{\text{un}}_{0,n}(S,D)\) consists of birational maps.

**Proof.** Since birationality can be detected after base change to the algebraic closure, it suffices to show this for geometric points of \(\mathcal{M}^{\text{un}}_{0,n}(S,D)\). Let \(f : (\mathbb{P}^1, p_*) \to S\) be a geometric point of \(\mathcal{M}^{\text{un}}_{0,n}(S,D)\). By the universal property of normalization, the map \(f : \mathbb{P}^1 \to f(\mathbb{P}^1)\) factors through the normalization \(\nu : Z \to f(\mathbb{P}^1)\). So, let \(g : \mathbb{P}^1 \to Z\) be the map such that \(\nu \circ g = f\). By Lüroth’s theorem, \(Z \cong \mathbb{P}^1\). Since \(f\) is unramified, the differential \(df_q : T_q\mathbb{P}^1 \to T_{f(q)}S\) is injective for all \(q \in \mathbb{P}^1\). It follows that \(dg_q : T_q\mathbb{P}^1 \to T_{g(q)}Z\) is injective for all \(q \in \mathbb{P}^1\) and thus \(g\) is unramified. Since \(g\) is a non-constant map of curves, whose codomain is normal, \(g\) is flat [49, Tag 0CCK]. Since \(g\) is flat, unramified, and of finite presentation, \(g\) is étale [49, Tag 02G3]. Since the étale fundamental group of \(\mathbb{P}^1\) is trivial (even in characteristic \(p > 0\)) [46, Théorème 2.6 Exposée X SGA1], \(g\) is an isomorphism. 

Our desired orientation of \(ev\) will be described in terms of singularities of the image curve \(f(\mathbb{P}^1)\) at \((f : \mathbb{P}^1 \to S, (p_1, \ldots, p_n))\) in \(\mathcal{M}_{0,n}(S,D)\), so we define certain singularities now. Suppose \(f : \mathbb{P}^1 \to S\) is an unramified map defined over an algebraically closed field. If the preimage of any point of \(S\) consists of at most two points and for all points with two inverse images \(p_1\) and \(p_2\), the subspaces \(df(T_{\mathbb{P}^1,p_i})\) are distinct for \(i = 1, 2\), then the image curve \(f(\mathbb{P}^1)\) has only ordinary double points. We can extend this notion to apply to a map \(f : \mathbb{P} \to S\) over an algebraically closed field, where \(\mathbb{P}\) has potentially multiple components. We say that \(f\) has only ordinary double points if \(\mathbb{P} \to f(\mathbb{P})\) is unramified, and if the map from the normalization \(\prod \mathbb{P}^1 \to S\) satisfies the property that the preimage of any point of \(S\) consists of at most two points and for all points with two inverse images \(p_1\) and \(p_2\), the subspaces \(df(T_{\mathbb{P}^i,p_i})\) are distinct for \(i = 1, 2\).

**Definition 2.9.** Let \(\mathcal{M}_{0,n}^{\text{odp}}(S,D)\) represent those \((f, p_*)\) in \(\mathcal{M}_{0,n}(S,D)\) such that \(f\) is unramified, and over every geometric point of the base, \(f(\mathbb{P}^1)\) has only ordinary double points. Dropping the assumption that the genus 0 curve \(\mathbb{P}\) be smooth, let \(\mathcal{M}_{0,n}^{\text{odp}}(S,D)\) represent those \(f\) in \(\mathcal{M}_{0,n}(S,D)\) such that \(f : \mathbb{P} \to f(\mathbb{P})\) is unramified, and has only ordinary double points over every geometric point of the base.

**Definition 2.10.** Let \(f : \mathbb{P} \to S\) be a geometric point of \(\mathcal{M}^{\text{bir}}_{0,n}(S,D)\). We say that \(f\) has a cusp or worse if there is point \(p \in \mathbb{P}\) such that \(T(f) : T_p(\mathbb{P}) \to T_{f(p)}(S)\) is the zero map; we say that \(f\) has a cusp if in addition \(f^{-1}(f(p)) = \{p\}\). We say a geometric point \(f\) of \(\mathcal{M}^{\text{un}}_{0,n}(S,D)\) has a tacnode or worse if there are points \(p \neq q \in \mathbb{P}\) such that \(f(p) = f(q)\) and \(T(f)(T_p(\mathbb{P})) = T(f)(T_q(S))\); if in addition \(f^{-1}(f(p)) = \{p, q\}\) we say \(f\) has a tacnode. We say that a geometric point \(f \in \mathcal{M}^{\text{un}}_{0,n}(S,D)\) has an \(m\)-fold point or worse if there are \(m\) points in \(\mathbb{P}, p_1, \ldots, p_m\), with \(f(p_i) = f(p_j)\) for all \(i, j\); we say that \(f\) has an \(m\)-fold point if in addition \(f^{-1}(f(p_i)) = \{p_i, \ldots, p_m\}\) for \(m = 3, 2\), we use the term triple point instead of \(m\)-fold point.

A cusp at \(p \in \mathbb{P}\) is ordinary if \(\dim_{f(p)}(\Omega_{\mathbb{P},p}/f^*\Omega_{S,f(p)}) = 1\) and \(f^{-1}(f(p)) = \{p\}\). An \(m\)-fold point is ordinary if the images \(df(T_{p_i}\mathbb{P})\) in of the tangent spaces in \(T_{S,f(p_i)}\) are pairwise distinct and \(f^{-1}(f(p_i)) = \{p_1, \ldots, p_m\}\). A tacnode is ordinary if \(f^{-1}(f(p)) = \{p, q\}\), and there are generators \(x, y\) for the maximal ideal in the
Lemma 2.12. Let ∆ be the completion of \( O \) to \( M_0^{unr} \). Assume that \( \pi : X_0 := M_0(1)(S, D) \rightarrow M_0(S, D) \) from the universal curve. Evaluation on the \((n+1)\)st point gives a map \( f := ev_{n+1} : M_{0, n+1}(S, D) \rightarrow S \). As usual, we write \( \pi : X_0 := M_{0, 1}(S, D) \rightarrow M_0(S, D) \) in case \( n = 0 \), and let \( \pi_{unr} : X_0^{unr} \rightarrow M_0^{unr}(S, D) \) be the restriction over \( M_0^{unr}(S, D) \).

Let \( \Delta S \subset S \times_k S \), \( \Delta_{X_0^{unr}} \subset X_0^{unr} \times M_0^{unr}(S, D) \) be the diagonals. Define the locally closed subset \( \mathcal{D}^{unr} \) of \( X_0^{unr} \times M_0^{unr}(S, D) \) by \( \mathcal{D}^{unr} := (f \times f)^{-1}(\Delta S) \setminus \Delta_{X_0^{unr}} \). Counter example, there is a point \((0, 0)\) such that \( \pi_{unr} \) is not closed in \( \mathcal{D}^{unr} \).

The projection \( \text{Spec} \mathcal{O} \rightarrow M_0^{unr}(S, D) \) gives us an unramified \( \mathcal{O} \)-morphism \( F_0 : \mathbb{P}_k \rightarrow S_0 \), with two sections \( s_1, s_2 : \text{Spec} \mathcal{O} \rightarrow \mathbb{P}_k \) such that \( f_0(s_1(\eta)) = f_0(s_2(\eta)) \). Since \( S \) is separated over \( k \), we have \( f_0 \circ s_1 = f_0 \circ s_2 \). Let \( q \in S(\mathcal{O}) \) be the \( \mathcal{O} \)-point \( f_0 \circ s_1 = f_0 \circ s_2 \). Consider the completion \( \mathcal{O}_{\Delta, q} \) of \( \mathcal{O}_{S_0} \) along \( q \). Since \( \mathcal{O} \) is complete and local, and \( S \) is smooth over \( k \), we can write the maximal ideal of \( \mathcal{O}_{\Delta, q} \) as \((y_1, y_2)\) and we have \( \mathcal{O}_{\Delta, q} = \mathcal{O}[[y_1, y_2]] := \lim_{\longrightarrow} \mathcal{O}[[y_1, y_2]/((t, y_1, y_2))^n] \).

Similarly, we have the k-point \( p = (a) = s_2(a) = \mathbb{P}^1(k) \), and we may assume that \( p = 0 \in \mathbb{P}^1 := \mathbb{P}^1 \setminus \{(0 : 1)\} \), with standard coordinate \( x := X_1/X_0 \). We pass to the completion of \( \mathcal{O}[x] \) at \((a, 0)\), which we identify with \( \mathcal{O}[[x]] \).

Thus on \( \text{Spec} \mathcal{O}[[x]] \), \( f_0 \) is given by two element \( f_i := f_0(y_i) \in \mathcal{O}[[x]] \), with \( f_i \equiv 0 \) mod \( x \), \( i = 1, 2 \). Similarly, the sections \( s_i \) are given by \( s_i \in \mathcal{O} \), with \( s_i \equiv 0 \) mod \( t \), and \( f_i(s_1) = f_i(s_2), i = 1, 2 \). Translating on \( \text{Spec} \mathcal{O}[[x]] \) by \( s_1 \), we may assume that \( s_1 = 0 \). Since \( s_1 = s_2 \) is not zero, and we may write \( s_2 = a_n t^n \mod t^{n+1} \) with \( a_n \in k \neq 0 \).

Since \( f_i \equiv 0 \mod x \), we may write \( f_i(x) = xh_i(x), i = 1, 2 \) for some \( h_i \in \mathcal{O}[[x]] \).

Since \( f_1(s_2) = 0 \), and \( s_2 \neq 0 \), we also have \( h_1(s_2) = 0 \), so \( h_1 \) is divisible by \( x - s_2 \) and \( f_1 \) is thus divisible by \( x^2 - x s_2 \). But then the image \( f_i(x) \in k[[x]] \) under the quotient map \( \mathcal{O}[[x]] \rightarrow k[[x]] \) is divisible by \( x^2 \), so

\[
\frac{d f_i}{dx}(0) = 0
\]

and thus \( f_k : \mathbb{P}^1 \rightarrow S \) is ramified at \((1 : 0)\), contrary to our assumption that \( \text{Spec} \mathcal{O} \) maps to \( M_0^{unr}(S, D) \).

We now return to our usual setting over the field \( k \).
Proof. We have the maps
\[ f \times f : X^{unr}_0 \times_{M^{unr}_0(S,D)} X^{unr}_0 \rightarrow S \times S \]
\[ f \times f \times f : X^{unr}_0 \times_{M^{unr}_0(S,D)} X^{unr}_0 \times_{M^{unr}_0(S,D)} X^{unr}_0 \rightarrow S \times S \times S \]
Let \( \Delta(f) \) denote the inverse image of \( \Delta_S \) under \( f \times f \). Then \( \Delta(f) \) is closed and contains \( \Delta_{X^{unr}_0} \); by Lemma 2.11 we have the closed subset \( D^{unr} := \Delta(f) \setminus \Delta_{X^{unr}_0} \) of \( X^{unr}_0 \times_{M^{unr}_0(S,D)} X^{unr}_0 \). Clearly \( D^{unr} \) parametrizes unramified maps \( f : \mathbb{P} \rightarrow S \) in \( M_0(S,D) \) together with a pair of points \( p \neq q \in \mathbb{P} \) such that \( f(p) = f(q) \).

Similarly, for \( m \geq 3 \) an integer, we have the \( m \)-fold fiber product \( (X^{unr}_0)^{\times m} \) and the morphism
\[ f \times \cdots \times f : (X^{unr}_0)^{\times m} \rightarrow S^m. \]
Let \( \Delta^{(m)} \subset S^m \) denote the (small) diagonal and let \( \Delta^{(m)}(f) := (f \times \cdots \times f)^{-1}(\Delta^{(m)}_S) \), a closed subset of \( (X^{unr}_0)^{\times m} \). For \( 1 \leq i < j \leq m \), let \( \Delta_{X^{unr}_0,i,j} \subset (X^{unr}_0)^{\times m} \) denote the \( i,j \)-diagonal. It follows from repeated applications of Lemma 2.11 that
\[ (m - \text{fold})^{unr} := \Delta^{(m)}(f) \setminus \bigcup_{1 \leq i < j \leq m} \Delta_{X^{unr}_0,i,j} \]
is a closed subset of \( (X^{unr}_0)^{\times m} \).

Since the projection
\[ \pi^{(m)}_{unr} : (X^{unr}_0)^{\times m} \rightarrow M^{unr}_0(S,D) \]
is proper, we have the closed subset
\[ D^{unr}_{m-fold} := \pi^{(m)}_{unr}((m - \text{fold})^{unr}) \]
of \( M^{unr}_0(S,D) \), parametrizing those \( f \in M^{unr}_0(S,D) \) having an \( m \)-fold point or worse.

For the case of a tacnode, let \( p : \mathbb{P}(T_S) \rightarrow S \) be the projectivization of the tangent bundle of \( S \) and let
\[ T_2 : D^{unr} \rightarrow \mathbb{P}(T_S) \times S \mathbb{P}(T_S) \]
be the map sending \( (f : \mathbb{P} \rightarrow S, p, q) \) to the pair of lines \( \{df(T_{\mathbb{P}, p}) \}, \{df(T_{\mathbb{P}, q}) \} \) viewed as a pair of points in \( \mathbb{P}(T_S) \). Let \( D^{unr}_{\mathbb{P}(T_S)} \subset D^{unr} = T_2^{-1}(\Delta_{\mathbb{P}(T_S)}) \), a closed subset of \( D^{unr} \), hence also closed in \( (X^{unr}_0)^{\times M^{unr}_0(S,D)^2} \). Letting
\[ \pi^{(2)}_{unr} : (X^{unr}_0)^{\times M^{unr}_0(S,D)^2} \rightarrow M^{unr}_0(S,D) \]
be the projection, we see as above that
\[ D^{unr}_{\mathbb{P}(T_S)} := \pi^{(2)}_{unr}(D^{unr}_{\mathbb{P}(T_S)} \setminus \Delta^{unr}) \]
is a closed subset of \( M^{unr}_0(S,D) \) that parametrizes maps \( f \in M^{unr}_0(S,D) \) with a tacnode or worse.

Finally, for the case of a cusp, we consider the universal curve \( \pi : X^{bir}_{0,1} \rightarrow M^{bir}_{0,1}(S,D) \) with section \( s : M^{bir}_{0,1}(S,D) \rightarrow X^{bir}_{0,1} \). Consider the universal map over \( M^{bir}_{0,1}(S,D) \),
\[ F : X^{bir}_{0,1} \rightarrow S \times_{k} M^{bir}_{0,1}(S,D) \]
and let $R \subset X^\text{bir}_{0,1}$ be the support of the cokernel of the map

$$dF : F^*\Omega_S \to \Omega_{X^\text{bir}_{0,1}/M^\text{bir}_{0,1}(S,D)}$$

Let $\tilde{R}_{0,1} := \pi(R \cap s(M_{0,1}(S,D)))$, a closed subset of $M^\text{bir}_{0,1}(S,D)$, and let $\tilde{R}$ be the image of $\tilde{R}_{0,1}$ under the projection $M^\text{bir}_{0,1}(S,D) \to M^\text{bir}_0(S,D)$. Noting that $\pi^\text{bir} : M^\text{bir}_{0,1}(S,D) \to M^\text{bir}_0(S,D)$ is the universal curve over $M^\text{bir}_0(S,D)$, so $\pi^\text{bir}$ is proper, and thus $\tilde{R}$ is closed in $M^\text{bir}_0(S,D)$. □

Relying on Lemma 2.12 we make the following definition.

**Definition 2.13.** We let $Z_{\text{cusp}} \subset M^\text{bir}_0(S,D)$ be the closed subset of stable maps with a cusp or worse. We let $Z_{\text{tac}} \subset M^\text{unr}_0(S,D)$, resp. $Z_{\text{trip}} \subset M^\text{unr}_0(S,D)$ be the closed set of stable maps with a tacnode or worse, resp. a triple point or worse.

**Lemma 2.14.** We have open subschemes

$$M^\text{opd}_{0,n}(S,D) \subset M^\text{unr}_{0,n}(S,D) \subset M_{0,n}(S,D)$$

**Proof.** Forgetting the last point defines a proper morphism

$$\pi : X_{0,n} := M_{0,n+1}(S,D) \to M_{0,n}(S,D)$$

from the universal curve. Evaluation on the $(n+1)$st point gives a map $f := ev_{n+1} : M_{0,n+1}(S,D) \to S$, which in turn induces a map of coherent sheaves $df : f^*\Omega_S \to \Omega_{M_{0,n+1}(S,D)/M_{0,n}(S,D)}$. The cokernel of $df$ has closed support on $X_{0,n}$, whence closed image under $\pi$. The open complement in $M_{0,n}(S,D)$ of this image is $M^\text{unr}_{0,n}(S,D)$ by Definition 2.6.

Geometric points of the complement of $M^\text{opd}_{0}(S,D)$ in $M^\text{unr}_{0}(S,D)$ are $f : \mathbb{P}^1 \to S$ with either three distinct points $p_1, p_2, p_3$ such that $f(p_1) = f(p_2) = f(p_3)$ or two distinct points $p_1, p_2$ with $f(p_1) = f(p_2)$ and $f_*(T_{\mathbb{P}^1}) = f_*(T_{\mathbb{P}^1})$. These are closed conditions as in Lemma 2.12. □

2.4. Some geometry of moduli stacks of birational and/or unramified maps. Here are two fundamental results.

**Theorem 2.15** (Göttsche-Pandharipande [13 Theorem 4.1]). Suppose $k$ is algebraically closed and of characteristic zero, $n = d - 1 \geq 1$ and $S$ is a general del Pezzo of degree $d_S$. Let $N_{D,S}$ be the Gromov-Witten invariant counting the number of rational curves in the curve class $D$ passing through $n$ general points of $S$ and suppose $N_{D,S} > 0$. Then $N_{D,S}$ is equal to the number of integral rational curves $C$ in the curve class $D$ passing through general points $p_1, \ldots, p_n$ of $S$. Moreover, for each such $C$, the associated morphism $f : \mathbb{P}^1 \to S$ with image $C$ is unramified.

This result can be interpreted as follows: let

$$ev : M_{0,n}(S,D) \to S^n$$

$$f : \mathbb{P} \to S, (p_1, \ldots, p_n) \mapsto (f(p_1), \ldots, f(p_n)) \in S^n$$

denote the evaluation map, where $\mathbb{P}$ is a semi-stable genus 0 curve with $n$ distinct points $p_1, \ldots, p_n$ and $f : (\mathbb{P}, p_1, \ldots, p_n) \to S$ is a stable map. For $S$ general, if $N_{D,S} > 0$, then $ev : M_{0,n}(S,D) \to S^n$ is surjective and étale over a dense open subset $U$ of $S^n$, moreover, for each $p_* := (p_1, \ldots, p_n) \in U$, we have $ev^{-1}(p_*) \subset M^\text{unr}_{0,n}(S,D)$.
Theorem 2.16 (Testa [50]). Suppose \( k \) is algebraically closed and of characteristic zero and that \( d_S \geq 2 \). Then \( \overline{M}_0^{\text{bir}}(S, D) \) is empty or is irreducible of dimension \( d - 1 \).

The analogous result in positive characteristic seems to be unknown. For \( S \) a toric del Pezzo in characteristic not 2, Theorem A.11 and Remarks A.13, A.14 characterize the \((S, D)\) so that \( M_0^{\text{bir}}(S, D) \neq \emptyset \).

Recall that \( \mathcal{N}_f \) denotes the normal sheaf defined by (2.1).

Lemma 2.17. Suppose that \( f \) is a geometric point of \( M_0(S, D) \) such that \( f : \mathbb{P}^1 \to S \) is a birational to the image curve \( f(\mathbb{P}^1) \) and \( H^1(\mathbb{P}^1, \mathcal{N}_f) = 0 \), for instance, \( f \) a geometric point of \( M_0^{\text{bir}}(S, D) \) or \( f \) unramified. Then \( M_0(S, D) \) is a smooth scheme over \( k \) of dimension \( d - 1 \) at \( f \).

Proof. Note that if \( f \) is unramified then \( \mathcal{N}_f \cong \mathcal{O}(d - 2) \) (Remark 2.2) and \( H^1(\mathbb{P}^1, \mathcal{N}_f) = 0 \), so we may assume \( f \) to be birational and \( H^1(\mathbb{P}^1, \mathcal{N}_f') = 0 \).

Since \( f \) is birational, \( f \) has no automorphisms, so \( M_0(S, D) \) is a \( k \)-scheme in a neighborhood of \( f \). Since \( H^1(\mathbb{P}^1, \mathcal{N}_f') = 0 \) and the morphism \( f : \mathbb{P}^1 \to S \) has no automorphisms, then by standard deformation theory, \( M_0(S, D) \) is smooth over \( k \) at \( f \), and the tangent space at \( f \) is isomorphic to \( H^0(\mathbb{P}^1, \mathcal{N}_f') \)

\[
H^0(\mathbb{P}^1, \mathcal{N}_f') \cong T_f M_0(S, D).
\]

By Remark 2.2, \( \mathcal{N}_f \cong \mathcal{O}_\mathbb{P}^1(m) \oplus \mathcal{N}_f^{\text{tor}} \) with \( m = d - 2 - \dim_F H^0(\mathbb{P}^1, \mathcal{N}_f^{\text{tor}}) \), where \( F \) denotes the field of definition of \( F \). Thus \( H^0(\mathbb{P}^1, \mathcal{N}_f) \) has dimension \( d - 1 \) over \( F \), which also proves that \( M_0(S, D) \) is a smooth scheme over \( k \) of dimension \( d - 1 \) at \( f \) as claimed.

Remark 2.18. More generally, consider a geometric point \((f, p_1, \ldots, p_n)\) of \( M_{0,n}(S, D) \) where \( f : \mathbb{P}^1 \to S \) is a stable map and \( p_1, \ldots, p_n \) are marked points on the domain curve \( \mathbb{P}^1 \). There is a canonical isomorphism

\[
T_f M_{0,n}(S, D) \cong H^1(\mathbb{P}^1, T\mathbb{P}^1(- \sum_i p_i) \xrightarrow{df} f^*TS)
\]

identifying the tangent space \( T_f M_{0,n}(S, D) \) with the hypercohomology of \( \mathbb{P}^1 \) with coefficients in the two-term complex \( T\mathbb{P}^1(- \sum_i p_i) \xrightarrow{Tf} f^*TS \), where \( T\mathbb{P}^1(- \sum_i p_i) \) is the sheaf of those tangent vector fields vanishing at the \( p_i \). See [6, p. 175].

When \( f \) is birational, the map \( df : T\mathbb{P}^1(- \sum_i p_i) \to f^*TS \) is injective, and there is a canonical quasi-isomorphism between \( T\mathbb{P}^1(- \sum_i p_i) \to f^*TS \) and the sheaf \( \mathcal{N}_{f,p} \) defined by

\[
0 \to T\mathbb{P}^1(- \sum_i p_i) \to f^*TS \to \mathcal{N}_{f,p} \to 0.
\]

Lemma 2.19. Suppose that \( f \) is a geometric point of \( M_{0,n}(S, D) \) with field of definition \( F \) such that \( f : \mathbb{P}^1 \to S \) is birational and \( H^1(\mathbb{P}^1, \mathcal{N}_{f,p}) = 0 \). Then \( M_{0,n}(S, D) \) is smooth at \( f \) of dimension \( d - 1 + n \) and there is a canonical isomorphism

\[
T_f M_0(S, D) \cong H^0(\mathbb{P}^1, \mathcal{N}_{f,p}).
\]

Proof. By Remark 2.18, there is a canonical quasi-isomorphism between \( \mathcal{N}_{f,p} \) and \( T\mathbb{P}^1(- \sum_i p_i) \to f^*TS \). Since \( H^1(\mathbb{P}^1, \mathcal{N}_{f,p}) = 0 \), it follows from [6, p. 175] and standard deformation theory that \( M_{0,n}(S, D) \) is smooth at \( f \) and \( T_f M_0(S, D) \cong H^0(\mathbb{P}^1, \mathcal{N}_{f,p}) \). Thus the dimension of \( M_{0,n}(S, D) \) at \( f \) is \( \dim_F H^0(\mathbb{P}^1, \mathcal{N}_{f,p}) \). We
have that \( \dim H^0(\mathbb{P}^1, \mathcal{N}_{f,p}) = d - 1 + n \) by the calculation \( \mathcal{N}_{f,p} \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{N}_{tor}^{i,j} \) with \( m = n + d - 2 - \dim_F H^0(\mathbb{P}^1, \mathcal{N}_{tor}^{i,j}) \) similarly to the above.

\[\square\]

**Proposition 2.20.** Let \( f : (\mathbb{P}, p_1, \ldots, p_n) \to S \) be a point of \( \tilde{M}_{0,n}(S, D) \) satisfying the following conditions.

1. \( \mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 \), with \( \mathbb{P}_i \cong \mathbb{P}^1 \) and \( \mathbb{P}_1 \cap \mathbb{P}_2 = \{p\} \).
2. \( f \) is unramified and \( f|_{\mathbb{P}_2} \) is transversal to \( f|_{\mathbb{P}_2} \) at \( p \).

Then \( \tilde{M}_{0,n}(S, D) \) is smooth at \( f \) and has dimension \( d - 1 + n \).

**Proof.** Let \( \mathcal{C} \) denote the mapping cone of \( f^*\Omega_S \to \Omega_{\mathbb{P}}(\sum_{i=1}^n p_i) \), or equivalently \( \mathcal{C} \) is the two-term complex

\[
\mathcal{C} = f^*\Omega_S \to \Omega_{\mathbb{P}}(\sum_{i=1}^n p_i)
\]

By [6, p. 175] the tangent space of \( \tilde{M}_{0,n}(S, D) \) at \( f \) is \( \text{Ext}^1_\mathbb{P}(\mathcal{C}, \mathcal{O}_\mathbb{P}) \) and the obstructions are \( \text{Ext}^2_\mathbb{P}(\mathcal{C}, \mathcal{O}_\mathbb{P}) \). It follows from stability [6, p. 175] that

\[
\text{Ext}^0_\mathbb{P}(\mathcal{C}, \mathcal{O}_\mathbb{P}) = 0.
\]

We show that

\[
\dim \text{Ext}^1_\mathbb{P}(\mathcal{C}, \mathcal{O}_\mathbb{P}) = d - 1 + n, \quad \dim \text{Ext}^2_\mathbb{P}(\mathcal{C}, \mathcal{O}_\mathbb{P}) = 0.
\]

By definition of \( \mathcal{C} \), there is long exact sequence

\[
0 \to \text{Ext}^0_\mathbb{P}(\mathcal{C}, \mathcal{O}_\mathbb{P}) \to \text{Ext}^0_\mathbb{P}(\Omega_{\mathbb{P}}(\sum_{i=1}^n p_i), \mathcal{O}_\mathbb{P}) \to \text{Ext}^0_\mathbb{P}(f^*\Omega_S, \mathcal{O}_\mathbb{P})
\]

\[
\to \text{Ext}^1_\mathbb{P}(\mathcal{C}, \mathcal{O}_\mathbb{P}) \to \text{Ext}^1_\mathbb{P}(\Omega_{\mathbb{P}}(\sum_{i=1}^n p_i), \mathcal{O}_\mathbb{P}) \to \text{Ext}^1_\mathbb{P}(f^*\Omega_S, \mathcal{O}_\mathbb{P})
\]

\[
\to \text{Ext}^2_\mathbb{P}(\mathcal{C}, \mathcal{O}_\mathbb{P}) \to \text{Ext}^2_\mathbb{P}(\Omega_{\mathbb{P}}(\sum_{i=1}^n p_i), \mathcal{O}_\mathbb{P}) \to \text{Ext}^2_\mathbb{P}(f^*\Omega_S, \mathcal{O}_\mathbb{P}) \to \ldots
\]

We show

\[
\dim \text{Ext}^1_\mathbb{P}(f^*\Omega_S, \mathcal{O}_\mathbb{P}) = 0, \quad \dim \text{Ext}^0_\mathbb{P}(f^*\Omega_S, \mathcal{O}_\mathbb{P}) = d + 2.
\]

Indeed, since \( f^*\Omega_S \) is locally free,

\[
\text{Ext}^1_\mathbb{P}(f^*\Omega_S, \mathcal{O}_\mathbb{P}) \cong H^i(\mathbb{P}, f^*T_S)
\]

for all \( i \).

Let \( i_j : \mathbb{P}_j \to \mathbb{P} \) denote the inclusion and let \( f_j = f \circ i_j \). Let \( D_j = (f_j)_*([\mathbb{P}^1]) \) and let \( d_j = -K_S \cdot D_j \). There is a short exact sequence

\[
0 \to f^*T_S \to (i_1)_*i_1^*f^*T_S \oplus (i_2)_*i_2^*f^*T_S \to (i_p)_*(i_p)^*f^*T_S \to 0.
\]

Since \( i_j \) is affine,

\[
H^k((i_1)_*i_1^*f^*T_S \oplus (i_2)_*i_2^*f^*T_S) = H^k(\mathbb{P}_1, f_1^*T_S) \oplus H^k(\mathbb{P}_2, f_2^*T_S), \quad k = 0, 1.
\]

So, by the long exact sequence in cohomology, it suffices to show that

\[
H^i(\mathbb{P}_1, f_1^*T_S) = 0, \quad \dim H^0(\mathbb{P}_i, f_i^*T_S) = d_i + 2,
\]

and that the map

\[
H^0(\mathbb{P}_1, f_1^*T_S) \oplus H^0(\mathbb{P}_2, f_2^*T_S) \to T_{S,f(p)}
\]
is surjective. To prove (2.4), consider the exact sequence
\[ 0 \to T_{P_1} \to f^*_i T_S \to \mathcal{N}_{f} \to 0. \]
Observe that \( \mathcal{N}_{f} \cong \mathcal{O}(d_i - 2) \). Moreover, since S is del-Pezzo, \( d_i = -K_S \cdot D_i > 0 \).
Thus \( \dim H^0(\mathcal{N}_{f}) = d_i - 1 \), \( \dim H^1(\mathcal{N}_{f}) = 0 \).
Since \( \dim H^0(T_{P_1}) = 3 \), \( \dim H^1(T_{P_1}) = 0 \), equation (2.4) follows.

To prove the surjectivity of (2.5), consider the commutative diagram
\[
\begin{array}{ccc}
H^0(T_{P_1}) & \longrightarrow & T_{P_1, p} \\
| & & | \\
\downarrow{df_i} & & \downarrow{(df_i)_p} \\
H^0(f^*_i T_S) & \longrightarrow & T_{S, f_i(p)}.
\end{array}
\]
Since the upper horizontal arrow is surjective, the image of the bottom horizontal arrow contains \((df_i)_p(T_{P_1, p})\). Since \( f_1 \) and \( f_2 \) are transversal at \( p \), the surjectivity of (2.5) follows.

Next, we calculate \( \text{Ext}^k(\Omega_{\mathbb{P}}(\bigoplus_{i=1}^n p_i), \mathcal{O}_{\mathbb{P}}) \). Indeed, since the dualizing sheaf \( \varpi \) of \( \mathbb{P} \) is a line bundle, Serre duality gives
\[
\text{Ext}^k(\Omega_{\mathbb{P}}(\bigoplus_{i=1}^n p_i), \mathcal{O}_{\mathbb{P}}) \cong \text{Ext}^k(\Omega_{\mathbb{P}}(\bigoplus_{i=1}^n p_i) \otimes \varpi, \varpi) \cong \text{Ext}^{1-k}(\mathcal{O}_{\mathbb{P}}, \Omega_{\mathbb{P}}(\bigoplus_{i=1}^n p_i) \otimes \varpi).
\]
This shows that \( \text{Ext}^2(\Omega_{\mathbb{P}}(\bigoplus_{i=1}^n p_i), \mathcal{O}_{\mathbb{P}}) = 0 \). It follows from the exact sequence (2.2) and (2.3) that \( \text{Ext}^2(C, \mathcal{O}_{\mathbb{P}}) = 0 \) and \( \mathcal{M}_{0,n}(S, D) \) is smooth at \( f \) as claimed.

On the other hand,
\[
\chi(\Omega_{\mathbb{P}}(\bigoplus_{i=1}^n p_i) \otimes \varpi) = \sum_{k=0}^1 (-1)^k \dim H^k(\Omega_{\mathbb{P}}(\bigoplus_{i=1}^n p_i) \otimes \varpi)
\]
is constant in flat families.

We smooth \( \mathbb{P} \) to a flat family \( \bar{\mathbb{P}} \to \text{Spec } k[[t]] \) with smooth generic fiber \( \mathbb{P}_{k((t))} \) and with sections \( p_i \) reducing to \( p_i \) over Spec \( k \), \( i = 1, \ldots, n \). More precisely, \( \bar{\mathbb{P}} \) is projective over Spec \( k[[t]] \), \( \bar{\mathbb{P}} \setminus \{ p_i \} \to \text{Spec } k[[t]] \) is smooth, and an open neighborhood of \( p \) in \( \bar{\mathbb{P}} \) is isomorphic to Spec \( k[[t]][x, y]/(xy - t) \) as \( k[[t]] \)-scheme.

An easy computation shows that \( \Omega_{\bar{\mathbb{P}}/k[[t]]} \) is flat over \( k[[t]] \); since \( \bar{\mathbb{P}} \to \text{Spec } k[[t]] \) is an snc morphism, the relative dualizing sheaf \( \varpi_{\bar{\mathbb{P}}/k[[t]]} \) is an invertible sheaf, hence is also flat over \( k[[t]] \). Since the sheaf Euler characteristic is locally constant in flat,
proper families, we have
\[
\sum_{k=0}^{1} (-1)^k \dim H^k(\Omega \sum_{i=1}^{n} p_i \otimes \varpi)
\]
\[= \sum_{k=0}^{1} (-1)^k \dim H^k(\mathbb{P}_1^{k((t))}, \Omega^{k((t))}/k((t))) (\sum_{i=1}^{n} p_i, k((t))) \otimes \Omega^{k((t))}/k((t)) \]
\[= \sum_{k=0}^{1} (-1)^k \dim H^k(\mathbb{P}_1^{k((t))}, \mathcal{O}^{k((t))}/k((t)))(n - 4) = n - 3.
\]
It follows from the exact sequence (2.2) and (2.3) that
\[
\dim \text{Ext}^1_{\mathbb{P}_1}(\mathcal{C}, \mathcal{O}) = d - 1 + n.
\]
as claimed.

\[\square\]

**Remark 2.21.** Let \((f, p_1, \ldots, p_n)\) be a geometric point of \(M_{0,n}(S, D)\) such that \(f : \mathbb{P}_1 \to S\) is birational and \(H^1(\mathbb{P}_1, N_{f,p}) = 0\). Let \(F\) be the field of definition of \((f, p_1, \ldots, p_n)\). Suppose additionally that \(d_f \otimes F : T_{\mathbb{P}_1} \otimes F \to T_{S,q_i} \otimes F\) is injective for all \(i\). For example, if \(f\) could be a geometric point of \(M_{0,n}^{unr}(S, D)\).

Then there is an additional description of the kernel and cokernel of \(d \text{ev}\) in terms of the exact sequence
\[
0 \to N_f(- \sum_{i} p_i) \to N_f \to \bigoplus_{i} (f^*T_{S,q_i} \otimes F)/d_f(T_{\mathbb{P}_1} \otimes F) \to 0,
\]
where \(q_i = f(p_i)\). We give this description now. Applying the snake lemma to the map of short exact sequences
\[
\begin{align*}
(2.6) & \quad 0 \to T_{\mathbb{P}_1}(- \sum_{i} p_i) \xrightarrow{1} f^*T_S \xrightarrow{N_{f,p}} 0 \\
& \quad 0 \to T_{\mathbb{P}_1} \xrightarrow{1} f^*T_S \xrightarrow{N_f} 0
\end{align*}
\]
defines a canonical isomorphism
\[
\ker(N_{f,p} \to N_{f}) \xrightarrow{\cong} \text{coker}(T_{\mathbb{P}_1}(- \sum_{i} p_i) \to T_{\mathbb{P}_1}) \cong \bigoplus_{i} \mathcal{O}_{p_i},
\]
where \(\mathcal{O}_{p_i} := (p_i)\). The pushforward of the structure sheaf of the points \(p_i\). We also deduce from (2.6) that \(N_{f,p} \to N_{f}\) is surjective, giving the short exact sequence
\[
0 \to \bigoplus_{i} \mathcal{O}_{p_i} \to N_{f,p} \to N_{f} \to 0.
\]
This gives rise to the map of short exact sequences
\[
\begin{align*}
0 \to H^0(\mathbb{P}_1, \mathcal{O}) & \to H^0(N_{f,p}) \xrightarrow{d \text{ev}} H^0(N_f) \to 0 \\
0 \to \bigoplus_{i} df(T_{\mathbb{P}_1,p_i} \otimes F) & \to \bigoplus_{i} (f^*T_{S,q_i} \otimes F) \to \bigoplus_{i} (f^*T_{S,q_i} \otimes F)/d_f(T_{\mathbb{P}_1,p_i} \otimes F) \to 0.
\end{align*}
\]
As the left vertical map is an isomorphism, the snake lemma gives canonical isomorphisms
\[ \ker \text{dev}_f \cong \ker(H^0(N_f) \to \bigoplus_i (f^* T_{S,q_i} \otimes F)/df(T_{F,p_i} \otimes F)) \cong H^0(N_f(-\sum_i p_i)) \]
and
\[ \coker \text{dev}_f \cong \coker(H^0(N_f) \to \bigoplus_i (f^* T_{S,q_i} \otimes F)/df(T_{F,p_i} \otimes F)) \cong H^1(N_f(-\sum_i p_i)). \]

**Definition 2.22.** Let \( F \) be an algebraically closed extension field of \( k \). For \( f : \mathbb{P}^1_F \to S_F \) a morphism and \( p \in \mathbb{P}^1(F) \), choose a uniformizing parameter \( t_p \in m_p \subset \mathcal{O}_{\mathbb{P}^1_F}(p) \) and coordinates \( (x,y) \) at \( q = f(p) \). Define the integer \( e_p \geq 0 \) by \( f^*(x,y)_p = (t^e_p) \); we call \( e_p \) the ramification index of \( f \) at \( p \).

**Definition 2.23.** If \( f^*(x,y)_{\mathcal{O}_{\mathbb{P}^1_F}} = (t^e_p) \), then after a linear of coordinates, we may assume that \( f^*(x) = u t^{e_p} \), \( f^*(y) = v t^{e_p+1} \) with \( u, v \in \mathcal{O}_{\mathbb{P}^1_F} \) and \( r > 0 \). Thus \( N_f^{\text{tor}} \otimes \mathcal{O}_{\mathbb{P}^1_F} \cong F^{*r} \), with \( t_p \geq e_p - 1 \), with equality if \( e_p \) is prime to the characteristic. Let \( t(f) = \sum_{p \in \mathbb{P}^1} t_p \). We call \( t(f) \) the torsion index of \( f \). For \( f : \mathbb{P} \to S \) a possibly reducible stable map, we define the torsion index \( t(f) \) to be the sum of torsion indices of the restrictions of \( f \) to each of the irreducible components.

**Remark 2.24.** Remark 2.7 generalizes as follows.

**Lemma 2.25.** Suppose \( k \) is a field of characteristic zero. Let \( V \subset M_0^{\text{bir}}(S,D) \) be an integral closed subscheme and let \( f \) be a geometric generic point of \( V \). Then the composition
\[ T_f V \to T_f M_0^{\text{bir}}(S,D) \cong H^0(\mathbb{P}^1_F, N_f) \to H^0(\mathbb{P}^1_F, N_f / N_f^{\text{tor}}) \]
of the displayed canonical maps is injective. Moreover, either
- \( d - 1 - \dim V \geq t(f) \)
- \( \dim V = 0 \).

**Proof.** We prove the first assertion following the proof of a closely related result by Tyomkin [51, Proposition 2.4].

Since \( M_{0,n}(S,D) \) is a separated Artin stack and \( M_0^{\text{bir}}(S,D) \) is an open subscheme of \( M_{0,n}(S,D) \), there is an étale dominant map \( \phi : V \to V \) and a morphism \( F : \bar{V} \times \mathbb{P}^1 \to \bar{V} \times S \) over \( V \) representing the inclusion \( V \to M_{0,n}(S,D) \). We consider \( f \) as a geometric point of \( V \).

Sending a point \( v \in V \) to the image curve \( F(v, \mathbb{P}^1) \subset v \times S \) defines a morphism \( \tilde{\alpha} : \bar{V} \to |D| \cong \mathbb{P}^N \); if \( F(v, \mathbb{P}^1) = F(v', \mathbb{P}^1) \), then since both \( F(v, -) \) and \( F(v', -) \) are birational maps to \( F(v, \mathbb{P}^1) \), there is a unique isomorphism \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) (defined over \( k(v) \otimes_{k(\tilde{\alpha}(v))} k(v') \)) with \( F(v', -) = F(v, -) \circ \phi \). Thus, since \( \bar{V} \to V \) is étale, \( \tilde{\alpha} \) descends to a morphism \( \alpha : V \to |D| \) and there is a dense open subscheme \( U \) of \( V \) over which the map \( \alpha \) is an isomorphism with an open subscheme of the image.
scheme $\tilde{\alpha}(V) \subset D$. For $v \in V$, let $C_v$ be the image curve $F(\tilde{v}, P^1)$ for $\tilde{v} \in \tilde{V}$ lying over $v$.

For $v = f$ a geometric generic point of $V$, the map $f : P^1 \to C := f(P^1)$ is birational and $C$ has only finitely many singularities. Choose a smooth curve $E$ on $S$ (defined over $k$) such that

1. $H^0(S, O_S(D - E)) = 0$
2. $E$ intersects $C$ transversely

Then (ii) also holds for $C_v$ for all $v$ in a dense open subset of $V$. Letting $N = \deg E : D$, this gives us the morphism $\beta : V_0 \to \Hilb_N(E)$, $\beta(v) = C_v \cap E$ for $V_0 \subset V$ a dense open subscheme. By (ii), we may assume that $\beta(V_0)$ is contained in the open subscheme $\Hilb_N^0(E)$ of $\Hilb_N(E)$ parametrizing reduced closed subschemes of $E$ of length $N$, which is a smooth scheme over $k$.

We claim that after shrinking $V_0$ is necessary, $\beta$ defines isomorphism of $V_0$ with its image in $\Hilb_N^0(E)$. Since the characteristic is zero, it suffices to show that $\beta$ is injective on geometric points of $V_0$. (Indeed, by [10, 8.10.5(i)] it suffices to show that the field extension $k(\beta(\eta_V)) \subset k(\eta_V)$ is an isomorphism, where $\eta_V$ denotes the generic point of $V$ or equivalently the image of $f$. Since $k$ is characteristic 0, this is equivalent to $\Gal(k(\eta_V)/k(\beta(\eta_V))) = 1$.)

Take $v \in V_0$ a geometric point, giving the curve $C_v$ on $S$. We have the exact sheaf sequence

$$0 \to O_S(D - E) \to O_S(D) \xrightarrow{i_E^*} O_E(E : D) \to 0;$$

since $H^0(S, O_S(D - E)) = 0$, $i_E^*$ induces an inclusion of linear systems $i_E^* : |D| \to |E|$. Thus, for $v, v'$ geometric points of $V_0$, if $C_v \cap E = C_{v'} \cap E$ then $C_v = C_{v'}$, and since $\alpha : U \to |D|$ is injective on geometric points and $\beta = i_E^* \circ \alpha$, we see that $\beta(v) = \beta(v')$.

On the other hand, let $W \subset \Hilb_N^0(E)$ be a smooth locally closed subscheme and let $w \in W$ be a geometric point. Then $w$ corresponds to $N$ distinct points $q_1, \ldots, q_N$ of $E$ and $T_w \Hilb_N^0(E)$ is isomorphic to $\bigoplus_{i=1}^N T_{q_i} E$. Taking $W = \beta(V_0)$ and $w = \beta(f)$, the points $q_i, q_j$ are the (transverse) intersection points of $C \cap E$, so at each $p_i$, we have $T_{S,q_i} = T_{C,q_i} \oplus T_{E,q_i}$. Since the points $q_i$ are all smooth points of $C = f(P^1)$, and $f : P^1 \to C$ is birational, we have $T_{C,q_i} \cong T_{P^1, p_i}$, where $p_i = f^{-1}(q_i)$, and the projection $T_{S,q_i} \to T_{E,q_i}$ defines an isomorphism $\pi_i : N_f \otimes k(p_i) \to T_{E,q_i}$. Since $\beta$ is an isomorphism $V_0 \to \beta(V_0) \subset \Hilb_N^0(E)$, sending $T_f(V) = H^0(P^1, N_f) \to \bigoplus_{i=1}^N T_{E,q_i} = T_{\beta(f)\Hilb_N^0(E)}$ via the composition

$$T_f(V) = H^0(P^1, N_f) \xrightarrow{\Res_{p_1, \ldots, p_N}} \bigoplus_{i=1}^N T_{E,q_i}$$

is injective. But as all the points $q_i \in C$ are smooth points, $f$ is unramified at each $q_i$, so this latter map factors through $H^0(P^1, N_f) \to H^0(P^1, N_f/N_f^{\text{cor}})$, so $T_f V \to H^0(P^1, N_f/N_f^{\text{cor}})$ is injective, as claimed.

As $H^0(P^1, N_f/N_f^{\text{cor}}) \cong F^{d-1-t(f)}$ for $d - 1 - t(f) \geq 0$ and is zero if $d - 1 - t(f) < 0$ (see Remark [2.24]), the second assertion follows from the first, which proves the lemma. In the first case, $M_0^\text{bir}(S, D)$ is smooth of $f$ of dimension $d - 1$ by obstruction theory because the obstruction $H^1(P^1, N_f) \cong H^1(P^1, N_f/N_f^{\text{cor}}) = 0$ and $H^0(P^1, N_f) \cong F^{d-1}$. In the second case, $\dim V = 0$ because $\dim V \leq \dim T_f V = 0$.  

$\square$
Remark 2.26. For $k$ of characteristic $p > 0$, the first assertion of Lemma 2.25 is false: Consider the family of maps $f_a : \mathbb{P}^1 \to \mathbb{P}^2$
\[f_a(t_0, t_1) = (t_0^{p^2} + at_0^3, t_0^2, t_0^3), \quad a \in k^1,\]
Fixing an $a$, take the tangent vector corresponding to the morphism $f_{a+i}$ over $k(a)[c]/(c^2)$. Then $f_a$ and $f_{a+i}$ have the same defining equation $y^2 = x^p - x^2$, so the corresponding section of $N_f$ vanishes away from $t = 0$.

Lemma 2.27. Let $k$ be a perfect field, $S$ a del Pezzo surface over $k$ and $D$ an effective Cartier divisor on $S$. Let $n = -\deg K_S \cdot D - 1$. Let $f : (\mathbb{P}^1, p_s) \to S$ be a geometric point of $M_{0,n}(S, D)^{unr}$. Then $M_{0,n}(S, D)$ is a smooth scheme of dimension $2n$ at $f$, and $ev : M_{0,n}(S, D) \to S^n$ is étale at $f$.

Proof. Since $f$ is birational, $f$ is birational by Lemma 2.28, so there are no automorphisms of $f$ and $M_{0,n}(S, D)$ is a scheme near $(f, p_s)$. Since $f : \mathbb{P}^1 \to f(\mathbb{P}^1)$ is unramified, we have $N_{f} \cong O_{\mathbb{P}^1}(n-1)$ (Remark 2.7) and $n \geq 0$, so $H^1(\mathbb{P}^1, N_f) = 0$ and $H^0(\mathbb{P}^1, N_f) \cong F^n$. Lemma 2.19 implies that $M_{0,n}(S, D)$ is smooth of dimension $2n$ at $(f, p_s)$. By Remark 2.7, the kernel and cokernel of $ev$ at $f$ are isomorphic to $H^0(\mathbb{P}^1, N_f (\sum_{i=1}^n p_i))$ and $H^1(\mathbb{P}^1, N_f (\sum_{i=1}^n p_i))$, respectively. Since $N_{f} \cong O_{\mathbb{P}^1}(n-1)$, it follows that $N_{f} (\sum_{i=1}^n p_i) \cong O_{\mathbb{P}^1}(-1)$ so both of these terms are zero. \qed

Lemma 2.28. Assume $d_S = 2$. Then, the anti-canonical map $\pi : S \to \mathbb{P}^2$ is a 2-1 finite morphism with branch divisor a smooth quartic curve.

Proof. This is [26] Theorem III.3.5 and proof]. \qed

Lemma 2.29. Assume $d_S = 2$ and char $k \neq 2, 3$. Let $\pi : S \to \mathbb{P}^2$ be the anti-canonical map as in Lemma 2.28. Let $f : \mathbb{P}^1 \to S$ be birational to its image $C = f(\mathbb{P}^1)$ and have $C \cdot (-K_S) = 2$. Then one of the following holds.

1. $\pi|_C : C \to \pi(C)$ is an isomorphism, $\pi(C)$ is a smooth conic, and $f : \mathbb{P}^1 \to C$ is an isomorphism.

2. $\pi|_C : C \to \pi(C)$ has degree 2 and one of the following holds.

(a) $f$ is unramified and $C$ has a single ordinary double point.

(b) $C$ has a single ordinary cusp and $f$ is ramified at a single point with $t(f) = 1$. Moreover, $\pi(C)$ is a line tangent to the branch curve of $\pi$ at a flex.

Proof. Since $\pi$ has degree 2, it follows that $\pi|_C : C \to \pi(C)$ is either birational or has degree 2. If $\pi : C \to \pi(C)$ is birational, then $\pi(C) \cdot O(1) = C \cdot (-K_S) = 2$, so $\pi(C)$ is a smooth conic. Hence, $f : \mathbb{P}^1 \to C$ and $\pi|_C : C \to \pi(C)$ are both isomorphisms.

If $\pi : C \to \pi(C)$ has degree 2, then
\[2(\pi(C) \cdot O(1)) = \pi_a(C) \cdot O(1) = C \cdot (-K_S) = 2,\]
so $\pi(C)$ is a line $\ell$. Let $E \subset \mathbb{P}^2$ be the branch curve of $\pi$. There are five possible cases:
\[\ell \cdot E = p_1 + p_2 + p_3 + p_4, \quad \ell \cdot E = 2 \cdot p_1 + p_2 + p_3, \quad \ell \cdot E = 3 \cdot p_1 + p_2, \quad \ell \cdot E = 2p_1 + 2p_2, \quad \ell \cdot E = 4 \cdot p.\]
with the \( p_i \) distinct in the first four cases. If \( \ell \cdot E \) were the sum of 4 distinct points, then \( C \) would be a smooth curve of genus 1 contrary to the hypothesis. In the last two cases, \( C \) would be geometrically reducible contrary to the hypothesis. In the second case \( C \) has an ordinary double point, so \( f \) must be unramified. In the third case, \( C \) has an ordinary cusp, and since \( \text{char } k \neq 3 \), it follows that \( f \) is ramified at a single point with \( t(f) = 1 \).

Often we will want to use the following assumption.

**Assumption 2.30.** For every effective Cartier divisor \( D' \) on \( S \), there is a geometric point \( f \) in each irreducible component of \( M_{0}^{\text{bir}}(S, D') \) with \( f \) unramified.

We prove in Appendix \( A \) that Assumption 2.30 holds for twists of toric del Pezzo surfaces in characteristic not 2. See Corollary A.15.

**Lemma 2.31.** If \( \text{char } k = 0 \) and \( d_S \geq 2 \), then Assumption 2.30 holds.

**Proof.** Let \( f \) be a geometric generic point of \( M_{0}^{\text{bir}}(S, D') \). By Theorem 2.16, the scheme \( M_{0}^{\text{bir}}(S, D') \) is irreducible of dimension \( \deg(-K_S \cdot D') - 1 \). Consequently, Lemma 2.25 implies that the map \( H^0(\mathbb{P}^1_F, \mathcal{N}_f) \to H^0(\mathbb{P}^1_F, \mathcal{N}_f') \) is injective. So, \( \mathcal{N}_f' \) is trivial and \( f \) is unramified. \( \square \)

**Proposition 2.32.** Suppose that \( k \) is a field of characteristic \( \neq 2, 3 \). Furthermore, suppose \( d_S \geq 2 \) and Assumption 2.30 holds. Let \( f \in M_{0}^{\text{bir}}(S, D) \) be a geometric generic point. Then \( f \) is in \( M_{0}^{\text{dp}}(S, D) \).

**Proof.** Since the condition to be unramified is open, Assumption 2.30 implies that \( f \) is unramified. By Remark 2.7, we have \( \mathcal{N}_f = \mathcal{O}(d-2) \). Since \( d \geq 1 \), it follows that \( H^1(\mathbb{P}^1, \mathcal{N}_f) = 0 \). Therefore, Lemma 2.19 gives \( \dim_f M_{0}^{\text{bir}}(S, D) = d - 1 \).

Let \( C := f(\mathbb{P}^1) \). Suppose first that \( d \geq 4 \). Since \( \dim_f M_{0}^{\text{bir}}(S, D) = d - 1 \), we may apply Theorem 2.8, which gives the result in this case. Since this result is proven under the assumption of characteristic zero, we give a quick sketch of the proof of the relevant portion of the result. Since \( f \) is unramified, we have \( \mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(d-2) \). We need to show that any point \( q \) of \( C \) has at most two preimages, and moreover, if two points of \( \mathbb{P}^1 \) have the same image under \( f \), the images of their tangent spaces are distinct. Suppose first that there are three distinct points \( p_1, p_2, p_3 \in \mathbb{P}^1 \) with \( f(p_i) = q \) for \( i = 1, 2, 3 \) for the sake of contradiction. Identify \( H^0(\mathbb{P}^1, \mathcal{N}_f) \) with \( T_f M_{0}^{\text{bir}}(S, D) \), and consider the first order deformation of \( f \) corresponding to \( s \in H^0(\mathbb{P}^1, \mathcal{N}_f) \). Since \( f \) is birational, there is an open neighborhood of \( q \) such that all other points of the neighborhood have at most one preimage under \( f \). Since \( f \) is a geometric generic point, the first order deformation must retain the property that there are three points mapping to one. Thus if \( s(p_1) = s(p_2) = 0 \), then \( s(p_3) = 0 \) as well. On the other hand, since \( d - 2 \geq 2 \) and \( \mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(d-2) \), we may find an \( s \in H^0(\mathbb{P}^1, \mathcal{N}_f) \) with \( s(p_1) = s(p_2) = 0 \) but \( s(p_3) \neq 0 \), which yields the desired contradiction.

We are now reduced to eliminating the possibility that we have points \( p_1, p_2 \in \mathbb{P}^1 \) with \( f(p_1) = q = f(p_2) \) and \( df(T_{\mathbb{P}^1, p_1}) = df(T_{\mathbb{P}^1, p_2}) \). Suppose that \( d \geq 5 \). Since \( f \) is unramified, \( \mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(d-2) \), and since \( d - 2 \geq 3 \), we can find a section \( s \in H^0(\mathbb{P}^1, \mathcal{N}_f) \) with \( s \) having a 2nd order zero at \( p_1 \), and a zero of order one at \( p_2 \). For the associated deformation \( f_u \) of \( f \) defined over \( F[[u]] \), we have, to first order, \( f_u(p_1) = f_u(p_2) \), \( df_u(T_{\mathbb{P}^1, p_1}) = df(T_{\mathbb{P}^1, p_1}) \) but \( df_u(T_{\mathbb{P}^1, p_2}) \neq df(T_{\mathbb{P}^1, p_2}) \), if we take analytic coordinates \((x, y)\) on \( S \) at \( q := f(p_1) = f(p_2) \) so that the image of
the branch of \( f \) around \( p_2 \) is defined by \( y = 0 \), then for a suitable local parameter \( t \) on \( \mathbb{P}^1 \) at \( p_2 \), we have \( f_u(t) = (t, aut) \) modulo terms of higher order in \( t \) and \( u \), with \( a \neq 0 \). Thus \( df_u(T_{\mathbb{P}^1,p_2}) \subseteq T_{S,q} \cong A^2 \) is the span of the vector \((1, au)\), while \( df_u(T_{\mathbb{P}^1,p_2}) \) is the span of \((1, 0)\), both modulo \( u^2 \). This eliminates the tacnode in \( f(\mathbb{P}^1) \) at \( q \) by taking the deformation \( f_u(\mathbb{P}^1) \); as above, this implies that there was no tacnode in \( f(\mathbb{P}^1) \) to begin with.

Suppose \( d = 4 \) and \( d_S \geq 3 \). The anti-canonical map embeds \( S \) in \( \mathbb{P}^{d_S} \), so we may consider \( f(\mathbb{P}^1) \) as a degree four rational curve in \( \mathbb{P}^{d_S} \) with a tacnode at \( q = f(p_1) = f(p_2) \). We claim that \( f(\mathbb{P}^1) \) is contained in a \( \mathbb{P}^2 \subseteq \mathbb{P}^{d_S} \). Since every degree four rational curve is the linear projection of degree four rational normal curve in \( \mathbb{P}^4 \), the fact that \( f(\mathbb{P}^1) \) is not smooth implies that \( f(\mathbb{P}^1) \) is contained in a \( \mathbb{P}^3 \subseteq \mathbb{P}^{d_S} \). Let \( \ell \) be the tangent line to the tacnode of \( f(\mathbb{P}^1) \) and consider a plane \( \Pi \) containing \( \ell \). If \( f(\mathbb{P}^1) \not\subseteq \Pi \), then since \( \ell \) is tangent to each of the two branches of \( f(\mathbb{P}^1) \) at \( q \), the intersection multiplicity at \( q \) of \( \Pi \) and \( f(\mathbb{P}^1) \) in \( \mathbb{P}^3 \) is at least \( 4 \), hence equal to \( 4 \) since \( f(\mathbb{P}^1) \) has degree \( 4 \). Taking a point \( q' \in f(\mathbb{P}^1) \), \( q' \neq q \), we can take \( \Pi' \) to be the plane spanned by \( \ell \) and \( q' \). But if \( f(\mathbb{P}^1) \not\subseteq \Pi' \), then \( \Pi' \cdot f(\mathbb{P}^1) \) has degree \( \geq 5 \), which is impossible, so \( f(\mathbb{P}^1) \) is contained in \( \Pi' \).

This implies that the intersection multiplicity in \( \Pi' \) at \( q \) with \( f(\mathbb{P}^1) \) is \( 4 \), and thus \( \ell \) has intersection multiplicity \( 2 \) with each of the two branches of \( f(\mathbb{P}^1) \) at \( q \). Using the fact that \( f(\mathbb{P}^1) \) has arithmetic genus \( 3 \), one sees that in local analytic coordinates at \( q \) on \( \Pi' \), \( f(\mathbb{P}^1) \) has equation of the form \((y-x^2)(y-ax^2-bx^3+\ldots) = 0\), where either \( a \neq 1 \) or \( a = 1 \) and \( b \neq 0 \) (here we are using the assumption that \( \text{char} k \neq 2, 3 \)). Consider now \( f(\mathbb{P}^1) \) as a smooth curve on \( S \). We claim there is a choice of analytic coordinates at \( q \) on \( S \) so that \( f(\mathbb{P}^1) \) also has equation of the same form in the completion of \( \mathcal{O}_{S,q} \). Indeed, \( \Pi' \) must be tangent to \( S \) at \( q \), because the Zariski tangent space of \( f(\mathbb{P}^1) \) at \( q \) has dimension \( 2 \) since \( q \) is a singular point, and therefore it is equal to the Zariski tangent space of \( S \) at \( q \). It follows that a projection from \( S \) to \( \Pi' \) is a local analytic isomorphism and the form of the equation of \( f(\mathbb{P}^1) \) at \( q \) is unchanged. We may assume that the branch through \( p_1 \) has the equation \( y = x^2 \) and the branch through \( p_2 \) the equation \( y = ax^2 + bx^3 + \ldots \). Since \( d = 4 \), we have \( \mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(2) \), so there is a section \( s \) of \( \mathcal{N}_f \) having a zero of order \( 2 \) at \( p_1 \) and with \( s(p_2) \neq 0 \). The resulting deformation \( f_u \) of \( f \) has image curve \( f_u(\mathbb{P}^1) \) with local analytic equation at \( q \) of the form \((y-x^2)(y-ax^2-bx^3+\ldots-u) = 0\), \( u \) modulo higher order terms in \( u \). Thus, intersection of the two local branches of \( f_u(\mathbb{P}^1) \) coming from a neighborhood of \( p_1 \) and a neighborhood of \( p_2 \) is of the form \((1-a)x^2-bx^3 = u \), which in characteristic \( \neq 2, 3 \) shows that the tacnode has separated into two ordinary double points if \( a \neq 1 \), respectively, three ordinary double points if \( a = 1, b \neq 0 \). As above, this shows that there was no tacnode on \( f(\mathbb{P}^1) \) to begin with.

Suppose that \( d_S \geq 3 \) and \( d \leq 3 \). We rule out multiple points and tacnodes by a global argument. When \( d = 3 \), as above, \( f(\mathbb{P}^1) \) is a rational cubic curve in \( \mathbb{P}^{d_S} \). Thus, \( f(\mathbb{P}^1) \) is either a smooth twisted cubic curve in \( \mathbb{P}^3 \subseteq \mathbb{P}^{d_S} \), or a singular cubic in \( \mathbb{P}^2 \subseteq \mathbb{P}^{d_S} \). In the first case, there is nothing to show, and in the second, since \( f \) is unramified, \( f(\mathbb{P}^1) \) has a single ordinary double point as singularity. If \( d_S \geq 3 \) and \( d = 1, 2 \), then \( f(\mathbb{P}^1) \) is a line (\( d = 1 \)) or a smooth conic (\( d = 2 \)). This completes the proof for \( d_S \geq 3 \).

If \( d_S = 2 \), we are in the situation of Lemma \( \ref{lem:anti-canonical} \) with anti-canonical double cover \( \pi : S \to \mathbb{P}^2 \) branched along a smooth degree four curve \( E \). We have handled
the case \( d \geq 5 \) above. We handle the case \( d = 4 \) as follows. Let \( C = f(\mathbb{P}^1) \). Then either \( \pi : C \to \pi(C) \) is a double cover, with \( \pi(C) \) a smooth conic, or \( \pi : C \to \pi(C) \) is birational, with \( \pi(C) \) a rational quartic curve. In the latter case, we need only eliminate the case of \( C \) having a tacnode. If \( C \) does have a tacnode, at say \( q' \in C \) then \( \pi(C) \) has a tacnode at \( q := \pi(q') \). Since \( \pi(C) \) is a quartic curve, the tacnode on \( \pi(C) \) has local analytic equation as above: \((y - x^2)(y - ax^3 - bx^3 + \ldots )\) with \( a \neq 1 \) or \( a = 1 \) and \( b \neq 0 \). If the map \( \pi \) is unramified at \( q' \), then \( C \) has the same local analytic equation at \( q' \) as does \( \pi(C) \) at \( q \), in suitable analytic coordinates \( x', y' \). If \( \pi \) is ramified at \( q' \), then a local analytic computation shows that \( C \) has local analytic equation at \( q' \) of the form \((y' - x^2)(y' - a'x^2 + \ldots )\) with \( a' \neq 1 \), again, in suitable analytic coordinates \( x', y' \). In either case, we proceed exactly as we did above in the case \( d_S \geq 3 \), \( d = 4 \).

If \( C \to \pi(C) \) is a double cover, then \( \pi(C) \) is a smooth conic and \( \pi(C) \cdot E \) must be of the form

\[ \pi(C) \cdot E = p_1 + p_2 + 2q_1 + 2q_2 + 2q_3 \]

with \( p_1 \neq p_2 \) and the \( p_i \) distinct from all the \( q_j \) (see the proof of Lemma 2.29). If all the \( q_j \) are distinct, then \( C \) is smooth outside of ordinary double points at the points \( q_j' \) over \( q_j \), \( j = 1, 2, 3 \). If however \( q_1 = q_2 \) then \( C \) acquires an ordinary tacnode at \( q_1' = q_2' \) and if \( q_1 = q_2 = q_3 \) then \( C \) acquires a higher order tacnode at \( q_1' = q_2' = q_3' \). Since \( d = 4 \), we have \( \text{dim}_f M^{\text{dio}}_0(S, D) = 3 \), so we need only show that the dimension of the space of smooth conics that have \( \pi(C) \cdot E \) of the form \( p_1 + p_2 + 2q_1 + 2q_2 + 2q_3 \) with at least two of the \( q_j \) equal is at most \( 2 \).

For this, fix \( q_1 = q_2 = q \) and consider the linear system on \( E \) cut out by degree two curves \( C' \) with \( C' \cdot E = 4q - 2q_3 > 0 \). This is the projective space on \( H^0(E, \mathcal{O}_E(2)(-4q - 2q_3)) \). If \( h^0(E, \mathcal{O}_E(2)(-4q - 2q_3)) > 0 \), then \( C' \cdot E = 4q - 2q_3 = p_1 + p_2 \) is effective divisor of degree two. By adjunction, the canonical class \( K_E \) is \( \mathcal{O}_E(1) \), so \( \mathcal{P}(H^0(E, K_E(-p_1 - p_2))) \) is the projective space of lines \( \ell \) in \( \mathbb{P}^2 \) with \( \ell \cdot E \geq p_1 + p_2 \), in other words, \( \mathcal{P}(H^0(E, K_E(-p_1 - p_2))) \) is the line through \( p_1 \) and \( p_2 \) if \( p_1 \neq p_2 \), or the line tangent to \( E \) at \( p \) if \( p = p_1 = p_2 \). Thus \( h^0(E, \mathcal{O}_E(2)(-4q - 2q_3)) = h^0(E, K_E - p_1 - p_2) = 1 \) and by Riemann-Roch, we have

\[ h^0(E, \mathcal{O}_E(2)(-4q - 2q_3)) = 2 + 1 - 3 + h^1(E, \mathcal{O}_E(2)(-4q - 2q_3)) = 1, \]

assuming that \( h^0(E, \mathcal{O}_E(2)(-4q - 2q_3)) > 0 \). In other words, for fixed \( q, q_3 \in E \), there is at most one smooth conic \( C' \) with \( C' \cdot E = 4q - 2q_3 = p_1 + p_2 \) with \( p_1 \neq p_2 \) and the \( p_j \) distinct from \( q, q_3 \). We can then vary the points \( q, q_3 \) over \( E \), to conclude that the space of smooth conics that have \( \pi(C) \cdot E \) of the form \( p_1 + p_2 + 4q + 2q_3 \) as above has dimension at most \( 2 \). The same argument shows that the space of smooth conics that have \( \pi(C) \cdot E \) of the form \( p_1 + p_2 + 6q \) with \( p_1 \neq p_2 \) and \( p_j \neq q \) for \( j = 1, 2 \) has dimension at most \( 1 \). This finishes the proof in case \( d_S = 2 \), \( d = 4 \).

It remains to handle the cases \( d = 1, 2, 3 \), \( d_S = 2 \), and with \( \text{char} \neq 2, 3 \).

If \( d = 3 \), then \( \pi : C \to \pi(C) \) is birational and \( \pi(C) \) is a degree 3 integral rational curve in \( \mathbb{P}^2 \), and hence has a single singularity, which is either an ordinary double point or an ordinary cusp. Thus \( C \) itself is either smooth or also has a single singularity, which is either an ordinary double point or an ordinary cusp. Since \( f \) is unramified, \( C \) cannot have an ordinary cusp. Thus \( f \) is in \( M^\text{dio}_0(S, D) \).

For \( d = 1 \), \( \pi(C) \) is a line and thus \( f : \mathbb{P}^1 \to C \) and \( \pi : C \to \pi(C) \) are isomorphisms. If \( d = 2 \), then we are in the situation of Lemma 2.29. In all cases of the
lemma except \([2(\text{b})]\) we see immediately that \(f\) is in \(M^\text{\text{ord}}_0(S, D)\). We show that case \([2(\text{b})]\) does not occur as follows. Either \(\pi : C \to \pi(C)\) is birational, in which case \(\pi(C)\) is a smooth conic and \(C\) is smooth, or \(\pi : C \to \pi(C)\) has degree two, in which case \(\pi(C)\) is a line \(\ell\). In this latter case, let \(E \subset \mathbb{P}^2\) be the smooth quartic branch curve of the map \(\pi : S \to \mathbb{P}^2\). Since \(f : \mathbb{P}^1 \to C\) is birational, there are three possible cases: either \(\ell \cdot E = 2 \cdot p_1 + p_2 + p_3\), \(\ell \cdot E = 3 \cdot p_1 + p_2\) or \(\ell \cdot E = 4 \cdot p\), with the \(p_i\) distinct in the first two cases; if \(\ell \cdot E\) were the sum of 4 distinct points, then \(C\) would be a smooth curve of genus 1. In the first case \(C\) has an ordinary double point, in the second, an ordinary cusp and in the third a tacnode (in this latter case, \(\pi^{-1}(\ell)\) is a union of two -1 curves, intersecting at a single point with multiplicity 2, but we will not need this fact).

By Lemma \(\text{2.33}\) there are only finitely many possibilities for \(\ell = \pi(C)\) if \(\ell \cdot E = 3 \cdot p_1 + p_2\) or \(\ell \cdot E = 4 \cdot p\). Since \(f\) is a generic point and \(\dim_f M^\text{\text{bir}}_0(S, D) = d - 1 = 1\), \(f\) is not of this form. This completes the proof. \(\square\)

**Lemma 2.33.** Let \(k\) be an algebraically closed field of characteristic \(\neq 2, 3\). Let \(E \subset \mathbb{P}^2\) be a smooth degree four curve. Then for all but finitely many points \(p \in E\), the tangent line \(\ell_p\) to \(E\) at \(p\) intersects \(E\) at \(p\) with multiplicity two. Moreover, \(E\) has only finitely many bi-tangents.

**Proof.** Since \(E\) is a smooth quartic curve, \(E\) has genus two; let \(J(E)\) denote the Jacobian of \(E\). We first show that for at most finitely many \(p\), one has \(\ell_p, E = 4 \cdot p\). Indeed, if this is the case for \(p, q\), we have \(\mathcal{O}_E(4 \cdot (p - q)) \cong \mathcal{O}_E\). Thus, if \(\ell_p \cdot E = 4 \cdot p\) for all but finitely many \(p \in E\), then the map \(\alpha : E(k) \times E(k) \to J(E)(k)\), \(\alpha(p, q) = [\mathcal{O}_E(p - q)] \in \text{Pic}(E)\), has image in the 4-torsion subgroup of \(J(E)(k)\), plus possibly finitely many additional points of \(J(E)(k)\). This is impossible, since the image of \(\alpha\) generates \(J(E)(k)\) and \(J(E)\) is an abelian variety of dimension two.

Suppose that for all \(p \in E\), we have \(\ell_p \cdot E = 3 \cdot p + p'\) (possibly \(p = p'\)) and choose \(q \in E\) such that \(\ell_q \cdot E = 3 \cdot q + q'\) with \(q' \neq q\) and such that a general line through \(q\) intersects \(E\) in four distinct points. Let \(\pi : C \to \mathbb{P}\) be the linear projection from \(q\), where \(\mathbb{P}\) is the \(\mathbb{P}^1\) of all lines in \(\mathbb{P}^2\) containing \(q\). For \(\ell\) such a line, \(\pi^{-1}(\ell) = \ell \cdot E - q\).

Then \(\pi\) has degree three and since we are assuming the characteristic is different from three, \(\pi\) is a separable morphism. For \(p \neq q\) in \(E\), \(\pi\) is ramified at \(p\) if and only if the tangent line \(\ell_p\) contains \(q\); at such \(p\), \(\pi\) has ramification index \(e_p(\pi) = 3\). For \(\ell = \ell_q\), \(\pi^{-1}(\ell) = \ell_q \cdot E - q = 2q + q'\), so \(e_q(\pi) = 2\); at all other points \(x \in E\), \(e_x(\pi) = 1\). Since the characteristic is \(\neq 2, 3\), \(\pi\) is everywhere tamely ramified. But then the Riemann-Hurwitz formula says

\[3 \cdot (-2) + \sum_{x \in E} (e_x(\pi) - 1) = 2g(E) - 2 = 4\]

which is not possible, since \(\sum_{x \in E} (e_x(\pi) - 1) = (e_q(\pi) - 1) + \sum_{p,e_p(\pi)=3} e_p(\pi) - 1\) is odd.

We finish by showing that \(E\) has only finitely many bi-tangents (we include as a bi-tangent a line \(\ell\) with \(\ell \cdot E = 4p\)). By what we have already shown, for all but finitely many points \(p \in E\), each line \(\ell\) through \(p\) that is also a tangent line to \(E\) at some point \(q \neq p\) satisfies \(\ell \cdot E = p + p' + 2q\) with \(p' \neq q\), and if \(\ell\) is the tangent line to \(E\) at \(p\), then \(\ell \cdot E = 2p + q + q'\) with \(p \neq q\), \(p \neq q'\). Taking such a point \(p\) and considering the projection from \(p\), \(\pi : E \to \mathbb{P}^1\) as above, we see that each ramified point \(q\) of \(\pi\) satisfies \(e_q(\pi) = 2\). By the Riemann-Hurwitz formula, this says that there are exactly 10 such points, so there are 10 lines \(\ell\) through \(p\).
with $\ell \cdot E = p + p' + 2q$ and with $p' \neq q$. At most one of these lines can be the tangent line to $E$ at $p$, so there exists at least nine points $q$ on $E$ such that the tangent line to $E$ at $q$ is not a bi-tangent. Since the set of bi-tangents is a closed subset of the dimension one variety of all tangent lines to $E$, $E$ has only finitely many bi-tangents. \hfill \Box

**Remark 2.34.** The Fermat quartic $E \subset \mathbb{P}^2$ defined by $\sum_{i=0}^{2} X_i^4 = 0$ is an example of a smooth quartic curve over a field of characteristic three such that each tangent line $\ell_p$ has at least a three-fold intersection at $p$: the Hessian matrix is identically zero. We don’t know an example in characteristic two.

**Lemma 2.35.** Suppose that $k$ is an algebraically closed field of characteristic \( \neq 2,3 \). Suppose $d \geq 2$ and $S$ satisfies Assumption 2.30. Let $f \in M_{O}^{\bir}((S, D))$ be a geometric generic point over $k$. Let $C \subset S$ be a reduced curve defined over $k$. Then each point $p \in f(\mathbb{P}^1) \cap C$ is a smooth point of $C$. Moreover, if $d \geq 3$, then each point $p \in f(\mathbb{P}^1) \cap C$ is a smooth point of $f(\mathbb{P}^1)$ and $f(\mathbb{P}^1)$ and $C$ intersect transversely at $p$.

**Remark 2.36.** The assumption that $d \geq 3$ is necessary. For example, let $C$ be the image of the map $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ given by $t \mapsto (t, t^p)$ and let $f$ be $\mathbb{P}^1 \times [1,0]$.

**Proof.** Let $C' = f(\mathbb{P}^1)$. By Proposition 2.32, the map $f : \mathbb{P}^1 \to C'$ is birational and unramified. Let $C'$ is birational and unramified and $C'$ has only ordinary double points as singularities. Since $k$ is free and birational, the degree $d := C' \cdot (-K_S)$ satisfies $d \geq 2$.

We first show that for $p \in S(k)$ a $k$-point, $p$ is not in $C'$. Let $F$ be an algebraically closed field of definition for $f : \mathbb{P}^1 \to S$ and suppose that $p$ is in $f(\mathbb{P}^1)$. There are two cases: $f^{-1}(p) = \{q_1, q_2\}$ with $q_1 \neq q_2$ in $\mathbb{P}^1(F)$, or $f^{-1}(p) = q$ is a single $F$-point of $\mathbb{P}^1$. In the first case, $p$ is an ordinary double point of $C'$; let $\ell_1, \ell_2$ be the two tangent lines, $\ell_i = df(T_{\mathbb{P}^1, q_i})$. Since $d \geq 2$ and $f$ is unramified, $\mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(d-2)$, so there is a section $s$ of $\mathcal{N}_f$ with $s(q_i) \neq 0$ for $i = 1, 2$. There are thus analytic coordinates $x, y$ for $S$ at $p$ such that $C'$ has equation $xy = 0$ at $p$ and a deformation $f_u$ corresponding to $s$, and defined over $F[[u]]$, has image curve $f_u(\mathbb{P}^1)$ with equation $(x - au)(y - bu)$ with $ab \neq 0$, modulo terms of order $\geq 2$. Considering $f_u$ as a morphism defined the algebraic closure $\overline{F((u))}$, this shows that $p$ is not in $f_u(\mathbb{P}^1)$. Since $f$ was already a geometric generic point of $M_{O}^{\bir}((S, D))$ over $k$, $p$ is not in $f(\mathbb{P}^1)$. A similar argument treats the case where $f^{-1}(p)$ is a single point.

In particular, this shows that $f(\mathbb{P}^1) \cap C$ contains no singular point of $C$.

Now suppose $d \geq 3$ and take $p \in f(\mathbb{P}^1) \cap C$. If $p$ is a singular point of $f(\mathbb{P}^1)$, we have $f^{-1}(p) = \{q_1, q_2\}$ with $q_1 \neq q_2$ in $\mathbb{P}^1(F)$. We first show that $df(T_{\mathbb{P}^1, q_1}) \neq T_{C, p}$ for $i = 1, 2$. For this, we already have $df(T_{\mathbb{P}^1, q_1}) \neq df(T_{\mathbb{P}^1, q_2})$, so we may assume that $df(T_{\mathbb{P}^1, q_1}) = T_{C, p}$, $df(T_{\mathbb{P}^1, q_2}) \neq T_{C, p}$, and that in the local analytic description of $f(\mathbb{P}^1)$ as $xy = 0$, $T_{C, p}$ is given by $x = 0$ and $df(T_{\mathbb{P}^1, q_2})$ is given by $y = 0$. This also identifies $s$ with $s$ having a zero of order one at $q_1$, $s(t) = at + \ldots, a \neq 0$, where $t$ is the local coordinate at $q_1$ given by the pullback of $y$ and we use a local trivialization of $\mathcal{N}_f$ at $q_1$ corresponding to a trivialization of $df(T_{\mathbb{P}^1, q_2})$. Letting $f_u$ be a deformation of $f$ over $F[[u]]$ with first order term given by $s$. This gives the equation for $f_u(\mathbb{P}^1)$ of the form $(x - au)(y - bu) = 0$, modulo terms of higher degree, which shows $df(T_{\mathbb{P}^1, q_1}) \neq T_{C, p}$. As $f$ is already a geometric generic point over $k$, this shows that $df(T_{\mathbb{P}^1, q_1}) \neq T_{C, p}$ to begin with.
A similar argument shows that \( df(T_{p,q}) \neq T_C \) if \( p \) is a smooth point of \( f(\mathbb{P}^1) \) and \( f(q) = p \).

Now suppose that there is a point \( p \in f(\mathbb{P}^1) \cap C \) that is a singular point of \( f(\mathbb{P}^1) \). Write \( f^{-1}(p) = \{ q_1, q_2 \} \). Since \( d \geq 3 \), there is a section \( s \) of \( N_f \cong \mathcal{O}_{\mathbb{P}^1}(d-2) \) such \( s(q_1) = 0 \), \( s(q_2) \neq 0 \). Since \( f(T_{p,q_i}) \neq T_C \) for \( i = 1, 2 \), we have analytic coordinates \( x, y \) at \( p \) such that \( f(\mathbb{P}^1) \) is defined by \( xy = 0 \) and \( C \) is defined by \( y = x + \ldots \). We identify \( N_f \otimes F(q_i) \) with \( T_{C,q_i} \), \( i = 1, 2 \) and use the pullback of \( y - x \) as a local coordinate at \( q_1 \). Letting \( f_u \) be a deformation of \( f \) corresponding to \( s \), we have the equation for \( f_u(\mathbb{P}^1) \) in these coordinates, modulo terms of higher order, and thus the tangent vector describing the 1st order movement of the double point of \( f(\mathbb{P}^1) \) is non-zero in the normal bundle of \( C \) at \( p \). This shows that the double point \( p \) of \( f(\mathbb{P}^1) \) moves away from \( C \) in \( f_u(\mathbb{P}^1) \) over \( F(u) \). As above, this shows that each point of \( f(\mathbb{P}^1) \cap C \) is smooth on \( f(\mathbb{P}^1) \).

Since \( df(T_{p,q}) \neq T_C,f(q) \) if \( p = f(q) \) is in \( C \), this implies that \( f(\mathbb{P}^1) \) and \( C \) intersect transversely at each intersection point \( p \).

\( \square \)

**Lemma 2.37.** Let \( k \) be a field of characteristic 0. Let \( V \subset M^{\text{bir}}(S,D) \) be an integral closed subscheme, let \( f \in V \) be a geometric generic point and let \( C := f(\mathbb{P}^1) \subset S \). Suppose that \( C \) has a cusp at \( q \) in \( S \) and let \( p \in \mathbb{P}^1 \) be the point lying over \( q \). Then:

1. \( \text{codim} \ V \geq 1 \).
2. If \( \text{codim} \ V = 1 \) and either \( d_S \geq 3 \) or \( d \geq 6 \), then \( q \) is an ordinary cusp and \( f \) is unramified on \( \mathbb{P}^1 \setminus \{ p \} \).

**Proof.** (1) Since \( C \) has a cusp, \( f \) is ramified, and thus \( N_f/N_f^{\text{tor}} \cong \mathcal{O}_{\mathbb{P}^1}(d-2-t(f)) \) with \( t(f) \geq 1 \) as in Remark 2.22. Since the map \( T_f V \to H^0(\mathbb{P}^1,N_f/N_f^{\text{tor}}) \) is injective (Lemma 2.25) and \( M^{\text{bir}}(S,D) \) is smooth of dimension \( d-1 \), we have \( \text{codim} \ V \geq 1 \).

(2) Now suppose that \( \text{codim} \ V = 1 \). By the computation above, we have \( t(f) = 1 = t_p(f) \) if \( f \) is unramified away from \( p \) and \( e_p(f) = 2 \). Suppose that the cusp at \( q \) is a higher order cusp; this implies that \( C \) is not a component of any \( H \subset | - K_S | \). Take a standard system of parameters \( t, (x, y) \) for the cusp, so \( C \) has local equation \( y^2 = x^{2n+1}, n \geq 2 \). If \( d_S \geq 3 \) there is a \( \mathbb{P}^1 \) of curves \( H \subset | - K_S | \) passing through \( q \) and tangent to the limit tangent line at \( q \). All such \( H \) have local equation of the form \( y = a_2x^2 + a_3x^3 + \ldots \), so there is at least one such \( H \) with \( a_2 = 0 \). This yields a multiplicity of at least 6 for \( q \) in \( H \cdot C \), so \( d \geq 6 \). Thus, in any case, we have \( d \geq 6 \).

We have \( N_f/N_f^{\text{tor}} \cong \mathcal{O}_{\mathbb{P}^1}(d-3) \), so as \( d \geq 3 \), there is a global section \( s \) of \( \mathcal{O}_{\mathbb{P}^1}(d-3) \) having a zero of order 3 at \( p \). In our standard coordinate system \( t, (x, y) \), we have \( f(t) = (t^2, t^{2n+1}) \) for some \( n > 1 \). Defining the invertible subsheaf \( \mathcal{L} \subset f^*T_S \) as the kernel of \( f^*T_S \to N_f/N_f^{\text{tor}} \), we have the injective map \( df : T_{\mathbb{P}^1} \to \mathcal{L} \) with image \( \mathcal{L}(-p) \subset \mathcal{L} \). Thus \( \mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(3) \), \( H^1(\mathbb{P}^1, \mathcal{L}) = 0 \), so \( H^0(\mathbb{P}^1, f^*T_S) \to H^0(\mathbb{P}^1, N_f/N_f^{\text{tor}}) \) is surjective and we may lift \( s \in H^0(\mathbb{P}^1, f^*T_S) \). With respect to our standard parameters \( t, (x, y) \), we have \( \mathcal{L} \otimes F(p) = F \cdot \partial/\partial x|_p \) and \( \partial/\partial y|_p \).
maps to a generator of \( \mathcal{N}_f/\mathcal{N}_f^{\text{ext}} \otimes F(p) \). Thus, in \( f^* T_S \otimes \mathcal{O}_{\mathbb{P}^1, p} \), we have
\[
\tilde{s} = a(t) \cdot \partial / \partial x + b(t) \cdot t^3 \partial / \partial y
\]
with \( a(t) \in \mathcal{O}_{\mathbb{P}^1, p} \) and \( b(t) \) a unit in \( \mathcal{O}_{\mathbb{P}^1, p} \).

The section \( \tilde{s} \) defines a 1st order deformation \( f_{c, 1} \) of \( f \), which one can lift to a deformation \( f_c \) over \( F[[\epsilon]] \), since \( H^1(\mathbb{P}^1, f^* T_S) = 0 \). From our description of \( \tilde{s} \), we have
\[
f_c \equiv f_{c, 1} (t) = (t^2, t^{2n + 1} + \epsilon (a(t), b(t) \cdot t^3) \mod c^2
\]
By a translation in \( x \) (\( \equiv \text{Id} \mod \epsilon \)) we may assume that \( a(t) = 0 \) and thus
\[
f_c = (t^2, t^{2n + 1} + b(t) \cdot t^3) \mod c^2
\]
Working over \( F((\epsilon)) \) we may replace \( y \) with \( (1/b(0)\epsilon) \cdot y - \sum_{j \geq 2} b_j x^j + y \cdot \sum_{j \geq 1} c_j x^j \) to form a standard coordinate system \( t, (x, y) \) with
\[
f_c^* (x) = t^2, f_c^*(y_t) = t^3
\]
Over the field \( F((\epsilon)) \), the image curve \( C_c := f_c(\mathbb{P}^1) \) has an ordinary cusp at \( f_c(p) \), which shows that \( V \) is a proper closed subscheme of an irreducible component of \( Z_{\text{cusp}} \); as codim \( Z_{\text{cusp}} \geq 1 \) by Lemma 2.25, this contradicts codim \( V = 1 \). \( \square \)

**Lemma 2.38.** Let \( k \) be a field of characteristic zero. Let \( V \subset M_{\text{bir}}(S, D) \) be an integral closed subscheme, let \( f \in V \) be a geometric generic point and let \( C := f(\mathbb{P}^1) \subset S \).

1. Suppose that \( d_S \geq 2 \) or \( d \geq 4 \) and that \( C \) has a tacnode. Then codim \( V \geq 1 \).
2. Suppose that \( d_S \geq 4 \), or \( d_S = 3 \) and \( d \not= 6 \), or \( d \geq 7 \). Suppose that \( C \) has a tacnode of order \( \geq 2 \). Then codim \( V \geq 2 \).

**Proof.** (1) By Lemma 2.25 we may assume that \( f \) is unramified, so \( \mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1} (d-2) \). Suppose that \( C \) has the tacnode at \( q \). We note that \( C \) is not a component of any \( H \in |-K_S| \); if \( S_k \not\cong \mathbb{P}^1 \times \mathbb{P}^1 \), then \( S_k \) is the blow-up of \( \mathbb{P}^2 \) at \( 9 - d_S \) points and thus each \( H \in |-K_S| \) projects to a cubic curve in \( \mathbb{P}^2 \) (containing those points). Since no irreducible component of a cubic plane curve has a tacnode, \( C \) cannot be a component of \( H \). In case \( S_k \cong \mathbb{P}^1 \times \mathbb{P}^1 \), take an \( H \in |-K_S| \) and blow up \( S \) at a smooth point of \( H \), \( \pi : S' \to S \). The proper transform of \( H \) to \( S' \) is in \( |-K_{S'}| \), so again, no component of \( H \) has a tacnode.

If \( d_S \geq 2 \), we may find an \( H \in |-K_S| \) containing \( q \) and if \( H \) is smooth at \( q \), with tangent \( T_{H, q} \) equal to the common tangent line of the tacnode: this represents at most two linear conditions on \( |-K_S| \cong \mathbb{P}^{d_S} \). These conditions imply that \( H \) intersects each of the two branches of \( C \) at \( q \) with multiplicity at least two, and thus \( d = \deg H \cdot C \geq 4 \), so in any case \( d \geq 4 \).

Let \( p_1, p_2 \in \mathbb{P}^1 \) be the pre-images of \( q \) under \( f \). Since \( q \) is a tacnode, we have \( df(T_{p_1, p_1}) = df(T_{p_2, p_2}) \), which gives a canonical isomorphism of the normal spaces \( \mathcal{N}_f \otimes k(p_1) \cong \mathcal{N}_f \otimes k(p_2) \).

Since the family of maps parametrized by \( V \) is equisingular on a dense open subset, \( V \) is equisingular on a neighborhood of \( f \). This implies that the tangent map \( T_f V \to H^0(\mathbb{P}^1, \mathcal{N}_f) = T_f M_{\text{bir}}(S, D) \) followed by the restriction map
\[
\text{Res}_{p_1, p_2} : H^0(\mathbb{P}^1, \mathcal{N}_f) \to \mathcal{N}_f \otimes k(p_1) \oplus \mathcal{N}_f \otimes k(p_2) \cong \mathcal{N}_f \otimes k(p_1)^2
\]
has image contained in the diagonal. Since \( \text{Res}_{p_1, p_2} \) itself is surjective \( (d - 2 \geq 2) \) it follows that codim \( V \geq 1 \).
For (2), we first consider the case of a tacnode of order $\geq 2$. Let $f^{-1}(q) = \{p_1, p_2\}$ and choose a standard system of parameters $t_1, t_2, (x, y)$ for the tacnode at $q$. This gives the local defining equation for $C$, $y(y - x^{n+1}) \in \hat{O}_{\mathbb{P}^1, q} \cong F[[x, y]]$, where $n \geq 2$. If $d_S \geq 3$, there is a $\mathbb{P}^1$ of curves $H \in |K_S|$ which contain $q$ and with tangent $T_{H, q}$ equal to the common tangent line of the tacnode (or are singular at $q$). Thus there is an $H \in |K_S|$ with local defining equation $g = y + bxy + cy^2 + \ldots$ and then $q$ appears with multiplicity $\geq 6$ in $H \cdot C$. Thus $d \geq 6$ if $d_S \geq 3$. If $d_S \geq 4$, there is a $\mathbb{P}^2$ of curves $H \in |K_S|$ which contain $q$ and with tangent $T_{H, q}$ equal to the common tangent line of the tacnode (or are singular at $q$). Arguing as above, there is a $\mathbb{P}^1$ of $H \in |K_S|$ such that $H$ intersects $C$ at $q$ with multiplicity $\geq 6$, and thus we can find such an $H$ that also intersects $C$ at a point $q' \neq q$, hence $d \geq 7$. Thus, in all cases, we have $d \geq 7$.

Suppose that $f$ is ramified and $d \geq 6$. Then by Lemma 2.25, codim $V \geq 1$ and if codim $V = 1$, then by Lemma 2.37, $t(f) = 1$ and $f$ is ramified at a single point $p_3$, with $f(p_3)$ a simple cusp. Thus $\mathcal{N}_f/\mathcal{N}_f^\text{tor} \cong \mathcal{O}_{\mathbb{P}^1}(d - 3)$. Consider the map $df : T_{\mathbb{P}^1} \to f^*T_S$. Since $f$ is ramified to first order at $p_3$, the kernel $\mathcal{L}$ of $f^*T_S \to \mathcal{N}_f/\mathcal{N}_f^\text{tor}$ is an invertible subsheaf of $f^*T_S$ containing $df(T_{\mathbb{P}^1})$ and with quotient $\mathcal{L}/df(T_{\mathbb{P}^1}) \cong \mathcal{O}_{\mathbb{P}^1}(3)$. Thus $df(T_{\mathbb{P}^1}) = \mathcal{L}(p)$ and $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(3)$.

Since $d \geq 6$, dim$_F H^0(\mathbb{P}^1, \mathcal{N}_f/\mathcal{N}_f^\text{tor}) \geq 4$, so the restriction map

$$\text{Res}_{p_1, p_2} : H^0(\mathbb{P}^1, \mathcal{N}_f/\mathcal{N}_f^\text{tor}) \to \mathcal{N}_{f, p_1}/(t_1^2) \oplus \mathcal{N}_{f, p_2}/(t_2^2)$$

is surjective. Thus there is a section $s$ of $\mathcal{N}_f/\mathcal{N}_f^\text{or}$ which in the local coordinates $t_1$ and $p$ and $t_2$ at $p_2$ has the form

$$s(t_1) = at_1 + c_1t_1^2 + \ldots, s(t_2) = c_2t_2^2 + \ldots$$

with $a \neq 0$. As in the proof of Lemma 2.37, this defines a 1st order deformation $f_{*, 1}$ of $f$, which we can lift to a deformation $f_\epsilon$ of $f$ defined over $F[[\epsilon]]$ with $q_\epsilon := f_\epsilon(p_1) = f_\epsilon(p_2)$ an ordinary double point (over $F((\epsilon))$). Thus $C_\epsilon := f_\epsilon(\mathbb{P}^1)$ is not an equisingular deformation at $q$. We claim that we can modify $f_\epsilon$ without changing the class of $C_\epsilon$ in the local deformation space at $q$, but such that there is an $F[[\epsilon]]$ point $p_3$ deforming $p_3$ such that $f_\epsilon$ is ramified at $p_3$. This will exhibit $V$ as a proper closed subscheme of an irreducible component of $Z_{\text{cusp}}$, hence codim $V > \text{codim } Z_{\text{cusp}} \geq 1$.

To verify the claim, write $f_\epsilon$ in the standard coordinate system $(s, (x', y'))$ for the ordinary cusp at $q' = f(p_3)$:

$$f_\epsilon(s) = (s^2 + \sum_{i=1}^{\infty} \epsilon^i \sum_{j=0}^{\infty} a_{i,j}s^j, s^3 + \sum_{i=1}^{\infty} \epsilon^i \sum_{j=0}^{\infty} b_{i,j}s^j)$$

In the basis $\partial/\partial x', \partial/\partial y'$ for $f_\epsilon^*T_{\mathbb{P}^1}$ near $p_3$, the line bundle $\mathcal{L}$ has generator $\lambda := (2, 3s)$ with $df(T_{\mathbb{P}^1}) \subset \mathcal{L}$ the $\mathcal{O}_{\mathbb{P}^2}$-submodule generated by $s \cdot \lambda$. We modify $f_{*, 1}$ first by adding $-\sum_{i=1}^{\infty} \epsilon^i \cdot (b_{i,1}/3) \cdot \lambda$ to eliminate the linear term in the $y'$ coordinate. Note that in a neighborhood of $p_1$ and $p_2$, $\mathcal{L} = df(T_{\mathbb{P}^1})$, so modifying by a section of $\mathcal{L}$ acts by a local automorphism of $\mathbb{P}^1$ in a neighborhood of $p_1, p_2$, which does not affect the class of $C_\epsilon$ in the local deformation theory of $C$ near $q$. Thus we may assume that $f_\epsilon$ is of the form

$$f_\epsilon(s) = (s^2 + \sum_{i=1}^{\infty} \epsilon^i \sum_{j=0}^{\infty} a_{i,j}s^j, s^3 + \sum_{i=1}^{\infty} \epsilon^i \sum_{j=0}^{\infty} b_{i,j}s^j)$$

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with $b_{i,1} = 0$ for all $i$. Making a similar modification by adding $- \sum_{i=1}^{\infty} \epsilon_i \cdot (a_{i,1}/2) \cdot s \cdot \lambda$ will eliminate that linear terms in the $x'$ coordinate, so we may assume that $a_{i,1} = 0$ for all $i$; this modification corresponds to a translation in $s$, so we have the new origin $p_{3r}$. Then it is clear that $f_e$ is ramified at $p_{3r}$.

Suppose now that $f$ is unramified. In this case, we use the assumption that $d \geq 7$. Then $\mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(d-2)$, so $d-2 \geq 5$ and $\dim_f H^0(\mathbb{P}^1, \mathcal{N}_f) \geq 6$ and thus the restriction map

$$\text{Res}_{p_1, p_2} : H^0(\mathbb{P}^1, \mathcal{N}_f/\mathcal{N}_f^{\text{cor}}) \to \mathcal{N}_{f,p_1}/(t_1^2) \oplus \mathcal{N}_{f,p_2}/(t_2^2)$$

is surjective. We take a global section $s$ of $\mathcal{N}_f$ with a zero of order three at $p_1$ and a zero of order two at $p_2$ and construct as above a deformation $f_e$ of $f$ with first-order deformation corresponding to $s$, and of the form

$$f_e(t_1) = (t_1, \epsilon t_1^2) \mod (\epsilon^2, \epsilon t_1^4 F[[t_1]]), \quad f_e(t_2) = (t_2, t_2^{n+1} + \epsilon t_2^2) \mod (\epsilon^2, \epsilon t_2^3 F[[t_2]])$$

and with $f_e(p_1) = f_e(p_2)$. This gives the image curve $C_e$ an ordinary tacnode at $q_e = f_e(p_1) = f_e(p_2)$, so $V$ is a proper closed subscheme of an integral component of $Z_{\text{tac}}$, hence $\text{codim} V \geq 2$.

\[\Box\]

**Lemma 2.39.** Let $S$ be a del Pezzo surface over a field $k$ of characteristic zero and let $V \subset M_0^{\text{brr}}(S, D)$ be an integral closed subscheme. Let $f \in V$ be a geometric generic point and let $C := f(\mathbb{P}^1) \subset S$.

1. Suppose that $d_S \geq 2$, $f \in M_0^{\text{unr}}(S, D)$ and $C$ has a singular point $q$ of order $m$. Then $\text{codim} V \geq m - 2$.
2. Suppose that $d_S \geq 2$ and $C$ has singular points $q \neq q'$. Suppose that $f$ is ramified at a point $p'$ with $f(p') = q'$, that $f$ is unramified at all points $p$ with $f(p) = q$ and that $C$ has multiplicity $m > 2$ at $q$. Then $\text{codim} V \geq 2$.
3. Suppose that $d_S \geq 3$ or $d \geq 6$. Suppose that $C$ has a singular point $q$, that $f$ is ramified at a point $p$ with $f(p) = q$, and that $f$ is unramified at a point $p' \neq p$ with $f(p') = q$. Then $\text{codim} V \geq 2$.
4. Suppose that $d_S \geq 3$, $f \in M_0^{\text{unr}}(S, D)$ and $C$ has singular points $q, q'$ of order $m, m'$, respectively, then $\text{codim} V \geq m + m' - 4$.
5. Suppose that $d_S = 2$, $f \in M_0^{\text{unr}}(S, D)$ and $C$ has singular points $q, q'$ of order $m, m'$, respectively. Then $\text{codim} V \geq m + m' - 5$. If $d \geq 7$ and $m \geq m' \geq 3$, then $\text{codim} V \geq 2$.

**Proof.** (1) We refer to the exact sequence (2.1). Since $f$ is unramified, $\mathcal{N}_f \cong sO_{\mathbb{P}^1}(d-2)$. Since $d \geq 1$, we have $H^1(\mathbb{P}^1, \mathcal{N}_f) = 0$, so following Lemma 2.19, $M_0(S, D)$ is smooth of dimension $d - 1$ at $f$.

For a general $H \in | - K_S |$ with $q \in H$, $H$ is integral and does not contain $C$ as a component. Since $| - K_S |$ has dimension $d_S \geq 2$, there is an $H \in | - K_S |$ with $H \cap C \supset \{ q, q' \}$, with $q \neq q'$, so $d = \deg( H \cdot \bar{C} ) \geq m + 1$.

Let $F$ be an algebraically closed field over which $f$ is defined. Since $f$ is unramified, $f^{-1}(q) = \{ x_1, \ldots, x_m \}$ with $x_i \neq x_j$ for $i \neq j$. Given $v \in T_{V,f}$, we have the corresponding first order deformation $f_v$ of $f$, defined over $F[\epsilon]/\epsilon^2$, the corresponding deformations $x_i$ of $x_i$ and $q_v$ of $q$, with $f_v(x_i) = q_v$ for all $i$. Let $L_i \subset T_{S,q}$ be the image of $df(T_{V,f})$. The deformation $f_v$ corresponds to a section $s_v$ of $N_f$, which gives us the affine subspaces $L_i + s_v(x_i) \subset T_{S,q}$, and the conditions $f_v(x_i) = q_v$, $i = 1, \ldots, m$ implies $\cap_{i=1}^m L_i + s_v(x_i) \neq \emptyset$. Let $W \subset \oplus_{i=1}^m N_f \otimes k(x_i)$ be the set of $(v_i) \in \oplus_{i=1}^m N_f \otimes k(x_i)$ satisfying $\cap_{i=1}^m L_i + v_i \neq \emptyset$; $W$ is a linear subspace of
codimension $\geq m - 2$. Since $\mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(d - 2)$ and $d - 2 \geq m - 1$, it follows that $H^0(\mathbb{P}^1, \mathcal{N}_f) \geq m$ and the product of restriction maps

$$\text{Res} := \prod_{i=1}^{m} \text{res}_{s_i} : H^0(\mathbb{P}^1, \mathcal{N}_f) \to \bigoplus_{i=1}^{m} \mathcal{N}_f \otimes k(x_i)$$

is surjective. Letting $W' \subset H^0(\mathbb{P}^1, \mathcal{N}_f)$ be the inverse image of $W$ under Res, we have $\text{codim} W' \geq m - 2$ and $s_v \in W'$ for all $v \in T_{V,f}$. Thus the image of $T_{V,f}$ under the map $v \mapsto s_v$ is a subspace of $H^0(\mathbb{P}^1, \mathcal{N}_f)$ of codimension $\geq m - 2$. Via the identification $H^0(\mathbb{P}^1, \mathcal{N}_f) = T_{\mathcal{M}_0(S,D), f}$, the map $v \mapsto s_v$ is just the inclusion of $T_{V,f}$ in $T_{\mathcal{M}_0(S,D), f}$, so $V$ has codimension $\geq m - 2$ in $\mathcal{M}_0(S,D)$, as claimed.

For (2), the fact that $f$ is ramified at $p'$ implies that $\mathcal{N}_f^{\text{tor}} \neq \{0\}$ and thus $\mathcal{N}_f/\mathcal{N}_f^{\text{tor}} \cong \mathcal{O}_{\mathbb{P}^1}(d-s)$ with $s \geq 3$. By Lemma 2.25, the map $T_{V,f} \to H^0(\mathbb{P}^1, \mathcal{N}_f/\mathcal{N}_f^{\text{tor}})$ is injective. If $\text{codim} V \leq 1$, then $\dim T_{V,f} \geq d-2$, and as $\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-s)) = d-s+1$, we have $s = 3$, $\text{codim} V = 1$ and $T_{V,f} \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-3))$ is an isomorphism.

Since $f$ is ramified at $p'$, and $d_S \geq 2$, there is an $H \in |- K_S|$ with $H \cdot C \geq m \cdot q + 2 \cdot q'$, so $d \geq m + 2$. If $m \geq 4$, then by (1), we have $\text{codim} V \geq 2$, so we may assume that $m = 3$, so $d - 3 \geq 2$. Letting $p_1, p_2, p_3$ be the points of $\mathbb{P}^1$ mapping to $q$, we see that the map

$$\text{Res}_{p_1, p_2, p_3} : H^0(\mathbb{P}^1, \mathcal{N}_f/\mathcal{N}_f^{\text{tor}}) \to \bigoplus_{i=1}^{3} \mathcal{N}_f \otimes k(p_i)$$

is surjective, and thus the composite $V \to \bigoplus_{i=1}^{3} \mathcal{N}_f \otimes k(p_i)$ is surjective as well. However, from the proof of (1), the condition that the triple point at $q$ deforms along a first order deformation corresponding to a section $s \in H^0(\mathbb{P}^1, \mathcal{N}_f)$ defines a proper subspace of $\bigoplus_{i=1}^{3} \mathcal{N}_f \otimes k(p_i)$, which contradicts the fact that $V \to \bigoplus_{i=1}^{3} \mathcal{N}_f \otimes k(x_i)$ is surjective.

For (3), we argue as for (2) to reduce to the case $m = 3, t(f) = 1$ and $f$ is unramified on $\mathbb{P}^1 \setminus \{p\}$. We consider an analytic neighborhood of $C$ near $q$ as the union of the branch $(C, p)$ corresponding to $p \in \mathbb{P}^1$ and the branch $(C, p')$ corresponding to $p' \in \mathbb{P}^1$. Since $m = 3$, $t(f) = 1$ and $f$ is unramified on $\mathbb{P}^1 \setminus \{p\}$, it follows that the branch $(C, p)$ is a cusp. Since $d_S \geq 3$, there is an $H \in | - K|$ containing $q$ and singular at $q$. As the multiplicity of $q$ in the branch $(C, p)$ is 2, this implies that $q$ occurs with multiplicity $\geq 6$ in $H \cdot C$, so $d \geq 6$. Also $\mathcal{N}_f/\mathcal{N}_f^{\text{tor}} \cong \mathcal{O}_{\mathbb{P}^1}(d-3)$ so $d - 3 \geq 3$. Moreover, if $(C, p)$ is not an ordinary cusp, then a small modification of the argument for Lemma 2.37(2) shows that $\text{codim} V \geq 2$, so we may assume that $(C, p)$ is an ordinary cusp.

Letting $t \in \mathcal{O}_{\mathbb{P}^1, p}$ be a local parameter, the restriction map

$$\text{Res}_{t, p} : H^0(\mathbb{P}^1, \mathcal{N}_f/\mathcal{N}_f^{\text{tor}}) \to \mathcal{N}_f/\mathcal{N}_f^{\text{tor}} \otimes \mathcal{O}_{\mathbb{P}^1, p}/(t^2) \cong \mathcal{N}_f \otimes k(p')$$

is surjective. There is thus a section $s \in H^0(\mathbb{P}^1, \mathcal{N}_f/\mathcal{N}_f^{\text{tor}})$ mapping to zero in $\mathcal{N}_f/\mathcal{N}_f^{\text{tor}} \otimes \mathcal{O}_{\mathbb{P}^1, p}/(t^2)$ and to a non-zero element in $\mathcal{N}_f \otimes k(p')$. Since $H^1(\mathbb{P}^1, \mathcal{N}_f) = 0$, the corresponding first-order deformation $f_{c, 1}$ is unobstructed. Arguing as for the proof of Lemma 2.37(2), we may extend $f_{c, 1}$ to a deformation $f_c$ over $F[[\epsilon]]$ so that $f_c$ (considered over $F((\epsilon))$) is still ramified at $p$. The conditions on $s$ imply that $f_c(p) \equiv p \mod t^2$, while the branch of $f_c$ through $p'$ does not pass through $p$, and thus the branch of $f_c$ through $p'$ does not pass through $f_c(p)$, and if $f_c(p)$ has multiplicity two on $C_e$. This implies that $V$ is a proper closed subscheme of an integral component of $Z_{cusp}$, hence $\text{codim} V \geq 2$. 

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The proof of (4) is similar: taking an $H \in |-K_S|$ passing through $q$ and $q'$ and tangent to one of the branches of $C$ at $q$, we see that $d \geq m + m' + 1$. We thus have $H^0(\mathbb{P}^1, \mathcal{N}_f) \geq m + m'$ and for $p_1, \ldots, p_m$ lying over $q$ and $p'_1, \ldots, p'_{m'}$ lying over $q'$, the evaluation map

$$\text{Res}_{p_i, p'_i} : H^0(\mathbb{P}^1, \mathcal{N}_f) \to \bigoplus_{i=1}^m \mathcal{N}_f \otimes k(p_i) \bigoplus \bigoplus_{i=1}^{m'} \mathcal{N}_f \otimes k(p'_i)$$

is surjective. Arguing as in (1), the subspace of $\bigoplus_{i=1}^m \mathcal{N}_f \otimes k(p_i) \bigoplus \bigoplus_{i=1}^{m'} \mathcal{N}_f \otimes k(p'_i)$ corresponding to 1st order deformations of the local germs of $f$ near $p_1, \ldots, p_m, p'_1, \ldots, p'_{m'}$ for which $q$ and $q'$ deform to singular points of order $m, m'$ respectively has codimension $\geq m + m' - 4$, and thus codim $V \geq m + m' - 4$.

For (5), we have the estimate $d \geq m + m'$. In this case, the map $\text{Res}_{p_i, p'_i}$ has image of codimension at most one, and the argument of (4) shows that codim $V \geq d - 5 \geq m + m' - 5$, in particular, if $m \geq m' \geq 3$ and $m + m' \geq 7$, then codim $V \geq 2$. If $m = m' = 3$ and $d \geq 7$, then as $d - 2 \geq 5$, the restriction map

$$\text{Res}_{p_i, p'_i} : H^0(\mathbb{P}^1, \mathcal{N}_f) \to \bigoplus_{i=1}^3 \mathcal{N}_f \otimes k(p_i) \bigoplus \bigoplus_{i=1}^3 \mathcal{N}_f \otimes k(p'_i)$$

is surjective and the argument of (4) shows that codim $V \geq 2$. \hfill \Box

**Proposition 2.40.** Let $k$ be a field of characteristic 0 and suppose that $d_S \geq 4$ or $d_S = 3$ and $d \neq 6$, or $d \geq 7$. Let $V \subset M^\text{birft}_0(S, D)$ be an integral closed subscheme, let $f \in V$ be a geometric generic point and let $C := f(\mathbb{P}^1) \subset S$. Suppose that codim $V = 1$.

1. If $C$ has a triple point at $q \in S$, then $q$ is the only triple point of $C$, $q$ is an ordinary triple point, $f$ is unramified, and all other singularities of $C$ are ordinary double points.
2. If $C$ has a cusp at $q \in S$, then $q$ is the only cusp of $C$, $q$ is an ordinary cusp and all other singularities of $C$ are ordinary double points.
3. If $C$ has a tacnode at $q \in S$, then $q$ is an ordinary tacnode, $f$ is unramified, and all other singularities of $C$ are ordinary double points.

**Proof.** (1) If $f$ is ramified at some point, then by Lemma 2.39(2,3), codim $V \geq 2$ so $f$ must be unramified. Similarly, if $C$ has a point $q'$ of multiplicity $\geq 4$, then by Lemma 2.39(4) codim $V \geq 2$, so $C$ has only triple points and double points.

We first show that $q$ is an ordinary triple point. Since $f$ is unramified, $f^{-1}(q)$ is three distinct points $p_1, p_2, p_3$ of $\mathbb{P}^1$. If $q$ is not ordinary, then (after reordering) $df(T_{p_1, p_2}) = df(T_{p_1, p_3})$.

As in the proof of Lemma 2.38(2), we have $d \geq 6$, and $\mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(d - 2)$. Moreover, there is a global section $s$ with $s(p_1) \neq 0$, and $s$ having a second order zero at $p_2$ and $p_3$. To first order, this preserves the tacnode at $q$ corresponding to the branches at $p_2, p_3$, but the branch at $p_1$ no longer passes through $q$. Since $H^1(\mathcal{N}_f) = 0$, this 1st order deformation is unobstructed, and arguing further as in the proof of Lemma 2.38(2), we can extend this to a deformation $f_\epsilon$ of $f$ over $F[[\epsilon]]$ with $f_\epsilon(\mathbb{P}^1)$ having a tacnode at $q$. Since $D_{\text{tac}}$ has codimension one, this implies that $V$ has codimension $\geq 2$, contrary to our assumption.

Now suppose $C$ has a double point at $q'$ and that $q'$ is not an ordinary double point. Since $f$ is unramified, $q'$ must be a tacnode; let $p'_1, p'_2 \in \mathbb{P}^1$ be the points lying over $q'$. If $d_S \geq 3$, there is an $H \in |-K_S|$ passing through $q'$ and $q$, and sharing the common tangent line at $q'$. Thus $d \geq 6$. If $d_S = 3$, then by assumption, $d \geq 7$. If $d_S \geq 4$, then we can find an $H$ as above and passing through an additional point of $C$, so again $d \geq 7$, and thus in all cases $d \geq 7$. 29
As $N_f \cong O_{\mathbb{P}^1}(d - 2)$, $H^0(\mathbb{P}^1, N_f)$ has dimension $\geq 6$ and

$$\text{Res} : H^0(\mathbb{P}^1, N_f) \to \oplus_{i=1}^3 N_f \otimes k(p_i) \otimes N_f \otimes k(p'_1) \otimes N_f \otimes O_{\mathbb{P}^1, p'_2}/m^2_{p'_2}$$

is surjective. Taking a section $s$ of $H^0(\mathbb{P}^1, N_f)$ with 1st order zeros at $p_1, p_2, p_3$, a second order zero at $p'_2$ but with $s(p'_1) \neq 0$, then $s$ defines a first order deformation $f_{s,1}$ that is equisingular at $q$ but not so at $q'$. Therefore, we can extend $f_{s,1}$ to a deformation $f_s$ over $F[[\epsilon]]$ so that $q$ deforms to a triple point on $f_s(\mathbb{P}^1)$, but the deformation near $q'$ is not equisingular. Thus $V$ is a proper closed subscheme of $D_{\text{trip}}$, so codim $V \geq 2$, contrary to assumption.

For (2), suppose $f$ is ramified at $p \in \mathbb{P}^1$ with $f(p) = q$. By Lemma 2.37, $q$ is an ordinary cusp and $f$ is unramified on $\mathbb{P}^1 \setminus \{p\}$. By Lemma 2.39, $C := f(\mathbb{P}^1)$ has only double points. If $q' \neq q$ is a double point of $C$, then as $f$ is unramified over $q'$, $q'$ must be a tacnode. By Lemma 2.38, $q'$ is a tacnode. Assuming $d_S \geq 3$, let $H \in \mid - K_S \mid$ be chosen so that $\{q, q'\} \subset H$ and that $H$ is tangent to the common tangent line at $q'$; as above, this actually implies that $d \geq 7$ and $N_f/N_f^{\text{tor}} \cong O_{\mathbb{P}^1}(d - 3)$, $d - 3 \geq 4$. Letting $p'_1, p'_2 \in \mathbb{P}^1$ be the points over $q'$, we may find a section $s \in H^0(\mathbb{P}^1, N_f/N_f^{\text{tor}})$ with a second order zero at $p$ and at $p'_1$ and $s(p'_2) \neq 0$. As in the proof of Lemma 2.37(2) and (1), the corresponding first order deformation $f_{s,1}$ of $f$ can be extended to a deformation $f_s$ over $F[[\epsilon]]$ so that $f_s$ is ramified at $p$, but the deformation $C_s := f_s(\mathbb{P}^1)$ is not equisingular at $q'$, which yields codim $V \geq 2$, contrary to assumption.

For (3), Lemma 2.38(2) implies that the tacnode at $q$ must be an ordinary tacnode, by (2) $f$ is unramified and by (1) and Lemma 2.39, all other singularities are double points. Applying Lemma 2.38(2), each double point $q' \neq q$ is either an ordinary tacnode or an ordinary double point, so suppose $q'$ is an ordinary tacnode. Suppose $d_S \geq 3$. Taking an $H \in \mid - K_S \mid$ passing through $q$ and $q'$ and tangent to the common tangent line at $q$, we see that $d \geq 6$ and $N_f \cong O_{\mathbb{P}^1}(d - 2)$; as above, this implies that $d \geq 7$ in all cases. Let $p_1, p_2 \in \mathbb{P}^1$ be the points lying over $q$ and $p'_1, p'_2 \in \mathbb{P}^1$ be the points lying over $q'$. Let $t_1, t_2, (x, y)$ be a standard system of parameters for $q$ and let $t'_1, t'_2, (x', y')$ be a standard system of parameters for $q'$.

Consider the restriction map

$$\text{Res} : H^0(\mathbb{P}^1, N_f) \to N_f \otimes O_{\mathbb{P}^1, p_1}/(t'_1) \otimes N_f \otimes O_{\mathbb{P}^1, p_2}/(t'_2) \otimes N_f \otimes k(p'_1) \otimes N_f \otimes k(p'_2) =: W$$

Since $d \geq 7$, Res is surjective. In particular, we may find a global section $s$ of $N_f$ that has a 2nd order zero at $p_1$ and $p_2$, a first order zero at $p'_1$ and is non-zero at $p'_2$. As $H^1(\mathbb{P}^1, N_f) = 0$, we may extend the corresponding first order deformation of $f$ to a deformation $f_s$ defined over $F[[\epsilon]]$.

Using the surjectivity of Res, we may take our extension $f_s$ of the 1st order deformation so that $q$ deforms to an ordinary tacnode $q_s$ on the image curve $C_s$, and in the coordinates $t'_1, t'_2, (x', y')$, we have

$$f_s(t'_1) = (t'_1 + \epsilon \cdot x_1(t'_1, \epsilon), \epsilon \cdot y_1(t'_1, \epsilon)), \quad f_s(t'_2) = (t'_2 + \epsilon \cdot x_2(t'_2, \epsilon), t'_2 + \epsilon \cdot y_2(t'_2, \epsilon))$$

with $y_1(0, \epsilon) = 0$, $y_2(0, \epsilon) \neq 0$. Translating in $x'$ and then in $t'_2$ (by translations $\equiv 0 \mod \epsilon$) we may rewrite this as

$$f_s(t'_1) = (t'_1, \epsilon \cdot y_1(t'_1, \epsilon)), \quad f_s(t'_2) = (t'_2, t'_2 + \epsilon \cdot y_2(t'_2, \epsilon))$$
still with $y_1(0, \epsilon) = 0, y_2(0, \epsilon) \neq 0$. Translating by replacing $y'$ with $y' - \epsilon \cdot y_2(t_2', \epsilon)$, we reduce to

$$f_\epsilon(t_1') = (t_1', \epsilon \cdot y_2(t_2', \epsilon)), \quad f_\epsilon(t_2') = (t_2', t_2'')$$

again with $y_2(0, \epsilon) \neq 0$. The image curve $C_\epsilon = f_\epsilon(\mathbb{P}^1)$ thus has defining equation $(y' - \epsilon \cdot y_2(t_2', \epsilon))((y' - x'^2) = f([x', y', \epsilon])$. By Lemma 2.41 $C_\epsilon'$ does not have a tacnode $q'_\epsilon$ specializing to $q'_\epsilon$. This exhibits $V$ as a proper closed subscheme of an integral component of $Z_{\text{tac}}$, forcing codim $V \geq 2$, contrary to assumption. □

**Lemma 2.41.** Let $f = y(y - x^2) \in F[[x, y]]$ define an ordinary tacnode over an algebraically closed field $F$ of characteristic $\neq 2$. Let

$$f_\epsilon(x, y) = (y - \sum_{i=1}^\infty \epsilon^i g_i(x))(y - x^2) \in F[[x, y, \epsilon]]$$

define a deformation of $f$ over $F[[\epsilon]]$ and suppose that $g_1(0) \neq 0$. Then $f_\epsilon$ is not equisingular: the curve $C_\epsilon := \text{Spec} F[[\sqrt{\epsilon}, x, y]]/(f_\epsilon)[1/\epsilon]$ has two ordinary double points specializing to $(0, 0)$ and no other singularities.

**Proof.** Write $\sum_{i=1}^\infty \epsilon^i g_i(x) = \epsilon \sum_{i,j \geq 0} a_{i,j} \epsilon^j x^j$ with $a_{i,j} \in F$. Then $a_{0,0} = g_1(0) \neq 0$, so there is an $h(x, \epsilon) \in F[[\epsilon, x]]$ with $h^2 = \sum_{i,j \geq 0} a_{i,j} \epsilon^j x^j$. The singular locus of $C_\epsilon$ is just the intersection of $y = \sum_{i=1}^\infty \epsilon^i g_i(x)$ with $y = x^2$, that is, the subscheme defined by $x^2 - \epsilon \cdot h^2$ or, over $F[[\sqrt{\epsilon}, x]]$, $(x - \sqrt{\epsilon} \cdot h)(x + \sqrt{\epsilon} \cdot h)$. Since $F[[\sqrt{\epsilon}, x]]/(x - \sqrt{\epsilon} \cdot h)$ and $F[[\sqrt{\epsilon}, x]]/(x + \sqrt{\epsilon} \cdot h)$ are both reduced, we have the desired description of $C_\epsilon$.

Note that $Z_{\text{cusp}}, Z_{\text{tac}}$ and $Z_{\text{trip}}$ each have only finitely many irreducible components, by Definition 2.13.

**Definition 2.42.** We define reduced codimension one subschemes $D_{\text{cusp}}, D_{\text{tac}}, D_{\text{trip}}$ on $M_0(S, D)$ as follows.

1. Let $D_{\text{cusp}}$ be the closure in $\bar{M}_0(S, D)$ of the union of the codimension one integral components $Z_{\text{cusp}} \subset M^b_{0,0}(S, D)$
2. Let $D_{\text{tac}}$ be the closure in $\bar{M}_0(S, D)$ of the union of the codimension one integral components $Z_{\text{tac}} \subset M^\text{unr}_{0,0}(S, D)$
3. Let $D_{\text{trip}}$ be the closure in $M_0(S, D)$ of the union of the codimension one integral components $Z_{\text{trip}} \subset M^\text{unr}_{0,0}(S, D)$

3. Non-birational and non-free maps

Having examined $M^b_{0, n}(S, D)$, we look more closely at the moduli stack of primary interest, $\bar{M}_{0, n}(S, D)$ with $n = -D \cdot K_S - 1$; set $d = -K_S \cdot D$. We have the “forget the marked points” map $\pi_{n/0} : \bar{M}_{0, n}(S, D) \to \bar{M}_0(S, D)$, which is a composition of the structure maps for the various universal curves $\pi_{i+1/j} : \bar{M}_{0, i+1}(S, D) \to \bar{M}_{0, i}(S, D)$, hence proper and flat.

**Definition 3.1.** The codimension one subschemes $D_{\text{cusp}}, D_{\text{tac}}, D_{\text{trip}}$ of $\bar{M}_{0, n}(S, D)$ are given by applying $\pi_{n/0}$ to the corresponding closed subschemes of $\bar{M}_0(S, D)$.

The results on $M^b_{0, n}(S, D)$ of the previous section carry over directly to $M^b_{0, n}(S, D)$, for instance, setting $d := -D \cdot K_S$, $M^b_{0, n}(S, D)$ is a smooth finite-type $k$-scheme with $\dim_k M^b_{0, n}(S, D) = 2d - 2$, or $M^b_{0, n}(S, D)$ is empty; this follows from Lemma 2.19

We proceed to study the complement $\bar{M}_{0, n}(S, D) \setminus M^b_{0, n}(S, D)$. 

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Lemma 3.3. Suppose $f$ is not birational to its image, we factor $f = f' \circ c$ where $c : \mathbb{P}^1 \to \mathbb{P}^1$ has degree $e$ and $f' : \mathbb{P}^1 \to S$ is birational to its image. Let $D' = f'_*([\mathbb{P}^1])$ and let $V'$ be the closure of $f'$. Then, $\text{im}V = e \text{im}V'$ and $\dim \text{im}V = \dim \text{im}V'$. Lemma 3.2 implies that $f'$ is not free.

Following the construction of $\bar{M}_{0,n}(S, D)$ given in [1], there is a quasi-projective scheme $\bar{M}_{0,n}(S, D)$ with PGL$_N$-action, presenting $\bar{M}_{0,n}(S, D)$ as quotient stack PGL$_N \backslash \bar{M}_{0,n}(S, D)$. For an $F$-point of $\bar{M}_{0,n}(S, D)$, $F \supset k$ an algebraically closed field, we have the corresponding morphism $f : \mathbb{P} \to S$, where $\mathbb{P}$ is a semi-stable genus $0$ curve. This gives us the image Cartier divisor $f_*([\mathbb{P}])$, which we may consider as an $F$-point of the projective space $|D|$: this extends to a morphism $\bar{im} : \bar{M}_{0,n}(S, D) \to |D|$. We note that $f_*([\mathbb{P}]) = (g f)_*([\mathbb{P}])$ for $g \in \text{PGL}_N$. It follows that we have the morphism $M_{0,n}(S, D) \to |D| \cong \mathbb{P}^N$, $N = (D^{(2)} + d)/2$, sending the equivalence class $[f]$ of a morphism $f : \mathbb{P} \to S$ to $f_*([\mathbb{P}])$. For $V \subset M_{0,n}(S, D)$ a locally closed substack, we have the constructible subset $\text{im}(V) \subset |D|$ and we may speak of the dimension $\dim \text{im}(V)$, which is at most $\dim V$.

Lemma 3.2. Let $f : \mathbb{P}^1 \to S$ factor as $f = g \circ q$ where $g : \mathbb{P}^1 \to S$ is birational onto its image and $q : \mathbb{P}^1 \to \mathbb{P}^1$ is a finite map. Then we have a short exact sequence

$$0 \to \text{coker}(dq) \to N_f \to q^* N_g \to 0.$$ 

Moreover, if $g$ is free then $f$ is free.

Proof. We have the following commutative diagram.

$$
\begin{array}{ccccccccc}
0 & \to & T \mathbb{P}^1 & \xrightarrow{df} & f^* TS & \xrightarrow{N_f} & 0 \\
& & \downarrow{dq} & & \downarrow{q} & & \\
0 & \to & q^* T \mathbb{P}^1 & \xrightarrow{q^* df} & q^* g^* TS & \xrightarrow{q^* N_g} & 0
\end{array}
$$

So, the short exact sequence follows by the snake lemma. Since $\text{char } k = 0$, it follows that $\text{coker}(dq)$ is torsion. Thus,

$$N_f/N_f^{tor} \cong q^*(N_g/N_g^{tor}).$$

If $g$ is free, then $N_g/N_g^{tor} \cong O(m)$ for $m \geq 0$, so $N_f/N_f^{tor} \cong O(\deg(q)m)$. So $f$ is free. \hfill \Box

Lemma 3.3. Suppose $k$ has characteristic zero. Let $V \subset \bar{M}_{0,n}(S, D)$ be an integral closed substack with geometric generic point $f$. Suppose that $f$ corresponds to a morphism $f : \mathbb{P}^1 \to S$ with image curve $C := f(\mathbb{P}^1)$ and that $f$ is non-free. Then $\dim \text{im}V = 0$. Moreover, one of the following cases holds:

1. $d = 1$, $n = 0$, $C$ is a -1 curve on $S$ and $f : \mathbb{P}^1 \to C$ is an isomorphism.
2. $d = 2$, $C$ is a -1 curve on $S$ and $f : \mathbb{P}^1 \to C$ is a 2-1 cover. In this case, $\dim ev(V) = 1$.
3. $d = 2$, $d_S = 2$, $f : \mathbb{P}^1 \to C$ is birational, $C$ has an ordinary cusp at $q \in C$, $f^{-1}(q)$ is a single point $p \in \mathbb{P}^1$ and $f : \mathbb{P}^1 \setminus \{p\} \to C \setminus \{q\}$ is an isomorphism. Moreover $\dim ev(V) = 1$.
4. $d \geq 3$ and $\text{codim } ev(V) \geq d - 1 \geq 2$.

In case (3), $V$ is dense in a component of $D_{cusp}$.

Proof. If $f$ is not birational to its image, we factor $f = f' \circ c$ where $c : \mathbb{P}^1 \to \mathbb{P}^1$ has degree $e$ and $f' : \mathbb{P}^1 \to S$ is birational to its image. Let $D' = f'_*([\mathbb{P}^1])$ and let $V'$ be the closure of $f'$. Then, $\text{im}V = e \text{im}V'$ and $\dim \text{im}V = \dim \text{im}V'$. Lemma 3.2 implies that $f'$ is not free.
Since \( f' \) is non-free, we have \( \mathcal{N}_f / \mathcal{N}^\text{tor}_f \cong \mathcal{O}_{\mathbb{P}^1} (m) \) with \( m < 0 \) and thus
\[
H^0 (\mathbb{P}^1, \mathcal{N}_f / \mathcal{N}^\text{tor}_f) = \{0\}.
\]
Since the tangent map \( T_f : V' \to H^0 (\mathbb{P}^1, \mathcal{N}_f / \mathcal{N}^\text{tor}_f) \) is injective (Lemma 2.25), this says that \( \dim \text{im} V = \dim \text{im} V' = 0 \), so \( f_* (\mathbb{P}^1) \) is the unique geometric point of \( \text{im} V \). Thus each element of \( \text{ev}(V) \) consists of a sequence of \( n \) points of \( C \). So, \( \dim \text{ev}(V) = n \) and \( \text{codim} \text{ev}(V) = 2n - n = d - 1 \). If \( d \geq 3 \), we are in case (4).

If \( d = 1 \), then \( f \) is birational and \( C \) is a line and thus a -1 curve, giving us case (1).

Suppose \( d = 2 \). It follows that \( n = 1 \). First assume \( d_S \geq 3 \), so \(-K_S \) embeds \( S \). So, either \( C \) is a conic and \( f : \mathbb{P}^1 \to C \) is an isomorphism or \( C \) is a line and \( f : \mathbb{P}^1 \to C \) is a double cover. If \( f : \mathbb{P}^1 \to C \) is an isomorphism, then \( \mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1} \), so \( f \) is free contrary to our hypothesis. If \( f \) is a double cover, we apply Lemma 3.2 to obtain \( \mathcal{N}_f / \mathcal{N}^\text{tor}_f \cong \mathcal{O}_{\mathbb{P}^1} (-2) \), so \( f \) is not free and we are in case (2).

If \( d = 2 \) and \( d_S = 2 \), then we are in the situation of Lemma 2.28. If \( f : \mathbb{P}^1 \to C \) has degree 2, then \( C \cdot (-K_S) = 1 \), so \( C \) is a -1 curve on \( S \) and we are again in case (2).

Suppose \( d = d_S = 2 \) and \( f : \mathbb{P}^1 \to C \) is birational. We are in the situation of Lemma 2.29. If \( \pi : C \to \pi (C) \) is birational, then \( \pi (C) \) is a smooth conic, \( C \) is smooth and \( f : \mathbb{P}^1 \to C \) is an isomorphism. Thus, \( \mathcal{N}_C = \mathcal{O}_{\mathbb{P}^1} \) and \( f \) is free, contrary to the hypothesis. If \( \pi : C \to \pi (C) \) is a double cover, then either \( f : \mathbb{P}^1 \to C \) is unramified, hence \( \mathcal{N}_C = \mathcal{O}_{\mathbb{P}^1} \) and \( f \) is free, or \( C \) has a single ordinary cusp, and \( t(f) = 1 \) so \( \mathcal{N}_C / \mathcal{N}^\text{tor}_C \cong \mathcal{O}_{\mathbb{P}^1} (d - 3) = \mathcal{O}_{\mathbb{P}^1} (-1) \). This is case (3).

\[ \square \]

**Lemma 3.4.** Let \( k \) be a field. Suppose Assumption 2.30 holds for \( S \). Let \( V \subset \mathcal{M}_{0,n}(S,D) \) be an integral closed substack with geometric generic point \( f \). Suppose that \( f \) corresponds to a morphism \( f : \mathbb{P}^1 \to S \) with image curve \( C := f (\mathbb{P}^1) \) and let \( d_f \) be the degree of \( f : \mathbb{P}^1 \to C \). Then \( \dim \text{im}(V) \leq (d/d_f) - 1 \).

**Proof.** By passing to an extension, we may assume that \( k \) is algebraically closed, and in particular perfect. Let \( d_C = C \cdot K_S \), so \( d_C \cdot d_f = d \) and \( d/d_f = d/d_C \geq 1 \); in particular, \( d \neq 1 \) if \( d_f > 1 \). Let \( F \) be an algebraically closed field of definition for \( C \) and \( f \). Let \( \hat{C} \) be the normalization of \( C \). Then \( \hat{C} \cong \mathbb{P}^1 \) (Lüroth’s theorem) and \( f \) factors as \( \mathbb{P}^1 \to \hat{C} \cong \mathbb{P}^1, C \) with \( g \) birational and \( \hat{f} \) of degree \( d_f \). By Assumption 2.30 there is an unramified map \( g_0 \) in the irreducible component of \( \mathcal{M}^\text{bir}_0 (S,C) \) containing \( g \). By Lemma 2.17 \( \dim_9 \mathcal{M}^\text{bir}_0 (S,C) = d_C - 1 \). Thus
\[
\dim \text{im}(V) \leq \dim_9 \mathcal{M}^\text{bir}_0 (S,C) \leq d_C - 1 = (d/d_f) - 1.
\]

\[ \square \]

**Definition 3.5.** Let \( U \) be a normal, integral scheme over a field \( F \) and let \( \mathbb{P} \to U \) be an \( n \)-marked semi-stable genus zero curve. We say that \( \mathbb{P} \) is ttreelike if the normalization \( \hat{\mathbb{P}} : \hat{\mathbb{P}} \to \mathbb{P} \) is a disjoint union with each component isomorphic to \( \mathbb{P}_{x} \).

**Lemma 3.6.** Let \( \mathbb{P} \to U \) be a ttreelike family over \( U \), with \( U \) normal and integral.
1. Suppose the number of irreducible components in the normalization \( \hat{\mathbb{P}} \) of \( \mathbb{P} \) is \( r \). Then for each geometric point \( x \) of \( U \), the fiber \( \mathbb{P}_x \) has exactly \( r-1 \) double points.
2. Let \( u \) be a geometric generic point of \( U \) and let \( y \) be a double point of \( \mathbb{P}_u \). Then there is a unique pair of components \( \hat{\mathbb{P}}_1, \hat{\mathbb{P}}_2 \) of \( \hat{\mathbb{P}} \) with \( y \) equal to the image of
By (2), there are sections \( \overline{r} \) so that \( \overline{U} \) is the closure of \( (\overline{P}_i \times_p \overline{P}_j)_\eta \) in \( \overline{P}_i \times_p \overline{P}_j \). Then the projection \\
\[ (\overline{P}_i \times_p \overline{P}_j)_\eta \to U \]
is an isomorphism.

3. Let \( \overline{P}_i, \overline{P}_j \) be distinct irreducible components of \( \overline{P} \) and let \( u \) be a geometric generic point of \( U \). If \( (\overline{P}_i \times_p \overline{P}_j) \times_U u = \emptyset \), then \( \overline{P}_i \times_p \overline{P}_j = \emptyset \).

4. For each pair of components \( \overline{P}_i, \overline{P}_j \) with \( \overline{P}_i \times_p \overline{P}_j \neq \emptyset \), the projection \( \overline{P}_i \times_p \overline{P}_j \to U \) is an isomorphism, defining two sections \( \sigma^{i,j}_1 : U \to \overline{P}_i, \sigma^{i,j}_r : U \to \overline{P}_j \) via the two projections \( \overline{P}_i \times_p \overline{P}_j \to \overline{P}_i, \overline{P}_i \times_p \overline{P}_j \to \overline{P}_j \).

Proof. For (1), recall that each connected genus zero semi-stable curve \( P \) over an algebraically closed field \( F \) has \( \dim_F \chi(O_P) = 1 \). Let \( \pi : \tilde{P} \to P \) be the normalization of \( P \), and let \( r \) be the number of irreducible components of \( \tilde{P} \). Then \( \chi(O_{\tilde{P}}) = r \). On the other hand, we have the exact sheaf sequence on \( \tilde{P} \),
\[ 0 \to O_{\tilde{P}} \to \pi_* O_{\tilde{P}} \to \bigoplus_{i=1}^r F(p_i) \to 0 \]
where \( \{p_1, \ldots, p_s\} \) are the double points of \( P \). Then
\[ r = \dim_F \chi(O_{\tilde{P}}) = \dim_F \chi(\pi_* O_{\tilde{P}}) = \dim_F \chi(O_{\tilde{P}}) + s = 1 + s, \]
so \( r = s + 1 \).

Now, since our family is treelike, each geometric fiber \( \overline{P}_x \) has normalization \( \overline{P}_x \cong \Pi_{i=1}^s \overline{P}^1_{x,i} \), so each geometric fiber \( \overline{P}_x \) has exactly \( r - 1 \) double points.

For (2), it follows directly from the definition of a semi-stable, genus zero curve, that there is a unique pair of components \( \overline{P}_i u, \overline{P}_j u \) of \( \tilde{P}_u \) with \( y \) equal to the image of \( (\overline{P}_i u \times P_{\tilde{P}_u} \overline{P}_j u) \) in \( \tilde{P}_u \). The first part of (2) thus follows from the fact that \( \tilde{P} \) is treelike. It is also obvious that the map \( \tilde{P}_i u \times P_{\tilde{P}_u} \overline{P}_j u \to P_u \to u \) is an isomorphism, so \( (\overline{P}_i \times_p \overline{P}_j)_\eta \to u \) is an isomorphism.

The morphism \( u \to \eta \) is qcqs and faithfully flat, so the map \( (\overline{P}_i \times_p \overline{P}_j)_\eta \to \eta \) is an isomorphism. For each geometric point \( x \) of \( U \), \( \Pi_{i=1}^r \overline{P}_{P_{\tilde{P}_u} x} \) is a finite set so \( (\overline{P}_i \times_p \overline{P}_j)_\eta \to U \) is birational, proper and quasi-finite. Since \( U \) is normal, and \( (\overline{P}_i \times_p \overline{P}_j)_\eta \) is integral, the projection is an isomorphism, by Zariski’s main theorem.

For (3), it follows from (1) that \( P_u \) has exactly \( r - 1 \) double points, \( y_1, \ldots, y_{r-1} \). By (2), there are sections \( \sigma_1, \ldots, \sigma_{r-1} : U \to \tilde{P}_x \times_p \tilde{P} \) with image sections \( \overline{\sigma}_1, \ldots, \overline{\sigma}_{r-1} : U \to \tilde{P} \) with \( \overline{\sigma}(\eta) = y_i \). By (1) again, for each geometric point \( x \) of \( U \), the points \( \overline{\sigma}_1(x), \ldots, \overline{\sigma}_{r-1}(x) \) are exactly the double points of the fiber \( \overline{P}_x \).

If now \( (\overline{P}_i \times_p \overline{P}_j) \times_U u = \emptyset \), but there is a \( z \in (\overline{P}_i \times_p \overline{P}_j)_x \), then the image of \( z \) in \( \tilde{P} \) is a double point of \( \tilde{P}_x \), so \( z = \overline{\sigma}(x) \) for some \( i \). But then there is a pair of components \( \tilde{P}_i', \tilde{P}_j' \) with \( \sigma_i \) corresponding to the non-empty fiber product \( \tilde{P}_i' \times_p \tilde{P}_j' \), and \( \{i, j, i', j'\} \) having size at least three. But then \( z \) is a point of \( \tilde{P}_x \) of multiplicity at least three, a contradiction.

For (4), it suffices by (2) to show that the inclusion
\[ (\overline{P}_i \times_p \overline{P}_j)_\eta \to \tilde{P}_i \times_p \tilde{P}_j \]
is an isomorphism. For this, we may take the basechange from \( k \) to its algebraic closure, so we may assume that \( k \) is itself algebraically closed. Since the \( k \)-points of \( U \) are dense, we need only check over a neighborhood of each closed point \( a \in U \).
Let \( b \in \tilde{\mathbb{P}}_i \times \mathbb{P} \tilde{\mathbb{P}}_j \) be the unique point lying over \( a \) (\( b \) is automatically a closed point); we consider \( b \) simultaneously as a closed point of \( \tilde{\mathbb{P}}_i, \mathbb{P}_j \) and \( \mathbb{P} \). We may also pass to the completions \( \tilde{\mathcal{A}} := \tilde{\mathcal{O}}_{\mathbb{P},a} \) and \( \tilde{\mathcal{B}} := \tilde{\mathcal{O}}_{\mathbb{P},b} \). As \( \mathcal{O}_{\mathbb{P},a} \) is an excellent normal local domain, \( \mathcal{A} \) is a complete normal local domain. Let \( \mathfrak{m} \subset \tilde{\mathcal{A}} \) be the maximal ideal.

We consider the versal deformation space of the singularity \( xy = 0 \), which has base \( \text{Spf} (k[[t]]) \) and versal family \( \text{Spf} (k[[t, x, y]]/(xy - t)) \). From this description of the versal family, we find there is an element \( f \in \mathfrak{m} \) such that \( \tilde{\mathcal{B}} \) is isomorphic as an \( \mathcal{A} \)-algebra to \( \mathcal{A}[[x, y]]/(xy - f) \). In addition, the section \( \sigma : \mathbb{P} \to \mathbb{P} \) with \( \sigma(a) = bt \) given by (2) defines a surjection \( \psi : \mathcal{A}[[x, y]]/(xy - f) \to \mathcal{A} \) splitting the inclusion \( \mathcal{A} \to \mathcal{A}[[x, y]]/(xy - f) \). Moreover, since \( \sigma(U) \) is contained in the relative singular locus of \( \mathbb{P} \to U \), the induced map \( \psi : \text{Spf} \mathcal{A} \to \text{Spf} \mathcal{A}[[x, y]]/(xy - f) \) is contained in the closed formal subscheme of \( \mathcal{A}[[x, y]]/(xy - f) \) defined by the vanishing of the section \( d(xy - f) \) of the completed relative Kähler differential \( \hat{\Omega} \mathcal{A}[[x, y]]/\mathcal{A} = \mathcal{A}[[x, y]] \cdot dx \oplus \mathcal{A}[[x, y]] \cdot dy \). Since \( d(xy - f) = xdy + ydx \), this says that the kernel of \( \psi \) contains the ideal \((x, y) + (xy - f)/(xy - f) \). But since \( \mathcal{A}[[x, y]]/(xy - f) = \mathcal{A} \), this says that \((x, y) \supset (xy - f) \) in \( \mathcal{A}[[x, y]] \), hence \( f = 0 \) and \( \mathcal{B} \cong \mathcal{A}[[x, y]]/(xy) \). This in turn implies that \( \hat{\mathcal{O}}_{\mathbb{P},b} \cong \mathcal{A}[[x, y]]/(x, y) \), \( \hat{\mathcal{O}}_{\mathbb{P},b} \cong \mathcal{A}[[x, y]]/(y) \) and thus

\[
\hat{\mathcal{O}}_{\mathbb{P},i \times \mathbb{P} \mathbb{P}_j,b} \cong \mathcal{A}[[x, y]]/(x, y) = \mathcal{A} = \hat{\mathcal{O}}_{(\mathbb{P}, \mathbb{P}_j)_n,b},
\]

which proves (4).

Because \((\mathbb{P}, p_*) \) is semi-stable, any marked point \( p_i : U \to \mathbb{P} \) lands in the smooth locus of \( \mathbb{P} \). Since normalization commutes with smooth base change, \( \pi : \tilde{\mathbb{P}} \to \mathbb{P} \) is an isomorphism over the smooth locus and \( p_i \) has a unique preimage under \( \pi(U) : \tilde{\mathbb{P}}(U) \to \mathbb{P}(U) \).

**Definition 3.7.** We define a tree associated to \( \mathbb{P} \) as follows. Let the vertices \( V(\mathbb{P}) \) be the set of components of \( \tilde{\mathbb{P}} \). Let the half-edges \( H(\mathbb{P}) \subset \tilde{\mathbb{P}}(U) \) be the preimage under \( \pi \) of the marked points and nodal points of \( \mathbb{P} \),

\[
H(\mathbb{P}) = \{ \pi(U)^{-1}(p_i) : i = 1, \ldots, n \} \cup \{ \sigma^i_{1,j}, \sigma^j_{1,i} : \tilde{\mathbb{P}}_i, \mathbb{P}_j \in V(\mathbb{P}) \text{ such that } \tilde{\mathbb{P}}_i \times \mathbb{P} \mathbb{P}_j \neq \emptyset \}
\]

where the \( \sigma^i_{1,j}, \sigma^j_{1,i} \) are the sections constructed in Lemma 3.6.4). Thus, there is a canonical map \( \nu : H(\mathbb{P}) \to V(\mathbb{P}) \). Let \( i : H(\mathbb{P}) \to H(\mathbb{P}) \) be the involution with orbits of length 1 corresponding to marked points and orbits of length 2 corresponding to nodal points. Let the edges \( E(\mathbb{P}) \subset H(\mathbb{P}) \times H(\mathbb{P}) \) be the subset consisting of orbits of \( i \) of length 2. Since \( \mathbb{P} \) has genus zero, the map \( \nu \) induces an inclusion \( E(\mathbb{P}) \to V(\mathbb{P}) \times V(\mathbb{P}) \). Thus, we obtain a tree \( T(\mathbb{P}) \). Below, by abuse of notation, we may use \( \mathbb{P} \) to refer to \( T(\mathbb{P}) \).

**Lemma 3.8.** Let \( \mathbb{P} \to U \) be a semistable genus zero curve with \( U \) integral. There exists a dense open subscheme \( U_0 \) and a surjective finite morphism \( W \to U_0 \) such that \( \mathbb{P} \times_U W \to W \) is treelike and \( W \) is smooth.

**Proof.** Let \( \eta \) be the generic point of \( U \), and let \( \tilde{\eta} : \bar{k}(\eta) \to U \) be a geometric point with image \( \eta \) and residue field an algebraic closure of \( k(\eta) \). The basechange \( \mathbb{P}_{\bar{k}} \) of \( \mathbb{P} \) to \( \bar{k} \) has a normalization \( \tilde{\mathbb{P}}_{\bar{k}} \to \mathbb{P}_{\bar{k}} \). Normal schemes are regular in codimension 1, whence the curve \( \tilde{\mathbb{P}}_{\bar{k}} \) is regular. Since \( \bar{k}(\eta) \) is algebraically closed, it follows that \( \tilde{\mathbb{P}}_{\bar{k}} \) is smooth \[ 038X \]. A smooth genus 0 curve over an algebraically closed field...
is isomorphic to a disjoint union of \( \mathbb{P}^1 \)'s. This isomorphism descends to a finite extension \( k(\eta) \to L \), giving a pullback diagram

\[
\begin{array}{c}
\mathbb{P}_\eta \\
\downarrow \\
\mathbb{P}_L \\
\downarrow \\
\text{Spec } k(\eta) \to \text{Spec } L
\end{array}
\]

\[
\begin{array}{c}
\prod_{i=1}^M \mathbb{P}^1_L \\
\downarrow \alpha \\
\prod_{i=1}^M \mathbb{P}^1_L \\
\end{array}
\]

By enlarging \( L \) if necessary, we may assume that \( \alpha : \prod_{i=1}^M \mathbb{P}^1_L \to \mathbb{P}_L \) is birational and finite, because the property of being an isomorphism and finite is detected after fpqc basechange. Since \( \prod_{i=1}^M \mathbb{P}^1_L \) is normal, it follows that \( \prod_{i=1}^M \mathbb{P}^1_L \to \mathbb{P}_L \) is canonically the normalization.

We may choose an open subset \( U_0 \) of \( U \) with \( U_0 \) affine. Let \( W = \text{Spec } \mathcal{O}(W) \) where \( \mathcal{O}(W) \) is the integral closure of \( \mathcal{O}(U_0) \) in \( L \). Since \( U_0 \) is a finite type \( k \)-algebra and an integral domain, \( \mathcal{O}(U_0) \) is a Nagata ring and N-2 \([30 \text{ p 240 } (31.H) \text{ Theorem 72 and Def N-2}]\). Thus \( W \to U_0 \) is a finite surjective map giving rise to the field extension \( k(\eta) \to L \) on generic points. \( \mathcal{O}(W) \) is an integral domain by construction. It follows that \( W \) is reduced, whence geometrically reduced because \( k \) is perfect \([49 \text{ tag 020I}]\). \( \mathcal{O}(W) \) is furthermore a finite type \( k \)-algebra because it is finite over \( \mathcal{O}(U_0) \). Thus by generic smoothness \([49 \text{ tag 056V}]\), there is a non-empty open subset \( W' \) of \( W \) which is smooth over \( k \). Replacing \( U_0 \) by the (non-empty, open) complement of the image of \( W - W' \), and then replacing \( W \) by its pullback over the new \( U_0 \), we obtain a finite surjective map \( W \to U_0 \) with \( W \) smooth over \( k \) \([49 \text{ tag 056V}]\). Passing to a further open subset of \( U_0 \), we may assume that \( \alpha \) extends to a map \( \alpha' : \prod_{i=1}^M \mathbb{P}^1_W \to \mathbb{P}_W \) which we may assume to be finite and birational. As above, it follows that \( \alpha' \) is canonically the normalization.

\[
\square
\]

**Definition 3.9.** Let \( U \) be a normal, integral scheme over a field \( F \) and let \( \mathbb{P} \to U \) be an \( n \)-marked semi-stable genus 0 curve which is treelike. Let \( f : (\mathbb{P}, p_*) \to S \) be a stable map of degree \( D \). We say that \( f \) is simple if for each geometric point \( u \) of \( U \), the restriction of \( f_u \) to any component of \( \mathbb{P}_u \) is either constant or birational and no two components of \( \mathbb{P}_u \) have the same image under \( f_u \) as a reduced closed subscheme.

**Lemma 3.10.** Let \( \mathbb{P} \to V \) be an \( n \)-marked semi-stable genus zero curve and let \( f : (\mathbb{P}, p_*) \to S \) be a stable map of degree \( D \). There exist

- a stratification \( V = V_0 \cup \ldots \cup V_N \);
- finite covers \( W_i \to V_i \);
- \( n \)-marked semi-stable genus zero curves \( \mathbb{P}_i \to W_i \);
- stable maps \( f_i : \mathbb{P}_i \to S \);

such that

(1) \( W_i \) is integral and normal, and \( \mathbb{P}_i \to W_i \) is treelike.

(2) \( f_i \) is simple;
(3) there exists a function \( m : V(P_i) \to \mathbb{Z}_{>0} \) such that
\[
\sum_{v \in V(P_i)} m(v)D_v = D,
\]
where \( D_v = (f_i)_s[v] \) is the Cartier divisor corresponding to the image of \( v \) under \( f_i \).

(4) Let \( a \in V_i \) be a geometric generic point, and let \( f_a \) denote the restriction of \( f \) to \( P_a \). If \( f_a \) is not simple, then there exists \( v \in V(P_i) \) such that \( m(v) \geq 2 \). Otherwise \( P_i \) is the image curve and non-birational components which are not contracted. We may assume that the components \( P_{ai}, \ldots, P_{ar} \) have different (reduced closed) images under \( f_a \) and the images of \( P_{a(s+1)}, \ldots, P_{ar} \) are all the same as one of the images of \( P_{a1}, \ldots, P_{as} \). For any \( i \) such that the restriction \( f|_{P_{ai}} : P_{ai} \to S \) of \( f \) to \( P_{ai} \) is not birational, let \( P_{ai}' \to f(P_{ai}) \) be defined to be the normalization of the reduced image curve \( f(P_{ai}) \). Since \( P_{ai} \cong \mathbb{P}^1 \) is normal, \( f|_{P_{ai}} \) factors
\[
P_{ai} \overset{\pi_i}{\to} P_{ai}' \overset{f'_i}{\to} f(P_{ai}).
\]
For notational simplicity, set \( P_{ai}' = P_{ai} \) if \( f|_{P_{ai}} : P_{ai} \to S \) is birational.

We will glue together the \( P_{ai}' \)'s for \( i = 1, \ldots, s \) to form a tree-like semi-stable genus \( 0 \) curve \( P' \). (We will not be gluing in the \( P_{a(s+1)}, \ldots, P_{ar} \).) Since the curve \( P_a \) is treelike, we have an associated tree \( T(P_a) \). Removing the vertices \( P_{a(s+1)}, \ldots, P_{ar} \) (and resulting edges) produces a forest \( F \). (By a forest, we mean a finite disjoint union of trees.) View \( P_{a1} \) as the root of \( T(P_a) \). Traveling out from \( P_{a1} \) defines a root of every tree of \( F \). Call the trees of \( F \) not containing \( P_{a1} \) the detached trees. Suppose there is a detached tree in \( F \) whose root \( r \) is attached to \( T(P_a) \) to a component with the same image as a component \( c \) on the tree of \( F \) containing \( P_{a1} \). Then attach \( r \) to \( c \). If there is no such root, then the component containing \( P_{a1} \) contains all the vertices and the forest is just a tree. Now \( P_{a1} \) is contained in a potentially larger component of a new forest. We again consider any detached tree of this new forest whose root \( r \) is attached to a component with the same image as a component \( c \) on the new tree containing \( P_{a1} \). Again attach \( r \) to \( c \). This process stops when we have a formed a new tree \( T' \) whose vertices are in canonical bijection with \( P_{a1}', \ldots, P_{as}' \).

We will modify this tree \( T' \) to a new tree \( T'' \) with some extra vertices. It will have an associated tree-like semi-stable genus \( 0 \) curve \( P' \) over \( a \). The extra vertices will correspond to contracted components. For each \( i = 1, \ldots, s \), let \( A_i \subseteq \{1, \ldots, r\} \) denote the subset those indices \( j \) such that \( P_{aj} \) has the same
reduced closed image curve as $P_{a_i}$. Let $H(P_{a_i})$ denote the half-edges associated to the tree-like $P_{a_i}$ (Definition 3.7). Define $H_i(P_{a_i}) \subset H(P_{a_i})$ to be those half-edges lying in $P_{a_j}$ for $j \in A_i$. In other words, $H_i(P_{a_i})$ contains the marked points and the points where two components are attached for every $P_{a_i}$. Because $v$ is a geometric point, we may choose a preimage under $f^\prime_a$ in $P^\prime_{a_i}$ for every point $f_a(p)$ with $p$ in $H_i(P_{a_i})$. Let $H_i^\prime$ denote the multiset of these preimages, i.e. the set of these preimages where repeated preimages are contained with the appropriate multiplicity.

We build $P^\prime_a$ by gluing $(P^1)$'s together. Start by putting the component $P^\prime_{a_1}$ in $P^\prime_a$. If $H_i^\prime$ has points with multiplicity greater than 1, attach a $P^1$ at the corresponding point $p$ and choose (arbitrarily) smooth points on the new $P^1$ in bijective correspondence with the multiple copies of $p$. For points of multiplicity equal to 1, mark the corresponding point on $P^\prime_{a_1}$. This builds a larger marked semi-stable genus 0 curve. The tree $T^\prime$ has a vertex for $P^\prime_{a_1}$ and each of the attached $(P^1)$'s.

Extend $f^\prime_a$ to this union by sending any attached $(P^1)$ to the corresponding $f_a(p)$ in $S$.

We continue to build $P^\prime_a$ and $f^\prime_a : P^\prime_a \rightarrow S$. For each edge in $T^\prime$ connected to the first vertex, attach the corresponding component $P^\prime_{a_i}$ for some $i = 1, \ldots, s$, at the appropriate point of the $P^\prime_a$ under construction. For each point $p$ of $H_i$ with multiplicity greater than one, attach a new $P^1$ to $P^\prime_a$ and choose and mark smooth points on the new $P^1$ in bijective correspondence with the multiple copies of $p$. Extend the definition of $f^\prime_a$ by mapping $P^\prime_{a_i}$ by $f^\prime_{a_i}$ and contracting the new $(P^1)$'s to the corresponding $f^\prime_a(p)$. Add the $P^\prime_{a_i}$ and new $(P^1)$'s to the tree $T^\prime$ and edges corresponding to the attachment points.

Running through the vertices of $T^\prime$, we obtain a treelike semistable $n$-marked curve $(P^\prime_a, P^\prime, p^\prime_a)$ with tree $T^\prime$ together with a stable map $f^\prime_a : (P^\prime_a, P^\prime, p^\prime_a) \rightarrow S$.

Define $m_a : V(P^\prime_a) \rightarrow \mathbb{N}$ by setting $m(P^\prime_{a_j})$ equal to the sum of the degrees of $f_a|_{P^\prime_{a_j}} : P^\prime_{a_j} \rightarrow S$ for each $j$ in $A_i$ and setting $m$ to be 0 on any contracted component. Thus

$$\sum_{j \in A_i} (f_a)_*[P^\prime_{a_j}] = m(P^\prime_{a_i})(f_a)_*[P^\prime_{a_i}],$$

and

$$\sum_{v \in V(P^\prime_a)} m(v)(f_a)_*[P^\prime_{a_i}] = D. \quad (3.1)$$

Since $P^\prime_a$ is treelike, there is a corresponding curve $P^\prime$ over $V$. Spreading out, there is an open neighborhood $W$ of $a$ in $V$ such that the marked points of $P^\prime_a$ come from sections $W \rightarrow P^\prime_W$. We thus obtain an $n$-marked semi-stable treelike $(P^\prime, p^\prime_a)$ over $W$. By potentially shrinking $W$ further, the stable map $f^\prime_a : (P^\prime_a, p^\prime_a) \rightarrow S$ spreads out to a stable map $f^\prime : (P^\prime, p^\prime_a) \rightarrow S$ over $W$. The property (3) follows from (3.1). This completes the proof.

Let $V$ be an integral finite type $k$-scheme, smooth over $k$, and let $P \rightarrow V$ be a tree-like family with two irreducible components, $P = P_1 \cup P_2$. Let $v \in V$ be a geometric generic point and let $f : P \rightarrow S$ be a morphism. Let $D_i = f_{v*}(P_{1, v})$ and let $d_i = f_{v*}(P_{i, v}) \cdot (-K_S)$. Let $D = D_1 + D_2$, $d = d_1 + d_2$.

By potentially shrinking $W$ further, the stable map $f^\prime_a : (P^\prime_a, p^\prime_a) \rightarrow S$ spreads out to a stable map $f^\prime : (P^\prime, p^\prime_a) \rightarrow S$ over $W$. The property (3) follows from (3.1). This completes the proof.

Let $V$ be an integral finite type $k$-scheme, smooth over $k$, and let $P \rightarrow V$ be a tree-like family with two irreducible components, $P = P_1 \cup P_2$. Let $v \in V$ be a geometric generic point and let $f : P \rightarrow S$ be a morphism. Let $D_i = f_{v*}(P_{1, v})$ and let $d_i = f_{v*}(P_{1, v}) \cdot (-K_S)$. Let $D = D_1 + D_2$, $d = d_1 + d_2$.

Sending $x \in V$ to the curve $f_{v*}(P_{1, x})$ defines morphisms $f_i : V \rightarrow |D_i|$, $i = 1, 2$, and we have $f : V \rightarrow |D|$.
with $\tilde{f}(x) = f_1(x) + f_2(x)$.

Let $F$ be an algebraic closed field and let $f : (\mathbb{P}, p_*) \to S$ be a simple stable map over $F$. Let $\mathbb{P}_{\text{cont}} \subset \mathbb{P}$ be the union of the irreducible components that get mapped to points by $f$ and let $\mathbb{P}_{\text{simp}} \subset \mathbb{P}$ be the union of the remaining irreducible components of $\mathbb{P}$. A connected component of $\mathbb{P}_{\text{cont}}$ is rigid if it intersects two or more components of $\mathbb{P}_{\text{simp}}$ and movable otherwise. Let $S_r(f)$ (resp. $S_m(f)$) denote the set of rigid (resp. movable) connected components of $\mathbb{P}_{\text{cont}}$ containing at least one marked point. Observe that each such component is a subtree of $\mathbb{P}$.

For $T \in S_m(f)$, let $n_T$ be the total number of marked points on the components of $T$. Let $S_r(f)$ be the set of irreducible components of $\mathbb{P}_{\text{simp}}$.

**Lemma 3.11.** Suppose that Assumption 2.30 holds for $S$. We take $n = d - 1$. Let $p : V \to M_{0,n}(S, D)$ be a map of an integral finite type $k$-scheme $V$ to $M_{0,n}(S, D)$, giving the stable map $f_V : (\mathbb{P}_V, p_{V*}) \to S$ of an $n$-pointed genus 0 curve $(\mathbb{P}_V, p_{V*})$ over $V$. Suppose $\mathbb{P}_V$ is treelike and $f_V$ is simple. Let $v$ be a geometric generic point of $V$, giving the stable map $f : (\mathbb{P}, p_*) \to S$. We consider the image $ev(V) \subset S^n$.

Then
\[
\dim_k ev(V) \leq d + n - |S_s(f)| - |S_m(f)| - |S_r(f)|,
\]
or more precisely,
\[
\dim_k ev(V) \leq d + n - |S_s(f)| - \sum_{T \in S_m(f)} (n_T - 1) - |S_r(f)|.
\]

**Proof.** Let $T \in S_m(f)$. We claim that $T$ contains a leaf of $\mathbb{P}$. Indeed, $T$ is a subtree of $\mathbb{P}$ that only connects to $\mathbb{P}\setminus T$ by a single edge. Therefore, $T$ contains at least two marked points. On the other hand, if $T \in S_r(f)$, the point $f(T)$ of $f(\mathbb{P})$ is in the intersection $f(P) \cap f(P')$ with $P \neq P' \in S_r(f)$.

Let $I \subset |D| \times S$ denote the incidence variety and let $V \subset I$ denote the preimage of $\text{im}(V)$. Let $\pi_V : \mathbb{P}_V \to V$ denote the projection. The map $f_V : \mathbb{P}_V \to S$ induces the morphism
\[
\tilde{f}_V : V \to |D|
\]
sending $g \in V$ to the divisor $f_{V*}(\mathbb{P}_{V,g})$ on $S$. We also have the relative evaluation map
\[
ev_{\mathbb{P}_V, |D|} = (\tilde{f}_V, ev_V) : V \to |D| \times S^n
\]
factoring the evaluation map $ev_V : V \to S^n$.

For $C \in S_s(f)$, define $d_C = f_s[C] \cdot (-K_S)$ to be the degree of $f$ restricted to $C$. By Lemma 3.4
\[
\dim \text{im}(V) \leq \sum_{C \in S_s(f)} (d_C - 1) \leq d - |S_s(f)|.
\]

Since $V$ is treelike, the decomposition of $\mathbb{P}_y$ into $\mathbb{P}_{y, \text{simp}}$ and $\mathbb{P}_{y, \text{cont}}$ and the further decomposition into the various trees in $S_m(f_y), S_r(f_y)$, is constant as $y$ varies over $V$. Namely, there are canonical bijections between, e.g. $S_m(f_y)$ and $S_m(f_{y'})$ for $y, y'$ in $V$. For $y \in V$ and $P$ a component of $\mathbb{P}_y$, let $n_P$ denote the number of marked points.

Fix a geometric point $x$ of $\text{im}(V)$ and let $V_x \subset V$ be the fiber $\tilde{f}_V^{-1}(x)$. We proceed to give a bound on $\dim_k(x) \text{ev}(V_x)$. For $y \in V_x$ and $P$ a component of $\mathbb{P}_{y, \text{simp}}$, we have $n_P$ marked points, each mapping via $f$ to the curve $f_y(P) \subset S$, so over all of $V_x$ these contribute at most $n_P$ to $\dim_k(x) \text{ev}(V_x)$. If $P$ is a component of some
rigid tree $T$, then each of the $n_P$ marked points of $P$ map to the intersection of two components of $f(P_{y,x},simp)$, so these contribute $0$ to $\dim_{\kappa(z)} ev(V_x)$. Finally, taking together all the components $P$ in some movable tree $T$, the sum $\sum_{P \in T} n_P \geq 2$ marked points in $T$ all map to the same point of the curve $f(P_T)$, where $P_T$ is the curve in $\mathbb{P}_{y,x,simp}$ intersecting $T$. So altogether, these marked points contribute at most $1$ to $\dim_{\kappa(z)} ev(V_x)$. We obtain the bound
\begin{equation}
\dim_{\kappa(z)} ev(V_x) \leq \sum_{P \in S(f)} n_P + \sum_{T \in S_m(f)} 1
\end{equation}
Combining the bounds (3.4) and (3.5), since $n_T \geq 2$ for each $T \in S_m(f)$, we get
\begin{equation}
\dim_k ev(V) \leq \dim_k ev_{V,|D|}(V) \leq \dim_k \im(V) + \max_{x \in \im(V)} \dim_{\kappa(x)} ev(V_x)
\leq d - |S_S(f)| + \sum_{P \in S(f)} n_P + \sum_{T \in S_m(f)} 1
= d + n - |S_S(f)| - \sum_{T \in S_m(f)} (n_T - 1) - |S_S(f)|
\leq d + n - |S_S(f)| - |S_m(f)| - |S_S(f)|.
\end{equation}

Lemma 3.12. Suppose that Assumption 2.30 holds for $S$. We take $n = d - 1$. Assume $D$ is not an $m$-fold multiple of a $-1$-curve for $m > 1$. Let $p : V \to M_{0,n}(S,D)$ be a map of an integral finite type $k$-scheme $V$ to $M_{0,n}(S,D)$, giving the stable map $f_V : (\mathbb{P}_V, p_{V_*}) \to S$ of an $n$-pointed genus $0$ curve $(\mathbb{P}_V, p_{V_*})$ over $V$. Let $v$ be a geometric generic point of $V$, giving the stable map $f : (\mathbb{P}, p_*) \to S$. We consider the image $ev(V) \subset S^n$.

1. Suppose that $\mathbb{P}$ has at least $3$ irreducible components. Then $\codim ev(V) \geq 2$.
2. Suppose that $f$ is non-birational. Then $\codim ev(V) \geq 2$.
3. Suppose that $\mathbb{P} = P_1 \cup P_2$ has $2$ irreducible components and $f$ is birational. Then $\codim ev(V) \geq 1$.
4. Suppose that $\mathbb{P} = P_1 \cup P_2$ has $2$ irreducible components, $f$ is birational and $\codim ev(V) = 1$. Suppose $\text{char } k = 0$, and either
   a) $d_S \geq 3$ or
   b) $d_S = 2$ and $d \neq 2,4$.
   Then $f$ is unramified on $\mathbb{P}$ and the image curve $C := f(\mathbb{P})$ has only ordinary double points as singularities. In particular, $f$ is an isomorphism to its image in a neighborhood of $\mathbb{P}_1 \cap \mathbb{P}_2$. Moreover, if $p_i$ has $n_i$ marked points, $i = 1, 2$, and $d_i = -K_S \cdot f_*(\mathbb{P}_i)$, then $d_i - 1 \leq n_i \leq d_i$, $i = 1, 2$.

Proof. We apply Lemma 3.10 to $f_V : (\mathbb{P}_V, p_{V_*}) \to S$. It suffices to prove the result for each stratum $V_i$ of $V$, so we may assume from the start that there is only a single stratum $V_1 = V$. Similarly since $k$ is perfect, we may assume that $W := W_1$ is smooth over $k$. Indeed, since $W_1$ is integral, it is reduced. So, since $k$ is perfect, it is geometrically reduced by [49 Tag 0201]. Thus, it is generically smooth by [49 Tag 056V], and we can apply Noetherian induction.

Denote the pullback family to $W$ by $(\mathbb{P}_W, p_{W_*})$ and the induced map to $S$ by $f_W : (\mathbb{P}_W, p_{W_*}) \to S$. Let $w$ be a geometric generic point of $W$ lying over
v, giving the stable map \( f_{W,w} : (\mathbb{P}_{W,w}, p_{W,w}) \to S \), with reduced image curve \( f_{W,w}(\mathbb{P}_{W,w}) = f(\mathbb{P})_{\text{red}} \) and with \( ev(W) = ev(V) \).

We now prove (1) in the case \( f \) is simple. If \( |S_s(f)| \geq 3 \), we are done by inequality (3.2). If \( |S_s(f)| = 2 \) then \( |S_s(f)| + |S_m(f)| \geq 1 \), so this case is also covered. If \( |S_s(f)| = 1 \) then \( S_s(f) = 0 \). If \( |S_m| \geq 2 \), we are again done. If \( |S_s(f)| = 1 \) and \( |S_m(f)| = 1 \), then for the unique \( T \in S_m \) we have \( n_T \geq 3 \) since \( T \) has at least two vertices. So, we are done by inequality (3.3). This completes the proof of (1) when \( f \) is simple.

It remains to discuss the case of non-simple \( f \). In this case, we consider the family \( \mathbb{P}_W \to W \) and map \( f_W : (\mathbb{P}_W, p_{W,*}) \to S \), with \( ev(W) = ev(V) \). If \( \mathbb{P}_W \) has at least 3 components, we are done by (1) applied to the family \( \mathbb{P}_W \). So, we can assume that \( \mathbb{P}_W \) has at most two components. By properties (3) and (4) of Lemma 3.10, since \( S \) is del Pezzo, the curve class \( D' := f_{W,w,*}(\mathbb{P}_{W,w}) \) has degree

\[
d' := D' \cdot (-K_S) < d.
\]

If only a single component \( P \) of \( \mathbb{P}_{W,w} \) is non-collapsed, then \( f(\mathbb{P})_{\text{red}} = f_{W,w}(P) \), hence is irreducible. Also, \( f_{W,w,*}(P) = D' \). By assumption \( f(\mathbb{P})_{\text{red}} \) is not a -1 curve, so \( D' \cdot (-K_S) \geq 2 \). So, by property (4) of Lemma 3.10, there exists \( m \geq 2 \) such that \( D = mD' \). Therefore, by property (3) of Lemma 3.10

\[
d = mD' \cdot (-K_S) \geq d' + 2.
\]

So, by inequality (3.2) applied to \( f_W \), we have

\[
\dim_k ev(W) \leq d' + n - 1 \leq d + n - 3.
\]

If \( \mathbb{P}_{W,w} \) has two non-collapsed components, then by inequalities (3.2) and (3.7), we obtain

\[
\dim_k ev(W) \leq d' + n - 2 \leq d + n - 3.
\]

This completes the proof of (1).

We now prove (2). By (1) we may assume \( \mathbb{P} \) has at most 2 components. So, if one component collapses to a point, that component has at least two marked points. It follows that the image under \( ev \) is contained in a diagonal of \( S^n \) whence has codimension at least \( \dim S = 2 \). Thus, it remains to consider the case when neither component collapses to a point. The assumption implies that there exists a geometric generic point \( a \in V \) such that \( f_a \) is not simple. Let \( d' \) be defined as in (3.7). Then \( d' < d \) by (3.7). Moreover, if \( \mathbb{P}_W \) has only one component, then \( d' < d - 1 \) by (3.8). Then apply (3.2) to \( f_W \). This completes the proof of (2).

We now prove (3). Define \( d' \) as in (3.7). If \( \mathbb{P}_W \) has only one component, then \( d' < d \) by (3.7). So, (3) follows by applying (3.2) to \( f_W \).

We now prove (4). So, we have codim \( ev(V) = 1 \). As we are only interested in the geometric generic point \( f \) and every geometric generic point of \( V \) lifts to a geometric generic point of \( W \), we may assume that the family \( \mathbb{P}_V \to V \) is tree-like with two irreducible components \( \mathbb{P}_{V,1}, \mathbb{P}_{V,2} \), and the map \( f_V : \mathbb{P}_V \to S \) decomposes as \( f_{V,1} \cup f_{V,2} \), with \( f_{V,i} : \mathbb{P}_{V,i} \to S \). Fixing a point \( v \in V \), let \( D_i \) be the curve class \( f_{V,i,*}(\mathbb{P}_{V,i,*}) \), let \( d_i = D_i \cdot (-K_S) \) and suppose that \( n_i \) of the \( n \) marked points of \( \mathbb{V}_V \) are in \( \mathbb{P}_{V,i} \), so \( d = d_1 + d_2, n = n_1 + n_2 \). The families \( f_{V,i} : \mathbb{P}_{V,i} \to S \) thus determine a morphism

\[
q : V \to M_{0,n_1}^{\text{bir}}(S, D_1) \times M_{0,n_2}^{\text{bir}}(S, D_2).
\]

The subset \( q(V) \) is a constructible subset of the product \( M_{0,n_1}^{\text{bir}}(S, D_1) \times M_{0,n_2}^{\text{bir}}(S, D_2) \). Let \( V' \) be a dense subset of \( q(V) \) such that \( V' \) is locally closed in the product. Then
$q^{-1}(V')$ is dense and open in $V$. Thus we may replace $V$ with $q^{-1}(V')$ and assume from the start that $V' = q(V)$ is locally closed in the product.

Let $ev_i : M_{0,n_i}^{\text{bir}}(S, D_1) \rightarrow S^n$ for $i = 1, 2$ denote the evaluation maps. The map $ev : V \rightarrow S^n$ factors through $q$ by $ev = (ev_1 \times ev_2) \circ q$. Moreover $M_{0,n_1}^{\text{bir}}(S, D_1) \times M_{0,n_2}^{\text{bir}}(S, D_2)$ is a fine moduli space, so the families $\mathbb{P}_{V,i}$ for $i = 1, 2$ are pulled back from $V'$. Let $g = (g_1, g_2)$ be a geometric generic point of $V'$. To prove (4) it suffices to show that $g_i$ is unramified for $i = 1, 2$ and the image curves of the $g_i$ have only ordinary double points and intersect transversally in $S$.

We claim that $V'$ is open in $M_{0,n_1}^{\text{bir}}(S, D_1) \times M_{0,n_2}^{\text{bir}}(S, D_2)$. By construction, $\text{codim}_{} ev_1 \times ev_2(V') = \text{codim}_{} ev(V) = 1$. Since $f$ is birational, neither component is contracted, so $d_1, d_2 \geq 1$. Thus by Assumption 2.30 and Lemma 2.19 we have $\dim M_{0,n_1}^{\text{bir}}(S, D_1) = n_1 + d_1 - 1$.

$$2n - 1 = \dim ev(V) = \dim ev_1 \times ev_2(V') \leq \dim V' \leq \dim M_{0,n_1}^{\text{bir}}(S, D_1) \times M_{0,n_2}^{\text{bir}}(S, D_2) = n_1 + d_1 - 1 + n_2 + d_2 - 1 = 2n - 1$$

Thus the inequalities are equalities. It follows that $V'$ is open in $M_{0,n_1}^{\text{bir}}(S, D_1) \times M_{0,n_2}^{\text{bir}}(S, D_2)$ as claimed.

We now prove the desired bounds on $n_i$. Indeed, $1 = \text{codim}_{} ev_1 \times ev_2(V') = \text{codim}_{} (M_{0,n_1}^{\text{bir}}(S, D_1)) + \text{codim}_{} (M_{0,n_2}^{\text{bir}}(S, D_2))$

So, for $i = 1, 2$.

$$1 \geq \text{codim}_{} ev_i (M_{0,n_i}^{\text{bir}}(S, D_i)) \geq 2n_i - (d_i + n_i - 1) = n_i - d_i + 1.$$

So, $n_i \leq d_i$. But $n_1 + n_2 = n = d - 1 = d_1 + d_2 - 1$, so $n_i \geq d_i - 1$.

It remains to prove that $C = f(\mathbb{P})$ has only ordinary double points. Since $V'$ is open in $M_{0,n_1}^{\text{bir}}(S, D_1) \times M_{0,n_2}^{\text{bir}}(S, D_2)$, it follows that $(g_1, g_2)$ is a geometric generic point of $M_{0,n_1}^{\text{bir}}(S, D_1) \times M_{0,n_2}^{\text{bir}}(S, D_2)$. By Assumption 2.30 $g_i$ is unramified for $i = 1, 2$. Let $C_i$ denote the image curve of $g_i$. By Proposition 2.32 $C_1$ and $C_2$ have only ordinary double points. Otherwise $d_i = 1$ and $g_i$ is an isomorphism to a $-1$ curve, which is smooth. Thus $C_1$ and $C_2$ have only ordinary double points. If $d_1$ or $d_2$ is at least 3, we apply Lemma 2.35 to conclude that $C_1$ and $C_2$ intersect transversally in $S$. Otherwise, $d_1, d_2 \leq 2$. We have three cases $(d_1, d_2) = (1, 1)$, $(1, 2)$, $(2, 2)$.

Suppose $(d_1, d_2) = (1, 1)$. Then $C_1$ and $C_2$ are distinct $-1$-curves because $f$ is birational. By hypothesis $d_S \geq 3$. Then $C_1$ and $C_2$ are embedded lines in $\mathbb{P}^d_S$ and must intersect transversally.

Suppose $d_S \geq 3$, so $S$ is anti-canonically embedded in $\mathbb{P}^d_S$. If $d = 3$, then we may assume $d_1 = 1, d_2 = 2$, so $C_1$ is a $-1$ curve and $C_2$ is a smooth conic. By the adjunction formula, we have $C_2 \cdot C_2 = 0$, and thus $C_2$ is a smooth rational curve on $S$ with trivial normal bundle. Noting the exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(C_2) \rightarrow i_{C_2}^{\mathcal{O}_S(C_2)} \rightarrow 0,$$

this implies that $h^0(S, \mathcal{O}_S(C_2)) = 2$. Thus the complete linear system $|C_2| \cong \mathbb{P}^1$ has dimension 1 and has no base-points. Moreover, there is open subset $U \subset |C_2|$ such that each $u \in U$ corresponds to a smooth rational curve $C_u$ in the curve class $|C_2|$. Since $\text{char } k = 0$, Bertini’s theorem applied to the linear system $|C_2| \cap C_1$ on $C_1$ implies that for all $u$ in a dense open subset $V$ of $U$, $C_u$ intersects $C_1$ transversely. Since $g_2$ is generic, this implies that $C_1$ and $C_2$ intersect transversely.
If \( d_1 = d_2 = 2 \), the same proof shows that \( C_1 \) and \( C_2 \) are smooth curves intersecting transversely.

The remaining case is \((d_1, d_2) = (1, 2)\), \( d = 3 \), \( d_S = 2 \). Let \( \pi : S \to \mathbb{P}^2 \) be the anti-canonical map, with smooth quartic branch curve \( E \). Thus \( C_1 \) is a \(-1\) curve. There are two possibilities for \( C_2 \): either \( \pi \) induces an isomorphism of \( C_2 \) with a smooth plane conic, or \( C_2 \to \pi(C_2) \) is a double cover, with \( \pi(C_2) \) a line \( \ell \) satisfying \( \ell \cdot E = p_1 + p_2 + 2p_3 \), with the \( p_i \) distinct points of \( E \) (see Lemma 2.29 and its proof); indeed, by Lemma 2.33 such lines are generic in the variety of tangent lines to \( E \). In the first case, we again have \( C_2 \cdot C_2 = 0 \) and we proceed as in the case \( d_S \geq 3 \), \( d = 3 \). Consider the second case. We note that the \(-1\) curve \( C_1 \) is one of the two components of \( \pi^{-1}(\ell') \), where \( \ell' \) is a line satisfying \( \ell' \cdot E = 2p_1 + 2p_2 \), with \( p_1, p_2 \) points of \( E \), not necessarily distinct. Since \( g_2 \) is generic, the lines \( \ell \) and \( \ell' \) intersect at a point \( q \) not on \( E \), so \( \pi \) is étale over a neighborhood of \( q \) and thus \( C_1 \) and \( C_2 \) intersect transversely.

Remark 3.13. If \( d_S = 2 \), then if the branch curve \( E \) of the anti-canonical map \( \pi : S \to \mathbb{P}^2 \) admits line \( \ell \) with a \( 4\)fold intersection at a single point \( q \) of \( E \), then \( \pi^{-1}(\ell) = C_1 \cup C_2 \), where the \( C_i \) are \(-1\) curves intersecting with multiplicity \( 2 \) at the point \( q' \) of \( S \) lying over \( q \). This is why we need to exclude the case \( d_S = d = 2 \) in (4) above.

Remark 3.14. Lemma 3.12 is false without the hypothesis that \( D \) is not an \( m\)-fold multiple of a \(-1\)-curve for \( m > 1 \). For example consider \( S = \text{Bl}_0 \mathbb{P}^2 \) over a field of characteristic \( 0 \), and let \( D \) be twice the exceptional divisor \( E \). Then \( d = 2 \).

The family over \( \mathbb{A}^1 \cong \text{Spec} k[a] \) given by \( f_{\mathbb{A}^1} : \mathbb{P}^1 \to \mathbb{P}^1 \) has \( \pi(\mathbb{A}^1) \) of codimension \( 1 \).

Corollary 3.15. Let \( k \) be a field of characteristic not \( 2 \) or \( 3 \) and let \( S \) be a del Pezzo surface over \( k \) with \( d_S \geq 2 \). Suppose that Assumption 2.30 holds for \( S \), and that \( D \) is a Cartier divisor on \( S \) which is not an \( m\)-fold multiple of a \(-1\)-curve for \( m > 1 \). Then \( \text{codim ev}(\overline{M}_{0, d-1}(S, D) \setminus M_{0, d-1}^{\text{op}}(S, D)) \geq 1 \).

Proof. By Lemma 3.12, \( \text{codim ev}(\overline{M}_{0, d-1}(S, D) \setminus M_{0, d-1}^{\text{bir}}(S, D)) \geq 1 \). By Proposition 2.32, the generic point of any irreducible component of \( M_{0, d}^{\text{bir}}(S, D) \) is in the ordinary double point locus. Thus the dimension of any irreducible component of \( M_{0, d}^{\text{bir}}(S, D) \) is \( 2(d - 1) \). See Lemmas 2.19 and Remark 2.7. Moreover, \( M_{0, d-1}^{\text{op}}(S, D) \) is dense in \( M_{0, d}^{\text{bir}}(S, D) \). Thus \( M_{0, d}^{\text{bir}}(S, D) \setminus M_{0, d-1}^{\text{op}}(S, D) \) is dimension \( < 2(d - 1) \), whence so is the closure of \( ev(M_{0, d}^{\text{bir}}(S, D) \setminus M_{0, d-1}^{\text{op}}(S, D)) \), which proves the claim.

4. Fine structure of the evaluation map

Let \( k \) be a perfect field, let \( S \) be a del Pezzo surface over \( k \) with effective Cartier divisor \( D \). Let \( d = -\deg K_S \cdot D \geq 1 \) and let \( n = d - 1 \). We introduce a list of assumptions, which will be convenient for future reference (but which are not running assumptions throughout this section).

Basic Assumptions 4.1.

1. \( \text{char } k = 0 \).
2. \( D \) is not an \( m\)-fold multiple of a \(-1\)-curve for \( m > 1 \).
3. One of the following holds.
\begin{itemize}
  \item $d_S \geq 4$
  \item $d_S = 3$ and $d \neq 6$
  \item $d_S = 2$ and $d \geq 7$
\end{itemize}

We do assume that $k$ is characteristic 0 in this section.

**Definition 4.2.** The locus of reducible stable maps is

$$\bar{M}_{0,n}(S,D)^{\text{red}} = \bar{M}_{0,n}(S,D) \setminus M_{0,n}(S,D).$$

The locus of balanced stable maps

$$\bar{M}_{0,n}(S,D)^{\text{bal}} \subset \bar{M}_{0,n}(S,D)^{\text{red}}$$

is the closure of the locus of stable maps $f : (\mathbb{P}, p_*) \to S$ satisfying the following conditions.

1. $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2$, with $\mathbb{P}_i \cong \mathbb{P}^1$ and $\mathbb{P}_1 \cap \mathbb{P}_2 = \{p\}$.
2. $f$ is unramified and $f|_{\mathbb{P}_i}$ is transversal to $f|_{\mathbb{P}_2}$ at $p$.
3. Letting $n_i$ denote the number marked points on $\mathbb{P}_i$, and letting
   $$d_i := -K_S \cdot f_*(\mathbb{P}_i)$$
   denote the degree of $f|_{\mathbb{P}_i}$, we have $d_i - 1 \leq n_i \leq d_i$ for $i = 1, 2$.

**Proposition 4.3.** Let $f : (\mathbb{P}, p_*) \to S$ be a stable map representing a geometric generic point of an irreducible component of $\bar{M}_{0,n}(S,D)^{\text{bal}}$. Then $ev$ is unramified at $f$.

**Proof.** Since $f$ is unramified, $f$ is birational (Lemma 2.8) and has no automorphisms, so $\bar{M}_{0,n}(S,D)$ is a scheme in a neighborhood of $f$. Applying [19] Tag 0B2G, the claim is equivalent to showing $ev$ is injective on tangent spaces. By Proposition 2.20 it thus suffices to show that $ev$ is surjective on tangent spaces.

We write $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2$, with $\mathbb{P}_i \cong \mathbb{P}^1$ and $\mathbb{P}_1 \cap \mathbb{P}_2 = \{p\}$. Let $n_i$ denote the number of marked points on $\mathbb{P}_i$, and let

$$D_i = f_*(\mathbb{P}_i), \quad d_i := -K_S \cdot D_i.$$ 

We may assume that $n_1 = d_1$ and $n_2 = d_2 - 1$ and $p_j$ lies on $\mathbb{P}_1$ for $j = 1, \ldots, n_1$.

Let $\nu : \bar{M}_{0,n}(S,D) \to \bar{M}_{0,n-1}(S,D)$ denote the map forgetting the first marked point and stabilizing. Let $f' = \nu(f)$. Let $n'_1 = n_1 - 1$ and $n'_2 = n_2$ be the number of marked points on the respective components of $f'$. We show that

$$de_{\nu} : T_{f'}\bar{M}_{0,n-1}(S,D)^{\text{red}} \to T_{\nu(f')}S^{n-1}$$

is surjective. Consider the following diagram.

\[
\begin{array}{ccc}
\bar{M}_{0,n'_1+1}(S,D_1) \times_S \bar{M}_{0,n'_2+1}(S,D_2) & \xrightarrow{\tau} & \bar{M}_{0,n-1}(S,D)^{\text{red}} \\
\downarrow \nu_1 \times \nu_2 & & \\
\bar{M}_{0,n'_1}(S,D_1) \times \bar{M}_{0,n'_2}(S,D_2) & \xrightarrow{ev_1 \times ev_2} & S^{n'_1} \times S^{n'_2} \xrightarrow{\sim} S^{n-1}.
\end{array}
\]

The fiber product over $S$ is taken with respect to the evaluation maps at the $(n'_i + 1)^{\text{th}}$ marked point on $\bar{M}_{0,n'_i+1}(S,D_i)$ for $i = 0, 1$. The map $\tau$ is the map that
attaches the domains at the \((n'_i + 1)\)th marked points forming a node. Observe that \(f'\) belongs to the image of \(\tau\). Let

\[ f'_i \in \bar{M}_{0,n'_i+1}(S,D_i) \]

be such that \(\tau(f'_1, f'_2) = f'\). By commutativity of the diagram, it suffices to show that \(d(\nu_1 \times \nu_2)(f'_1, f'_2)\) is surjective. By Lemma \[2.27\] \(d(\nu_i(v_i))\) is surjective for \(i = 1, 2\). So, using the commutativity of the diagram, it suffices to show that \(d(\nu_1 \times \nu_2)(f'_1, f'_2)\) is surjective. Indeed, let \(\xi_i \in \Gamma(P_i, \mathcal{N}_{\nu_i(f'_i)})\) represent tangent vectors at \(\nu(f'_i)\).

By condition \((2)\) of Definition \[4.2\],

\[ d f'_1(T_{p_{n'_i+1}} \mathbb{P}_1) \oplus d f'_2(T_{p_{n'_i+1}} \mathbb{P}_2) = T_{f(p)} S. \]

So, projecting along the respective summands, we obtain canonical isomorphisms

\[ (\mathcal{N}_{f'_1})_{p_{n'_i+1}} \cong T_{p_{n'_i+1}} \mathbb{P}_2, \quad (\mathcal{N}_{f'_2})_{p_{n'_i+1}} \cong T_{p_{n'_i+1}} \mathbb{P}_1. \]

Let \(v_i \in T_{p_{n'_i+1}} \mathbb{P}_i\) be the tangent vector corresponding to \(\xi_j(p_{n'_j+1})\) for \(j = 3 - i\). Then the tangent vectors at \(f'_i\) corresponding to \(\xi_i\) and \(v_i\) lift the tangent vectors corresponding to \(\xi_i\). Thus, we have established the surjectivity of \(dev_{f'}\).

We show that there exists a tangent vector \(v \in T_f \bar{M}_{0,n}(S,D) \setminus T_f \bar{M}_{0,n}(S,D)_{\text{red}}\) such that \(dev_f(v) = 0\) for \(j = 2, \ldots, n\).

\[
\begin{array}{ccc}
T_f \bar{M}_{0,n}(S,D)_{\text{red}} & \xrightarrow{dev_f} & T_{ev(f)} S^n \\
\downarrow{df} & & \downarrow{d\pi} \\
T_f \bar{M}_{0,n-1}(S,D)_{\text{red}} & \xrightarrow{dev_{f'}} & T_{ev(f')} S^{n-1}.
\end{array}
\]

Since \(f\) is generic, the domain curve of \(\nu(f)\) does not undergo stabilization. It follows that \(df\) is surjective. By the proceeding claim, \(dev_{f'}\) is surjective. So, for any \(v' \in T_f \bar{M}_{0,n}(S,D) \setminus T_f \bar{M}_{0,n}(S,D)_{\text{red}}\), we may pick \(w \in T_f \bar{M}_{0,n}(S,D)_{\text{red}}\) such that

\[ dev_{f'} \circ df(v) = d\pi \circ dev_f(v'). \]

Thus, we take \(v = v' - w\). To complete the construction of \(v\), we show that such \(v'\) exists. Indeed, by Proposition \[2.20\]

\[ \dim T_f \bar{M}_{0,n}(S,D) = d - 1 + n. \]

On the other hand, we compute

\[ \dim T_f \bar{M}_{0,n}(S,D)_{\text{red}} = d - 2 + n \]

as a transverse fiber product over \(S\). The dimension of the factors are given by Lemma \[2.19\]. The transversality over \(S\) follows from Lemma \[2.27\].

We claim that to complete the proof of the proposition it suffices to show that

\[
(4.1) \quad dev_1(v) \notin \text{im} d f_{p_1}(T_{p_1} \mathbb{P}).
\]

Indeed, let

\[ V = \text{span}\{v, T_{p_1} \mathbb{P}\} \subset T_f \bar{M}_{0,n}(S,D). \]
So, by construction of $v$, we have a diagram with short exact rows as follows.

$$
\begin{array}{ccc}
T_f \tilde{M}_{0,n}(S, D)^{\text{red}} & \xrightarrow{\text{dev}_f \circ \text{dev}_j} & \text{coker} \\
V & \xrightarrow{\text{dev}_f|_V} & T_f \tilde{M}_{0,n}(S, D) \\
T_f(p_1)S & \xrightarrow{\text{dev}_f} & \oplus_{i=1}^n T_f(p_i)S \\
\end{array}
$$

(Here coker denotes the quotient vector space $T_f \tilde{M}_{0,n}(S, D)/V$.) We showed above that $\text{dev}_f \circ \text{dev}_j$ is surjective, so $\text{dev}_f \circ \text{dev}_j$ is surjective. Thus $c$ is surjective. It follows from (4.1) that $\text{dev}_f|_V$ is surjective. So, $\text{dev}_f$ is surjective as desired.

Finally, we show (4.1). Let $F$ be the field of definition of $f$. Let $\xi : (\mathfrak{P}, p_s) \to S$ denote a stable map over $F[\epsilon]/(\epsilon^2)$ corresponding to $v$. We identify $\mathfrak{P}$ with the closed fiber of $\mathfrak{P}$, so $f|_p = f$. Choose an open set $p \in U \subset \mathfrak{P}$ with an open immersion $U \subset \text{Spec}(F[\epsilon]/(\epsilon^2))$. Choose an open set $f(p) \in V \subset S$ and maps $x, y : V \to \mathbb{A}^1$ such that $x \times y : V \to \mathbb{A}^2$ is étale and

(4.2) \[x \circ f \in s + (s^2, st, t^2), \quad y \circ f \in t + (s^2, st, t^2).\]

Consider the open subscheme

$$U' = \text{Spec}(F[\epsilon]/(\epsilon^2)) \cap U \subset U.$$  

The advantage of $U'$ is that it carries a well-defined vector field $\frac{\partial}{\partial \epsilon}$ along the locus $\{ \epsilon = 0 \}$. Such a vector field cannot exist on $U$ because the map $U \to \text{Spec}(F[\epsilon]/(\epsilon^2))$ is not smooth. We claim it suffices to show that the sections

$$\xi = df \left( \frac{\partial}{\partial \epsilon} \right) \bigg|_{\epsilon=0} \in \Gamma(U' \cap \mathbb{P}, f^* TS), \quad \eta = df \left( \frac{\partial}{\partial s} \right) \in \Gamma(U' \cap \mathbb{P}, f^* TS),$$

are generically linearly independent. Indeed, since $\text{dev}_1(v) = \xi(p_1)$, $\text{dev}_p(T_p \mathbb{P}) = \text{span}_F(\eta(p_1))$, and $p_1$ is generic, we obtain (4.1).

We show the generic linear independence of $\xi$ and $\eta$ as follows. Observe that

$$x \circ f|_U \in s + (\epsilon, s^2, st, t^2), \quad y \circ f|_U \in t + (\epsilon, s^2, st, t^2).$$

Let $Q \subset F(s)[\epsilon]/(\epsilon^2)$ denote the regular functions on $U'$, let $Z = U \cap \{ t = 0 \}$ and let $R \subset F[s](s)$ denote the regular functions on $Z$. Observe there is an inclusion map $R \to Q$ induced by the inclusion $F[s](s) \subset F(s)[\epsilon]/(\epsilon^2)$. For $a_1, a_2 \in Q$, we denote by $(a_1, a_2)_R \subset Q$ the set of all functions obtained by linear combinations of $a_1$ and $a_2$ with coefficients in $R$. Restricting to $U'$ amounts to replacing $t \mapsto \epsilon s^{-1}$, so we obtain,

$$x \circ f|_{U'} \in s + (\epsilon, s^2)_R, \quad y \circ f|_{U'} \in \epsilon s^{-1} + (\epsilon, s^2)_R.$$  

So,

$$dx(\xi) = d(x \circ f|_{U'}) \left( \frac{\partial}{\partial \epsilon} \right) \bigg|_{\epsilon=0}$$

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is regular at $p$, and
\[ dy = d(y \circ f|_{U'}) \left( \frac{\partial}{\partial x} \right)_{x=0} \]
has a simple pole at $p$. On the other hand, by equation (4.2) we have
\[ dx = d(x \circ f|_{Z}) \left( \frac{\partial}{\partial s} \right) \in 1 + (s), \quad dy = d(y \circ f|_{Z}) \left( \frac{\partial}{\partial s} \right) \in (s). \]
It follows that
\[ \det \begin{pmatrix} dx(\xi) \\ dy(\eta) \\ \ldots \end{pmatrix} \]
has a simple pole at $p$, so $\xi, \eta$, are generically linearly independent as desired. \(\square\)

**Lemma 4.4.** Suppose Basic Assumptions 4.1 hold for $k, S, D$, and suppose $M^\bir_{0, n}(S, D) = \emptyset$. Then $\text{codim } ev(M_{0, n}(S, D)) \geq 2$.

**Proof.** Since char $k = 0$, it follows by Lemma 2.31 that Assumption 2.30 holds. By Lemma 3.4 and Basic Assumption 4.1 (2), we conclude that $\text{codim } ev(M_{0, n}(S, D)) \geq 2$. Suppose by way of contradiction that
\[ c := \text{codim } ev(M_{0, n}(S, D) \setminus M_{0, n}(S, D)) \leq 1. \]
By Lemma 3.12 and Basic Assumption 4.1 (3), we see that $c = 1$ and $M_{0, n}(S, D)^{\text{bal}} \neq \emptyset$. So, Proposition 4.3 shows that $c = 0$, which is a contradiction. \(\square\)

**Theorem 4.5.** Suppose Basic Assumptions 4.1 hold for $k, S, D$. Then there is a closed subset $A \subset S^n$ with codim $A \geq 2$ such that the complement of the inverse image $M_{0, n}(S, D)^{\text{good}} := M_{0, n}(S, D) \setminus ev^{-1}(A)$ satisfies the following.

1. $M_{0, n}(S, D)^{\text{good}} = \emptyset$ if and only if $M^\bir_{0, n}(S, D) = \emptyset$. If $M^\bir_{0, n}(S, D) \neq \emptyset$, then the moduli space $M_{0, n}(S, D)^{\text{good}}$ is a geometrically irreducible smooth finite-type $k$-scheme, and the restriction of $ev$ to $ev : M_{0, n}(S, D)^{\text{good}} \rightarrow S^n \setminus A$ is a finite, flat, dominant morphism.

2. The evaluation map $ev$ is étale in a neighborhood of each $f \in M_{0, n}(S, D)^{\text{good}}$ with $t(f) = 0$.

3. $M_{0, n}(S, D)^{\text{good}}$ contains a dense open subset of $M^\odp_{0, n}(S; D)$.

4. Geometric points of $M^\odp_{0, n}(S, D)^{\text{good}}$ correspond to birational maps.

5. Let $f$ be a geometric point of $M_{0, n}(S, D)^{\text{good}} \setminus M^\odp_{0, n}(S; D)$, which we consider as a morphism $f : \mathbb{P} \rightarrow S$ for some genus $0$ semi-stable curve $\mathbb{P}$. Then $f$ satisfies:

   (i) If $\mathbb{P} = \mathbb{P}^1$ is irreducible, then the image curve $C := f(\mathbb{P}^1)$ has one singular point at $q$ that is not an ordinary double point, and $C$ has either an ordinary cusp, an ordinary tacnode or an ordinary triple point at $q$. Moreover, the marked points do not map to $q$ and $f$ is free.

   (ii) If $\mathbb{P}$ is not irreducible, then $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2$, with $\mathbb{P}_i \cong \mathbb{P}^1$. The image curve $C := f(\mathbb{P})$ has only ordinary double points as singularities. Moreover, if $n_i$ of the $n$ marked points of $\mathbb{P}$ are in $\mathbb{P}_i$, and $C_i := f(\mathbb{P}_i)$ has degree $d_i := -K_S \cdot C_i$, then $d_i - 1 \leq n_i \leq d_i$ for $i = 1, 2$.

**Remark 4.6.** In particular, if $M_{0, n}(S, D)^{\text{good}} \neq \emptyset$ then $ev : M_{0, n}(S, D)^{\text{good}} \rightarrow S^n \setminus A$ is dominant.

---

1. See Definition 2.29 for the definition of the torsion index $t(f)$. 

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Proof. As the assertions are all detected after a field extension of \( k \), we may assume that \( k \) is algebraically closed. The moduli stack \( \bar{M}_{0,n}(S, D) \) is a proper Artin stack over \( k \), so the morphism \( ev : \bar{M}_{0,n}(S, D) \to S^n \) is a proper morphism.

If \( M^\text{bir}_{0,n}(S, D) = \emptyset \), by Lemma 3.12, we can take \( A = ev(\bar{M}_{0,n}(S, D)) \) and

\[
\bar{M}_{0,n}(S, D)^\text{good} = \emptyset.
\]

Since \( M^\text{ord}_{0,n}(S, D) \subset M^\text{bir}_{0,n}(S, D) \) part (3) holds. The rest of the Proposition is immediate. Thus, for the remainder of the proof, we assume \( M^\text{bir}_{0,n}(S, D) \neq \emptyset \).

By Lemma 2.31 and Basic Assumption 4.1(2), we may apply Lemma 3.12. By Lemma 3.12(2), we may choose \( \bar{M}_{0,n}(S, D)^\text{good} \) so that geometric points \( f \) of \( M^\text{ord}_{0,n}(S, D)^\text{good} \) correspond to birational maps, showing (4).

We claim that we may choose \( \bar{M}_{0,n}(S, D)^\text{good} \) so that geometric points \( f : \mathbb{P}^1 \to S \) in \( M^\text{ord}_{0,n}(S, D)^\text{good} \setminus M^\text{ord}_{0,n}(S, D) \) are free. By semi-continuity of cohomology, the non-free locus is closed, and therefore has a finite number of irreducible components. By Lemma 3.3 with \( V \) the closure of a geometric point with the reduced substack structure, this will be accomplished by eliminating cases 1, 2, and 3 of Lemma 3.3. Case 3 of Lemma 3.3 contradicts Basic Assumption 4.1(3). We now eliminate case 2. Since \( f : \mathbb{P}^1 \to C \) is a two-to-one cover, we have \( D = f_*[\mathbb{P}^1] = 2C \). By assumption, we have that \( M^\text{bir}_{0,n}(S, D) \neq \emptyset \). Since \( d = 2 \), it follows from Lemma 2.31 Remark 2.7, and the vanishing of \( H^1(\mathbb{P}^1, \mathcal{O}(-1)) \) that \( M^\text{bir}_{0,n}(S, D) \neq \emptyset \). Choose a geometric point \( f' \) of \( M^\text{bir}_{0,n}(S, D) \). Let \( C' \) be \( f'(\mathbb{P}^1) \). We can not have \( C' = C \) because then \( f' \) would not be birational. Thus \( C' \cdot C \) must be positive, as it is the intersection of two distinct irreducible curves. On the other hand, \( C' \cdot C = 2C \cdot C = -2 \) because \( C \) is a 1 curve. In case 1, the map \( f \) belongs to \( M^\text{ord}_{0,n}(S, D)^\text{good} \), which we do not consider.

By the preceding claims of birationality and freeness,

\[
(\bar{M}_{0,n}(S, D)^\text{good} \cap \bar{M}_{0,n}(S, D)) \setminus M^\text{ord}_{0,n}(S, D) \subset M^\text{bir}_{0,n}(S, D)^{\text{fond}}.
\]

So Lemma 2.40 applies and (5) follows except for the claim about the marked points. However, the locus in \( (\bar{M}_{0,n}(S, D)^\text{good} \cap \bar{M}_{0,n}(S, D)) \setminus M^\text{ord}_{0,n}(S, D) \) where one of the marked points coincides with \( q \) is codimension 1, so we can redefine \( A \) to remove it.

Since we have chosen \( \bar{M}_{0,n}(S, D)^\text{good} \) so that geometric points are birational, we may apply Lemma 3.12 part (4). Note that by Basic Assumption 4.1(3), we have \( d_\ell \geq 2 \). Thus when the domain curve \( \mathbb{P} \) of \( f : \mathbb{P} \to S \) is reducible, we have \( \mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 \), with \( \mathbb{P}_i \cong \mathbb{P}^1 \), and the image curve \( C := f(\mathbb{P}) \) has only ordinary double points as singularities, and \( f : \mathbb{P} \to C \) is an isomorphism in a neighborhood of \( \mathbb{P}_1 \cap \mathbb{P}_2 \). Since \( \mathbb{P} \) has arithmetic genus 0, the intersection \( \mathbb{P}_1 \cap \mathbb{P}_2 \) consists of a single point.

For the bounds \( d_i - 1 \leq n_i \leq d_i \) in (5), proceed as follows. Let \( V \subset \bar{M}_{0,n}(S, D)^\text{good} \setminus \bar{M}_{0,n}(S, D) \) be an irreducible component. If codim \( ev(V) \geq 2 \), add \( ev(V) \) to \( A \). So, we may assume codim \( ev(V) \leq 1 \). Applying Lemma 3.12 parts (3) and (4) to the generic point of \( V \) proves the desired bounds and thus completes the proof of (5).

Next, we prove (3). By assumption, we have \( M^\text{bir}_{0,n}(S, D) \neq \emptyset \). So, it follows from Proposition 2.32 that \( \bar{M}_{0,n}(S, D)^\text{ord} \neq \emptyset \). Since \( d_\ell \geq 2 \), apply Lemma 2.27 to deduce that \( ev \) is étale on \( M^\text{ord}_{0,n}(S, D) \). Thus, for any proper closed subset \( A \subset S^n \), the complement of the preimage \( M^\text{ord}_{0,n}(S, D) \setminus ev^{-1}(A) \) is dense in \( M^\text{ord}_{0,n}(S, D) \).
So, \( \tilde{M}_{0,n}(S, D)^{\text{good}} = ev^{-1}(S^n \setminus A) \) contains a non-empty dense open subset of \( M_{0,n}^{\text{odp}}(S; D) \) proving part (3).

Since \( ev \) is unramified on the non-empty space \( M_{0,n}^{\text{odp}}(S; D) \), and \( M_{0,n}^{\text{odp}}(S; D) \) is smooth of dimension \( 2n \) by Lemma 2.27, it follows that \( ev : M_{0,n}^{\text{odp}}(S; D) \to S^n \) is dominant. By part (3), it follows that \( ev : \tilde{M}_{0,n}(S, D)^{\text{good}} \to S^n \setminus A \) is dominant as claimed in part (1).

We now show part (2). Let \( f : \mathbb{P} \to S \) with \( t(f) = 0 \) represent a point of \( \tilde{M}_{0,n}(S, D)^{\text{good}} \). In the case \( \mathbb{P} = \mathbb{P}^1 \), part (2) follows from Lemma 2.27. In the case \( \mathbb{P} \neq \mathbb{P}^1 \), it follows from (5)(ii) that \( f \) is balanced, so we may apply Proposition 4.3.

The geometric irreducibility in (1) is proved as follows. Since birationality is an open condition, property (4) implies \( \tilde{M}_{0,n}(S, D)^{\text{good}} \subset \bar{M}_{0,n}^{\text{bar}}(S, D) \). Since the inclusion \( \tilde{M}_{0,n}(S, D)^{\text{good}} \subset M_{0,n}(S, D) \) is open and dense, so is the inclusion \( \tilde{M}_{0,n}(S, D)^{\text{good}} \subset \bar{M}_{0,n}^{\text{bar}}(S, D) \). So geometric irreducibility follows from Theorem 2.16. The fact that \( ev^{-1}(S^n \setminus A) \) is a scheme as in (1) follows from the fact that by construction each \( f \in ev^{-1}(S^n \setminus A) \) is birational and hence has no automorphisms.

We now show the smoothness claim of (1). Let \( f \in \tilde{M}_{0,n}(S, D)^{\text{good}} \). If \( f \) is irreducible, by property (5)(i) it is free, so Lemma 2.19 asserts that \( \tilde{M}_{0,n}(S, D)^{\text{good}} \) is smooth at \( f \) of dimension \( 2n \). If \( f \) is reducible, then property (5)(ii) and Proposition 2.20 imply that \( \tilde{M}_{0,n}(S, D)^{\text{good}} \) is smooth at \( f \) of dimension \( 2n \).

To finish the proof of (1) and thus the proof of the proposition, we need to show that the map \( ev : ev^{-1}(S^n \setminus A) \to S^n \setminus A \) is finite and flat. By (2), it is generically finite. It is proper because \( \tilde{M}_{0,n}(S, D) \) and \( S \) are proper and properness is preserved under pull-back. Since \( S \) is smooth, after potentially adding to \( A \) a subset of codimension at least \( 2 \) in \( S \), we can assume that \( ev : ev^{-1}(S^n \setminus A) \to S^n \setminus A \) is finite and flat as desired. See, for example, Proposition 3.8 of [22].

From Theorem 4.5, we see that \( ev : \tilde{M}_{0,n}(S, D)^{\text{good}} \to S^n \setminus A \) is unramified on \( \tilde{M}_{0,n}(S, D)^{\text{good}} \setminus D_{\text{cusp}} \). We conclude our discussion of the structure of the evaluation map by computing the ramification index along \( D_{\text{cusp}} \).

Let \( f \) be a geometric point of \( D_{\text{cusp}} \cap \tilde{M}_{0,n}(S, D)^{\text{good}} \). We construct a linear map \( F : T_f \tilde{M}_{0,n}(S, D)^{\text{good}} \to T_f \tilde{M}_{0,n}(S, D)^{\text{good}} \), which in fact we will show to be injective for any such \( f \), even a geometric generic point. First, we construct vectors \( \mathcal{V}_a \) in the tangent space \( T_f \tilde{M}_{0,n}(S, D)^{\text{good}} \) of the base change to \( F \) of \( \tilde{M}_{0,n}(S, D)^{\text{good}} \) at the canonical point \( \hat{f} \) corresponding to \( f \); our map sends \( a \) in \( F \) to the image of \( \mathcal{V}_a \) in \( T_f \tilde{M}_{0,n}(S, D)^{\text{good}} \). This allows us to use the cohomological description of the tangent space to \( \tilde{M}_{0,n}(S, D)^{\text{good}} \) at a closed point while constructing tangent vectors at potentially non-closed points.

By Theorem 4.5, the point \( \hat{f} \) corresponds to a morphism \( \hat{f} : \mathbb{P}^1_F \to S_F \) which is birational onto its image \( C = f(\mathbb{P}^1_F) \) together with \( n = d - 1 \) distinct points \( p_1, \ldots, p_n \in \mathbb{P}^1(F) \). Moreover, there is a unique point \( p \in \mathbb{P}^1 \) where \( \hat{f} \) is ramified, and \( \hat{f}(\mathbb{P}^1) \) has a simple cusp at \( q = \hat{f}(p) \). We may assume \( p = 0 := [1 : 0] \in \mathbb{P}^1 \). Let \( t \) be the standard coordinate on \( \mathbb{P}^1 \setminus \{\infty\} \).

**Lemma 4.7.** We can choose a system of analytic coordinates \((x, y)\) at the ordinary cusp \( q \in C \subset S \) such that \( f \) is analytically locally of the form

\[
    f(t) = (t^2, t^3).
\]

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Proof. Since \( q \) is a simple cusp, we can find \( x, y \in \hat{\mathcal{O}}_{S,q} \) and \( u, v \in \hat{\mathcal{O}}^*_p \) such that \( f^*(x) = ut^2 \) and \( f^*(y) = vt^3 \). After rescaling by constants, we can assume \( u, v \in 1 + m_p \). So,

\[
f^*(x) = t^2 + \sum_{i \geq 3} a_i t^i, \quad f^*(y) = t^3 + \sum_{j \geq 4} b_j t^j.
\]

If \( a_i \neq 0 \) for some \( i \), we proceed by induction. Let \( k \) be the minimal integer such that \( a_k \neq 0 \). If \( k \) is even, we change coordinates on \( S \) by

\[
x \mapsto x - a_k x^{k/2}, \quad y \mapsto y.
\]

If \( k \) is odd, we change coordinates by

\[
x \mapsto x - a_k x^{k-3 \over 2} y, \quad y \mapsto y.
\]

After this procedure, the minimal integer \( k \) such that \( a_k \neq 0 \) increases. The infinite composition of these coordinate changes converges formally. Thus, we may assume that \( f^*(x) = t^2 \).

Similarly, if \( b_j \neq 0 \) for some \( j \), we proceed by induction. Let \( l \) be the minimal integer such that \( b_l \neq 0 \). If \( l \) is even, we change coordinates on \( S \) by

\[
x \mapsto x, \quad y \mapsto y - b_l x^{l/2}.
\]

If \( l \) is odd, we change coordinates by

\[
x \mapsto x, \quad y \mapsto y - b_l x^{l-3 \over 2} y.
\]

Again, an infinite composition of such coordinate changes converges formally, so we may assume \( f^*(y) = t^3 \). \( \square \)

We proceed with \((x, y)\) as in the preceding lemma. Fix an \( a \in F^* \) and let \( t' = X_0/X_1 = 1/t \) be the standard coordinate on \( U_1 := \mathbb{P}^1 \setminus \{0\} \). Since \( \partial / \partial t = -t^2 \partial / \partial t' \), the (rational) vector field \( (1/t) \cdot \partial / \partial t \) extends to a global section of \( \mathcal{T}_{\mathbb{P}^1}(p) \). Let

\[
\psi_a := (a/t) \partial / \partial t \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}(p)).
\]

Since \( df \) has a zero at \( p = 0 \), we have \( df(\psi_a) \in H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}(S)) \). Thus \( df(\psi_a) \) induces a tangent vector to \( \mathcal{M}_0(S,D)_{\text{good}} \) corresponding to its image in \( H^0(\mathbb{P}^1, \mathcal{N}_{\tilde{f}}) \) along with a first order deformation \( \tilde{f}_{t \epsilon} \) of \( \tilde{f} \). Let \( f_{t \epsilon} \) denote the first order deformation of \( f \) given by projecting \( f_{t \epsilon} \) along \( S_F \to S \).

**Lemma 4.8.** Suppose Basic Assumptions \([4.7]\) hold for \( k, S, D \). Let \( f \) be a geometric point of \( D_{\text{cusp}} \cap \bar{M}_{0,n}(S,D)_{\text{good}} \) with field of definition \( F \). Then

1. The first order deformation \( f_{t \epsilon} \) defined above extends to a deformation \( f_{t \epsilon} : \mathbb{P}^1 \otimes F[[\epsilon]] \to S_F \) such that there are local analytic coordinates \((x, y)\) on \( S \) such that near \( p \) the map \( f_{t \epsilon} \) is of the form

\[
f_{t \epsilon}(t) = (t^2, t^3) + \epsilon \cdot (2a, 3at) \mod \epsilon^2.
\]

(2) There is a deformation of marked points \((p_{1 \epsilon}, \ldots, p_{n \epsilon})\) such that the tangent vector \( V_{a \epsilon} \) corresponding to the deformation \((f_{t \epsilon}, p_{1 \epsilon}, \ldots, p_{n \epsilon})\) is in \( \ker d(ev) \).

(3) Let \( \nu : \text{Spec}(F[[\epsilon]]) \to \bar{M}_{0,n}(S,D)_{\text{good}} \) be the morphism corresponding to \((f_{t \epsilon}, p_{1 \epsilon}, \ldots, p_{n \epsilon})\). The composition of \( ev \circ \nu \) has ramification index \( 2 \), i.e., \( e_f(ev \circ f_{t \epsilon}) = 2 \).

In particular, the map \( ev : \bar{M}_{0,n}(S,D)_{\text{good}} \to S^n \) has ramification index two along \( D_{\text{cusp}} \), i.e., for \( f \in D_{\text{cusp}} \cap \bar{M}_{0,n}(S,D)_{\text{good}} \) a geometric generic point, \( e_f(ev) = 2 \).
Proof. We may assume that \( f \) is a closed point, so \( \hat{f} = f \). Let \( C, p, q, t, x, y, p_1, \ldots, p_n \) be as in the above discussion. Let \( \mathcal{L} \subset f^* T_S \) be the kernel of the quotient map \( f^* T_S \to N_f / N_{f, \text{tor}} \). Then \( \mathcal{L} \) is an invertible subsheaf of \( f^* T_S \) containing the image \( df(T_{p_1}) \). By the diagram

\[
\begin{array}{c}
0 \longrightarrow T_{p_1} \longrightarrow f^* T_S \longrightarrow N_f \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 \longrightarrow \mathcal{L} \longrightarrow f^* T_S \longrightarrow N_f / N_{f, \text{tor}} \longrightarrow 0
\end{array}
\]

and the snake lemma, we have \( N_{f, \text{tor}} = \mathcal{L} / df(T_{p_1}) \). Since \( df \) vanishes to first order at \( p \) and nowhere else, the map \( \mathcal{L} \to T_{p_1} \) identifies \( \mathcal{L} \) with \( \mathcal{L} \simeq T_{p_1}(p) \simeq \mathcal{O}_{p_1}(3) \).

Since \( \det(f^* T_S) \simeq \mathcal{O}(d) \), it follows from the exact sequence

\[
0 \to \mathcal{L} \to f^* T_S \to N_f / N_{f, \text{tor}} \to 0,
\]

that

\[
N_f / N_{f, \text{tor}} \simeq \mathcal{O}_{p_1}(d - 3).
\]

If \( d_S \geq 3 \), then \( S \) embeds into \( \mathbb{P}^N \) by \( -K_S \), and one can choose a hyperplane in \( \mathbb{P}^N \) passing through the cusp \( q \) and one other point of \( C = f(\mathbb{P}^1) \). This hyperplane has intersection multiplicity \( d \geq 4 \) with \( D \). So \( d \geq 4 \) for any \( (d_S, d) \) allowed by the hypothesis. Thus \( N_f / N_{f, \text{tor}}(-1) \) is generated by global sections and \( H^1(\mathbb{P}^1, N_f(-1)) = 0 \); in particular, \( f \) is free.

For a morphism \( \phi : Y \to X \) of smooth varieties over \( k \) and a geometric point \( y \in Y \) with image \( x = \phi(y) \), we have the differential

\[
d\phi_y : T_{Y,y} \to T_{X,x}.
\]

The second order differential is the map

\[
d^2 \phi_y : \text{Sym}^2_k(y)(T_{Y,y} \otimes k(y)) \to \text{coker}(d\phi_y \otimes k(y)),
\]

defined as follows. The map \( \phi^* : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \) induces the map on jet spaces

\[
\mathcal{J}^2 \phi^* : \mathcal{J}^2 \mathcal{O}_{X,x} = \mathcal{O}_{X,x}/m_x^2 \to \mathcal{J}^2 \mathcal{O}_{Y,y} = \mathcal{O}_{Y,y}/m_y^2,
\]

and thus \( \mathcal{J}^2 \phi^* \) induces a map of the kernel of \( d\phi^* : \Omega^1_{X,x} \otimes k(x) \to \Omega^1_{Y,y} \otimes k(y) \) to the subspace

\[
\text{Sym}^2 \Omega^1_{Y,y} \otimes k(y) \cong \text{Sym}^2(m_y^1/m_y^2) \cong m_y^2/m_y^3 \subset \mathcal{J}^2 \mathcal{O}_{Y,y}.
\]

The map \( d^2 \phi_y \) is the dual of this map.

We show that \( d^2 ev_f \) is nonvanishing. Let \( q_i = f(p_i) \in S(F) \). The sheaf sequence

\[
0 \to N_f(-\sum_i p_i) \to N_f \to \oplus f^* T_{S,q_i} / df(T_{p_1,p_i}) \to 0
\]

identifies the cokernel of \( d ev_f : T_f(M_{0,n}(S, D)) \to T_{ev(f)}(S^n) \) with \( H^1(\mathbb{P}^1, N_f(-\sum_i p_i)) \).

See Remark 2.21. We will show \( d^2 ev_f \neq 0 \), by showing its restriction to \( \ker d ev_f \) is nonvanishing. Since \( N_f(-\sum_i p_i) \cong N_{f, \text{tor}} \oplus \mathcal{O}_{p_1}(-2) \), the sequence (4.4) similarly gives an identification

\[
\ker d ev_f \cong H^0(N_{f, \text{tor}}) \cong H^0(i_{p_1} F).
\]
We are interested in showing that the map
\[
d^2ev_f : \text{Sym}^2 H^0(N_f^{\text{tor}}) \cong F \to H^1(\mathbb{P}^1, N_f(-\sum p_i)) \cong F.
\]
is nonzero. We will consider \(d^2ev_f\) as a quadratic form on \(F\), sending \(a \in F\) to \(d^2ev_f(V_a)\) where \(V_a\) is the tangent vector \(V_a \in T_f(M_{0,n}(S,D)^{\text{good}})\) corresponding to \(i_p(a)\).

Noting that \(H^1(\mathbb{P}^1, N_f(-\sum p_i)) \cong H^1(\mathbb{P}^1, N_f^\text{tor}(-\sum p_i))\), we will compute \(d^2ev_f\) by composing with the Serre duality pairing
\[
H^1(\mathbb{P}^1, N_f^\text{tor}(-\sum p_i)) \times H^0(\mathbb{P}^1, K_{\mathbb{P}^1} \otimes (N_f^\text{tor})^\vee(\sum p_i)) \to F.
\]
In fact, since \(N_f^\text{tor} \cong O_{\mathbb{P}^1}(d-3) = O_{\mathbb{P}^1}(n-2)\), we have
\[
K_{\mathbb{P}^1} \otimes (N_f^\text{tor})^\vee(\sum p_i) \cong O_{\mathbb{P}^1}.
\]
so there is a unique (up to a non-zero scalar) section \(\omega \in H^0(\mathbb{P}^1, K_{\mathbb{P}^1} \otimes (N_f^\text{tor})^\vee(\sum p_i))\).

We will show that \((\omega, d^2ev_f(V_a^2)) \neq 0\) for nonzero \(a\).

We construct a deformation \((f, p_1, \ldots, p_n)\) of \((f, p_1, \ldots, p_n)\) over \(F[[\epsilon]]\) with first order deformation corresponding to \(V_a\) as follows. The invertible sheaf \(L\) has local generator \(\lambda := 2 \cdot \partial/\partial x + 3t \cdot \partial/\partial y\) near \(p\), with \(t \cdot \lambda = df(\partial/\partial t)\). Recall above we set
\[
v_a := (a/t)\partial/\partial t \in H^0(\mathbb{P}^1, T_{\mathbb{P}^1}(p))
\]
for \(a \in F^*\), so \(df(v_a) = a \cdot \lambda\) in \(L \otimes \Omega_{\mathbb{P}^1,0}\). Let \(s_a := df(v_a) \in H^0(\mathbb{P}^1, L)\). In particular, \(s_a \in H^0(\mathbb{P}^1, \mathcal{L})\) is in \(H^0(\mathbb{P}^1, N_f) \subseteq H^0(\mathbb{P}^1, N_f^{\text{tor}}) \subseteq H^0(\mathbb{P}^1, N_f)\).

Fixing the isomorphism \(i_{p^*} F \cong N_f^{\text{tor}}\) to be given by \(a \mapsto a[\lambda]\), we see that
\[
(4.6) \quad i_{p^*} a = [s_a] \in H^0(N_f^{\text{tor}}).
\]
Thus \(V_a\) corresponds to the first order deformation \(f_{1,\epsilon}\) of \(f\) corresponding to the image of \(s_a\) in \(H^0(\mathbb{P}^1, N_f)\) equipped with appropriate marked points. Since \(H^1(\mathbb{P}^1, N_f) = \{0\}\), the 1st order deformation \(f_{1,\epsilon}\) extends to a deformation \(f_{\epsilon}\) over \(F[[\epsilon]]\). Locally in the coordinate system \((t, x, y)\), the map \(f_{\epsilon}\) is of the form
\[
f_{\epsilon}(t) = \left(\frac{t^2}{t^3} + \epsilon \cdot (2a, 3at) \mod \epsilon^2\right)
\]
This shows \([1]\).

We choose points of \(\mathbb{P}^1(F[[\epsilon]])\) deforming the \(p_i\) such that \((f_{\epsilon}, p_{1,\epsilon}, \ldots, p_{n,\epsilon})\) determines an element of \(\ker d(ev)\) as follows. The global vector field \(v_a\) on \(U_1\) gives us the automorphism \(\phi_{\epsilon}\) of \(U_1 \times \text{Spec } F[[\epsilon]]\)
\[
\phi_{\epsilon}(t') = t' - \epsilon \cdot at^3.
\]
In coordinates \((t, \epsilon)\), this is
\[
\phi_{\epsilon}(t) = \frac{1}{1/t - \epsilon \cdot at^3} = t + \epsilon \cdot \frac{a}{t} + \epsilon^2 \cdot \frac{a^2}{t^2} \mod \epsilon^3.
\]
Let \(p_{1,\epsilon} = \phi_{-\epsilon}(p_i) \in \mathbb{P}^1(F[[\epsilon]])\). Although the \(t, (x,y)\) coordinate system is not necessarily valid near \(p_i\), we have
\[
f_{\epsilon}(p_{1,\epsilon}) \equiv f(p_i) \mod \epsilon^2
\]
because
\[
\frac{\partial}{\partial \epsilon} f_\epsilon(p)_{\epsilon=0} = \frac{\partial}{\partial \epsilon} f_{\epsilon=0}(p) + df \left( \frac{\partial}{\partial \epsilon} p_{\epsilon=0} \right)
\]
\[
= s_\alpha(p) + df \left( \frac{\partial}{\partial \epsilon} \phi_{-\epsilon}(p)_{\epsilon=0} \right)
\]
\[
= s_\alpha(p) + df(-v_\alpha(p))
\]
\[
= s_\alpha(p) - s_\alpha(p) = 0.
\]

Thus the \( F[\epsilon] \) point \( v : \text{Spec}(F[\epsilon]) \to M_{0,n}(S, D)^{\text{good}} \) given by \((f_\epsilon, p_1, \ldots, p_n)\) determines a tangent vector which is in ker \( d(\epsilon v) \). It follows from (4.5) that ker \( d(\epsilon v) \)
is 1-dimensional. Therefore, by (4.6) the tangent vector corresponding to \( \epsilon \) equals \( \nu_\alpha \in T_f(M_{0,n}(S, D)^{\text{good}}) \). This shows (2).

We now give a cocycle representing \( d^2 ev_f(V_a^2) \) in \( H^1(\mathbb{P}^1, \mathcal{N}_f / \mathcal{N}_f^{\text{tor}}(\sum_{i}(-p_i))) \) in terms of a morphism \( h : U_1[[\epsilon]] \to S \), defined by

\[ h(t', \epsilon) := f_\epsilon \circ \phi_{-\epsilon}(t'). \]

Note that \( dh(t',0)(\partial/\partial \epsilon) = 0 \) by a chain rule calculation similar to the above, whence the canonical map \( \text{coker}(d\epsilon f_\epsilon) \to \text{coker}(dh(t',0)) \) is an isomorphism. It follows from (4.3) that \( d^2h(\partial/\partial \epsilon_{|\epsilon=0})^2 \) is thus a section of \( H^0(U_1, \mathcal{N}_f)^2 \). We let \( \partial^2 h/\partial \epsilon^2_{|\epsilon=0} \in H^0(U_1, \mathcal{N}_f) \) denote this section \( d^2h(\partial/\partial \epsilon_{|\epsilon=0})^2 \).

Let \( t'_i \) be the \( t' \) coordinate of \( p_i \). We compute

\[ (4.7) \quad \partial^2 h(t'_i, \epsilon)/\partial \epsilon^2_{|\epsilon=0} = \frac{d^2 f_\epsilon(p_{\epsilon=0})}{d\epsilon^2}_{|\epsilon=0} \]
in \( f^* T_{S, q_i} / T_{P_1, p_i} = \mathcal{N}_f \otimes F(p_i) \).

On the other hand,

\[ (4.8) \quad d^2 ev_f(V_a^2) = \partial(\ldots, \frac{d^2 f_\epsilon(p_{\epsilon=0})}{d\epsilon^2} |_{\epsilon=0}, \ldots), \]

where

\[ \partial : \bigoplus_{i=1}^n f^* T_{S, q_i} / df(T_{P_1, p_i}) \to H^1(\mathbb{P}^1, \mathcal{N}_f(-\sum_{i=1}^n p_i)) \]
is the boundary map in the cohomology sequence associated to (4.4).

Let \( U \) be the cover of \( \mathbb{P}^1 \) given by \( U_1 = \mathbb{P}^1 \setminus \{0\}, U_0 = \mathbb{P}^1 \setminus \{p_1, \ldots, p_n\} \). This is indeed a cover because the \( p_i \) do not coincide with the point \( p = 0 \) where \( f \) is ramified by Theorem 4.3 (5) (i). By (4.8), (4.7), we can represent \( d^2 ev_f(V_a) \) as the 1-cocycle in \( C^1(U, \mathcal{N}_f / \mathcal{N}_f^{\text{tor}}(\sum_{i}(-p_i))) \) given by

\[ [\partial^2 h/\partial \epsilon^2_{|\epsilon=0}] \in H^0(U_0 \cap U_1, \mathcal{N}_f(\sum_{i}(-p_i))). \]

Moreover, the trace map \( H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1/F}) \xrightarrow{\text{Res}} F \) sends a class \([\eta] \in H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1/F})\) represented by some \( \eta \in C^1(U, \omega_{\mathbb{P}^1/F}) = H^0(U_0 \cap U_1, \omega_{\mathbb{P}^1/F}) \) to the residue Res_{0}\eta.

---

2Another point of view on this is that for any vector field \( v \) on \( U_1[[\epsilon]] \) extending \( \partial/\partial \epsilon |_{\epsilon=0} \), including the examples \( v = \partial/\partial \epsilon v \) and \( v = \partial/\partial \epsilon + \epsilon\partial/\partial t' \), one obtains a section in \( H^0(U_1, h^* T S) \); because the section \( dh(v) \in H^0(U_1[[\epsilon]], h^* T S) \) vanishes along \( \epsilon = 0 \) and a section of a vector bundle has a first derivative which lies in the same vector bundle restricted to the vanishing locus of the section by taking the derivative in some local trivialization. The image of this section in \( H^0(U_1, \mathcal{N}_f) \) is independent of the choice of \( v \).
For $\omega \in H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1/F} \otimes (\mathcal{N}_f/\mathcal{N}_f)^\vee(\sum_i p_i))$, the pairing $\langle \omega, d^2 ev_t(V_0) \rangle$ is therefore given by

$$
\langle \omega, d^2 ev_t(V_0) \rangle = \text{Res}_0 \partial^2 h/\partial x^2|_{x=0} \cdot \omega
$$

where $\partial^2 h/\partial x^2|_{x=0} \cdot \omega$ is to be considered as a section of $\omega_{\mathbb{P}^1/F}$ over $U_0 \cap U_1$ via the pairing

$$
\omega_{\mathbb{P}^1/F} \otimes (\mathcal{N}_f/\mathcal{N}_f)^\vee(\sum_i p_i) \otimes \mathcal{N}_f/\mathcal{N}_f(-\sum_i p_i) \to \omega_{\mathbb{P}^1/F}.
$$

To make the computation of $\text{Res}_0$, we use a trivialization of $\mathcal{N}_f/\mathcal{N}_f^\text{tor}$ in a neighborhood of 0 given as follows: Since $\mathcal{L}$ has local generator $\lambda = 2 \partial/\partial x + 3t \partial/\partial y$, and $\mathcal{N}_f/\mathcal{N}_f^\text{tor} = f^* T_{\mathbb{P}^1} \otimes \mathcal{L}$, sending a section $\alpha \partial/\partial x + \beta \partial/\partial y$ of $f^* T_{\mathbb{P}^1}$ to $2\beta - 3at$ descends to give an isomorphism of $\mathcal{N}_f/\mathcal{N}_f^\text{tor}(-\sum_i p_i)$ with $\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{1_\mathbb{P}}$ over $\mathbb{P}^1 \setminus \{p_1, \ldots, p_n, \infty\}$.

Define $\gamma \in \mathcal{O}_{\mathbb{P}^1,0}$ so that $\omega$ will transform to a 1-form $\gamma(t)dt$ via this isomorphism.

Combining the previous local expressions for $f_\sigma(t)$ and $\phi_\tau(t)$, we have

$$
h(t, \epsilon) = (t^2, t^3) + t^2 \left( \frac{3a^2}{t^2} - \frac{3a^2}{t} \right) \mod t^3
$$
in local coordinates $t, (x, y), \epsilon$. Thus $\frac{\partial^2 h}{\partial x^2}|_{x=0}$ maps to $-6a^2/t$ under this local trivialization of $\mathcal{N}_f/\mathcal{N}_f^\text{tor}(-\sum_i p_i)$. Thus

$$
\frac{\partial^2 h}{\partial x^2}|_{x=0} \cdot \omega = -6a^2/t \cdot \gamma(t)dt,
$$

which yields

$$
\text{Res}_0 \frac{\partial^2 h}{\partial x^2}|_{x=0} \cdot \omega = -6\gamma(0) \cdot a^2.
$$

Thus, the quadratic form $d^2 ev_t$ is nonzero as claimed and hence the ramification index is two. \hfill \Box

**Proposition 4.9.** Let $k$ be a field of characteristic 0. Suppose Basic Assumptions 4.1 hold for $k, S, D$. Take $ev : \bar{M}_{0,n}(S, D)^\text{good} \to S^n \setminus A$ as given by Theorem 4.5. We suppose that $\bar{M}_{0,n}(S, D)^\text{good}$ is non-empty.

(1) $ev$ is ramified along $D_{\text{cusp}}$ with ramification index two: at each geometric generic point $f$ of $D_{\text{cusp}}$, there are local analytic coordinates $t_1, \ldots, t_{2n}$ for $\bar{M}_{0,n}(S, D)^\text{good}$ at $f$ and $s_1, \ldots, s_{2n}$ for $S^n$ at $ev(f)$ such that $D_{\text{cusp}}$ has local defining equation $t_1$ and $ev^*(s_i) = t_i^2$, $ev^*(s_i) = t_i$ for $i = 2, \ldots, 2n$.

(2) The determinant of the section $d(ev) : \mathcal{O}_{\bar{M}_{0,n}(S, D)^\text{good}} \to \omega_{ev}$ of the relative canonical sheaf $\Omega_{ev} := \Omega_{\bar{M}_{0,n}(S, D)^\text{good}/k} \otimes ev^*(\Omega_{S^n/k})^{-1}$ has divisor $1 \cdot D_{\text{cusp}}$, and thus determines an isomorphism $\det d(ev) : \mathcal{O}_{\bar{M}_{0,n}(S, D)^\text{good}/k} \otimes ev^*(\Omega_{S^n/k})^{-1} \to \omega_{ev}$, where $\omega_{ev} \cong \omega_{\bar{M}_{0,n}(S, D)^\text{good}/k} \otimes ev^*(\omega_{S^n/k})^{-1}$.

**Proof.** Noting that $\bar{M}_{0,n}(S, D)^\text{good}$ and $S^n$ are both smooth $k$-schemes, it follows that $\omega_{ev}$ is an invertible sheaf. Moreover, $ev$ is flat and is unramified over $\bar{M}_{0,n}(S, D)^\text{good} \setminus D_{\text{cusp}}$, so we need only show that the ramification index of $ev$ along $D_{\text{cusp}}$ is two, which is Lemma 4.8. (2) is an immediate consequence of (1) and the theorem on purity of the branch locus. \hfill \Box
5. The double point locus

We define the double point locus using ideas from [11 Chapter 9.3].

**Definition 5.1.** Given a composition of closed immersions \( Z \subset W \subset X \), we define the subscheme of \( W \) residual to \( Z \) to be the subscheme defined by the ideal sheaf \( (I_W : I_Z) \). Recall that this is the ideal sheaf of all local sections \( s \) of \( O_X \) such that \( s \) lies in \( I_W \) for all local sections \( t \) of \( I_Z \).

Let \( S \) be a smooth del Pezzo surface over a perfect field \( k \) equipped with an effective Cartier divisor satisfying Assumption 4.1(2)(3). We work mostly in characteristic 0, but also have some analysis in characteristic \( p \). Throughout this section, we assume that \( \bar{M}_{0,n}(S,D)_{odp} \) is non-empty, which implies that \( M_{0,n}^{bir}(S,D) \) is non-empty.

**Remark 5.2.** In characteristic 0, \( \bar{M}_{0,n}(S,D)_{odp} \) is non-empty if and only if \( M_{0,n}^{bir}(S,D) \) is non-empty, and these are both equivalent to \( \bar{M}_{0,n}(S,D)_{good} \) being non-empty by Theorem 4.5(1)(3). In characteristic \( p \) under Assumption 2.30 we also have that \( \bar{M}_{0,n}(S,D)_{odp} \) is non-empty if and only if \( M_{0,n}^{bir}(S,D) \) is non-empty by Proposition 2.32. So we could equally well assume that \( M_{0,n}^{bir}(S,D) \) is non-empty. When \( M_{0,n}^{bir}(S,D) \) is empty but \( (k,S,D) \) satisfies Assumption 4.1(2)(3) and Assumption 2.30, the associated Gromov-Witten invariants are defined to be 0. Note the consistency with Lemma 4.4 and Corollary 3.15 over a dense open of \( S^n \), \( ev \) has empty fiber in this case, so it makes sense to define the degree and the count to be 0.

Let \( d = -\deg K_S \cdot D \geq 1 \). Let \( n = d - 1 \). Let \( \bar{X}_{0,n} \to \bar{M}_{0,n}(S,D) \) denote the universal curve, \( \bar{X}_{0,n} := \bar{M}_{0,n+1}(S,D) \) and let \( ev: \bar{X}_{0,n} \to S \times \bar{M}_{0,n}(S,D) \) denote the universal map, or in other words, the product of evaluation on the \((n+1)\)st marked point with the canonical projection. If the characteristic of \( k \) is 0, we may apply Theorem 4.5 and obtain the smooth \( k \)-scheme \( \bar{M}_{0,n}(S,D)_{good} \). Let \( \bar{X}_{0,n}^{good} \to \bar{M}_{0,n}(S,D)_{good} \) denote the pullback of \( \bar{X}_{0,n} \to \bar{M}_{0,n}(S,D) \) to \( \bar{M}_{0,n}(S,D)_{good} \). In particular, \( \bar{X}_{0,n}^{good} \) is a scheme. In positive characteristic, the map \( \bar{X}_{0,n}^{good} \to \bar{M}_{0,n}(S,D)_{good} \) will be replaced by the pullback \( \bar{X}_{0,n}^{odp} \to \bar{M}_{0,n}(S,D)_{odp} \) of \( \bar{X}_{0,n} \to \bar{M}_{0,n}(S,D) \) to the locus \( M_{0,n}(S,D)_{odp} \) of parametrized curves with only ordinary double points.

Work in schemes over \( \bar{M}_{0,n}(S,D)_{good} \). So for example, we have \( S \times \bar{M}_{0,n}(S,d)_{good} \) and 

\[
\Delta_S \hookrightarrow (S \times \bar{M}_{0,n}(S,d)_{good}) \times_{\bar{M}_{0,n}(S,d)_{good}} (S \times \bar{M}_{0,n}(S,d)_{good}) \cong S \times S \times \bar{M}_{0,n}(S,d)_{good}
\]

Consider the product of the universal maps 

\[
ev \times ev: \bar{X}_{0,n}^{good} \times_{\bar{M}_{0,n}(S,d)_{good}} \bar{X}_{0,n}^{good} \to S \times S \times \bar{M}_{0,n}(S,d)_{good}.
\]

The preimage \( (ev \times ev)^{-1}(\Delta_S) \) of the diagonal \( \Delta_S \subset S \times S \times \bar{M}_{0,n}(S,d)_{good} \) contains the diagonal \( \Delta_{\bar{X}_{0,n}^{good}} \subset \bar{X}_{0,n}^{good} \times_{\bar{M}_{0,n}(S,d)_{good}} \bar{X}_{0,n}^{good} \) as one irreducible component.

**Definition 5.3.** Under the hypotheses of Theorem 4.5, the double point locus is the subscheme 

\[
D \subset \bar{X}_{0,n}^{good} \times_{\bar{M}_{0,n}(S,d)_{good}} \bar{X}_{0,n}^{good}
\]

defined to be the subscheme of \( (ev \times ev)^{-1}(\Delta_S) \) residual to \( \Delta_{\bar{X}_{0,n}^{good}} \). Let 

\[
\pi: D \to \bar{M}_{0,n}(S,D)_{good}
\]

55
denote the canonical map.

Now drop the assumption that the characteristic of $k$ is 0. For $S$ a smooth del Pezzo surface over $k$ and $D$ an effective Cartier divisor, define
\[
\mathcal{D}_{\text{odp}} \subset \mathcal{X}_{0,n}^{\text{odp}} \times \bar{M}_{0,n}(S,D)^{\text{odp}} \mathcal{X}_{0,n}^{\text{odp}}
\]
to be the subscheme of \((ev \times ev)^{-1}(\Delta_S)\) residual to $\Delta_{\mathcal{X}_{0,n}^{\text{odp}}}$ and let $\pi$ denote the projection map $\pi : \mathcal{D}_{\text{odp}} \to \bar{M}_{0,n}(S,D)^{\text{odp}}$.

**Lemma 5.4.** Let $k$ be a perfect field. Let $S$ be a smooth del Pezzo surface over $k$ equipped with an effective Cartier divisor $D$. The projection from the double point locus $\pi : \mathcal{D}_{\text{odp}} \to \bar{M}_{0,n}(S,D)$ is étale.

**Proof.** Over $\bar{M}_{0,n}(S,D)$ the product of universal evaluation maps
\[
ev \times ev : \mathcal{X}_{0,n}^{\text{odp}} \times \bar{M}_{0,n}(S,D)^{\text{odp}} \mathcal{X}_{0,n}^{\text{odp}} \to S \times S \times \bar{M}_{0,n}(S,D)^{\text{odp}}
\]
is transverse to $\Delta_S$ over $\bar{M}_{0,n}(S,D)^{\text{odp}}$. So, the morphism
\[
\mathcal{D}_{\text{odp}} \setminus \Delta_{\mathcal{X}_{0,n}^{\text{odp}}} = (ev \times ev)^{-1}(\Delta_S) \setminus \Delta_{\mathcal{X}_{0,n}^{\text{odp}}} \xrightarrow{\pi} \bar{M}_{0,n}(S,D)^{\text{odp}}
\]
is smooth of relative dimension zero and thus étale. A straightforward argument based on Remark 5.6 shows that
\[
\mathcal{D}_{\text{odp}} \cap \Delta_{\mathcal{X}_{0,n}^{\text{odp}}} = \emptyset.
\]
Alternatively, this follows from Lemma 2.11.

For the remainder of this section, we assume $k$ has characteristic zero. Note that we have
\[
\mathcal{D}_{\text{odp}} = \pi^{-1}(\bar{M}_{0,n}(S,D)) \subset \mathcal{D}.
\]
We will need the following special loci in $\mathcal{D}$.

**Definition 5.5.** Let $\mathcal{D}_{\text{cusp}} \subset \Delta_{\mathcal{X}_{0,n}^{\text{good}}}$ denote the locus with geometric points given by a map $f : \mathbb{P}^1 \to S$ and a point $p \in \mathbb{P}^1$ where $f$ has a simple cusp, together with marked points $(p_1, \ldots, p_n)$ on $\mathbb{P}^1$ such that $(f, p_1, \ldots, p_n)$ is in $\bar{M}_{0,n}(S,D)^{\text{good}}$. Let $\mathcal{D}_{\text{tac}} \subset \mathcal{D}$ denote the locus with geometric points given by $(f : \mathbb{P}^1 \to S, p_1, \ldots, p_n)$ a geometric point of $\bar{M}_{0,n}(S,D)^{\text{good}}$ and a pair of points $p, q \in \mathbb{P}^1$ where $f$ has a simple tacnode. Let $\mathcal{D}_{\text{trip}} \subset \mathcal{D}$ denote the locus with geometric points given by a geometric point $(f : \mathbb{P}^1 \to S, p_1, \ldots, p_n)$ of $\bar{M}_{0,n}(S,D)^{\text{good}}$ and a pair of points $p, q \in \mathbb{P}^1$ where $f$ has a triple point.

**Remark 5.6.** Let $\mathcal{F} : B \to \bar{M}_{0,n}(S,D)^{\text{good}}$ be a family of stable maps corresponding to a curve $\mathbb{P} \to B$ and a map $f : \mathbb{P} \to S$. Let
\[
\tilde{\mathcal{F}} : \mathbb{P} \times_B \mathbb{P} \to \mathcal{X}_{0,n}^{\text{good}} \times \bar{M}_{0,n}(S,D)^{\text{good}} \mathcal{X}_{0,n}^{\text{good}}
\]
be the induced map. Let $\mathcal{I}_D$ denote the ideal sheaf of $\mathcal{D}$. Let $(p_1, p_2)$ be a point of $\mathbb{P} \times_B \mathbb{P}$ such that $f(p_1) = f(p_2)$. Let $t_1, t_2$, be local coordinates on $\mathbb{P}$ at $p_1, p_2$, respectively. So, locally the ideal sheaf of the diagonal $\Delta_\mathbb{P} \subset \mathbb{P} \times_B \mathbb{P}$ is generated by $t_1 - t_2$. Let $s = (s_1, s_2)$ be local coordinates at $f(p_1) = f(p_2)$. Then, since $t_1 - t_2$ is not a zero divisor, the pull-back of the colon ideal sheaf $\mathcal{I}_D$ is given by
\[
\tilde{\mathcal{F}}^* \mathcal{I}_D = \left( \frac{s \circ f(t_1) - s \circ f(t_2)}{t_1 - t_2} \right) .
\]
Lemma 5.7. We have $\mathcal{D}_{\text{cusp}} \subset \mathcal{D}$.

Proof. Let $f : \mathbb{P}^1 \to S$ be a map with a single simple cusp at $p \in \mathbb{P}^1$. So $(f, p, p) \in \Delta_{X_{0, n}}^{\text{good}}$ is an $F$ point of $\mathcal{D}_{\text{cusp}}$. Let $\mathcal{F} : \text{Spec}(F) \to \overline{M}_{0, n}(S, D)^{\text{good}}$ be the corresponding map, so we get an induced map

$$\mathcal{F} : \mathbb{P}^1 \times \mathbb{P}^1 \to \overline{X}_{0, n}^{\text{good}} \times_{\overline{M}_{0, n}(S, D)^{\text{good}}} \overline{X}_{0, n}^{\text{good}}.$$

Choose local coordinates on $\mathbb{P}^1$ at $p$ and on $S$ at $f(p)$ as in Lemma 4.7. Let $t_1, t_2$, be copies of the local coordinate on $\mathbb{P}$. In particular, $t_i$ vanishes at $p$. By Remark 5.6 we have locally at $p$,

$$\mathcal{F}^* \mathcal{I}_D = \left( \frac{(t_1^2, t_1^3) - (t_2^2, t_2^3)}{t_1 - t_2} \right) = (t_1 + t_2, t_1^2 + t_1 t_2 + t_2^2) = (t_1 + t_2, t_1^2),$$

and it is clear these equations vanish at $t_1 = t_2 = 0$. □

Lemma 5.8. The loci $\mathcal{D}_{\text{cusp}}, \mathcal{D}_{\text{tac}}, \mathcal{D}_{\text{trip}} \subset \mathcal{D}$ are closed.

Proof. This follows from Theorem 4.5(5)(i). □

In light of Lemma 5.8, we equip $\mathcal{D}_{\text{cusp}}, \mathcal{D}_{\text{tac}},$ and $\mathcal{D}_{\text{trip}}$ with the reduced induced subscheme structure.

Lemma 5.9. The projection from the double point locus $\pi : \mathcal{D} \to \overline{M}_{0, n}(S, D)^{\text{good}}$ is étale over a neighborhood of $\mathcal{D}_{\text{trip}}$.

Proof. The proof is the same as that of Lemma 5.4. □

To prove the double point locus $\mathcal{D}$ is smooth at points of $\mathcal{D}_{\text{cusp}}$ and $\mathcal{D}_{\text{tac}}$, we introduce the following lemma.

Lemma 5.10. Let $k$ be a field, and let $X$ and $Y$ be smooth, integral, finite type $k$-schemes. Let $Z \subset Y$ be a closed subscheme and take $z \in Z$. Suppose that there is a morphism $f : X \to Y$ with $z \in f(X)$, and an integer $\ell$ such that

1. there is an irreducible component $Z_0$ of $Z$ containing $z$ and of codimension $\leq \ell$ on $Y$,
2. The closed subscheme $X \times_Y Z$ of $X$ is smooth of pure codimension $\ell$ on $X$.

Then $Z$ is smooth of pure codimension $\ell$ in a neighborhood of $z$.

Proof. We may assume $k$ is algebraically closed and $z$ is a $k$-point of $Z$. Since $X$ is smooth, $Z/k$ is smooth in a neighborhood of $z$ if and only if $X \times_k Z$ is smooth over $X$ in a neighborhood of $X \times_k z$. Let $\Gamma_f \subset X \times_k Y$ be the graph of $f$. Note that $X \times_Y Z$ is isomorphic to the intersection $\Gamma_f \cap X \times_k Z$. So changing notation, we may assume that $f : X \to Y$ is a closed immersion, that is, we may assume that $X$ is a smooth closed subscheme of $Y$.

Since the assertion is local on $Y$ for the étale topology, we may assume that $Y$ is an principal open subscheme of $\mathbb{A}^n_k$. Since $X$ is smooth, we may assume that $X$ is a smooth complete intersection in $Y$, with ideal $I_X = (g_1, \ldots, g_m)$, where $m$ is the codimension of $X$ in $Y$.

Let $i : X \cap Z \to Z$ be the inclusion. We have the exact sequence of $\mathcal{O}_{X \cap Z, z}$-modules

$$I_{X,z}/I_{X,z}^2 \otimes_{\mathcal{O}_{X,z}} \mathcal{O}_{X \cap Z, z} \to i^* \Omega_{Z/k, z} \to \Omega_{X \cap Z, z} \to 0$$
Since $X \cap Z$ is smooth of dimension $n - m - \ell$ at $z$, $\Omega_{X \cap Z, z} \cong \mathcal{O}_{X \cap Z, z}^{n-m-\ell}$. Thus the sequence splits and
\[ i^* \Omega_{Z/k, z} \cong \Omega_{X \cap Z, z} \oplus \text{im}(d). \]
Since the images of $g_1, \ldots, g_m$ in $d(I_{X,z}/I_{Y,z}^2)$ generate $\text{im}(d)$, we have a surjection $\mathcal{O}_{X \cap Z, z}^n \to \text{im}(d)$. Applying $- \otimes_{\mathcal{O}_{X \cap Z, z}} k(z)$, we see that
\[
\dim_{k(z)} \Omega_{Z/k, z} \otimes_{\mathcal{O}_{Z, z}} k(z) = \dim_{k(z)} i^* \Omega_{Z/k, z} \otimes_{\mathcal{O}_{X \cap Z, z}} k(z) = \\
\dim_{k(z)} \Omega_{X \cap Z, z} \otimes_{\mathcal{O}_{X \cap Z, z}} k(z) + \dim_{k(z)} \text{im}(d) \otimes_{\mathcal{O}_{X \cap Z, z}} k(z) \\
\leq (n - m - \ell) + m = n - \ell.
\]
Choose generators $f_1, \ldots, f_s$ for $I_{Z,z} \subset \mathcal{O}_{Y,z}$ and let $x_1, \ldots, x_n$ be the standard coordinates on $\mathbb{A}^n$. Then
\[
\dim_{k(z)} \Omega_{Z/k, z} \otimes_{\mathcal{O}_{Z, z}} k(z) = n - \text{rank}\left(\partial f_i/\partial x_j\right)(z)
\]
Since
\[
n - \ell \geq \dim_{k(z)} \Omega_{Z/k, z} \otimes_{\mathcal{O}_{Z, z}} k(z)
\]
by the previous, it follows that
\[
\text{rank}\left(\partial f_i/\partial x_j\right)(z) \geq \ell
\]
After reordering the $f_i$, we may assume that the matrix
\[
\left(\partial f_i/\partial x_j\right)(z)_{1 \leq i \leq \ell}
\]
has rank $\ell$, which implies that (after shrinking $Y$ if necessary) the closed subscheme $Z' \subset Y$ defined by $f_1, \ldots, f_s$ is smooth of codimension $\ell$; shrinking $Y$ again if necessary, we may assume that $Z'$ is integral. But then $Z_0$ is a closed subscheme of $Z'$ of the same dimension, so $Z_0 = Z'$ and $Z_0 \subset Z \subset Z'$, so $Z = Z'$ and $Z$ is smooth of codimension $\ell$ on $Y$.

**Lemma 5.11.** The double point locus $D$ is smooth of dimension $d - 1 + n$ at the geometric points of the cuspidal locus of $M_{0,n}(S,D)^{\text{good}}$ has ramification index 2 at $D_{\text{cusp}}$. The restriction of $\pi$ to a map $D_{\text{cusp}} \to D_{\text{cusp}} \cap M_{0,n}(S,D)^{\text{good}}$ is birational.

**Proof.** Let $\tilde{\mathfrak{p}} = (f, p) = ((f, p), (f, p))$ represent a geometric point
\[
\text{Spec } F \to D_{\text{cusp}} \subset \bar{X}_{0,n} \times_{M_{0,n}(S,D)^{\text{good}}} \bar{X}_{0,n}.
\]
Let $q = f(p)$. We may assume $p = 0 \in \mathbb{P}^1$. Let $t$ be the standard coordinate on $\mathbb{P}^1 \setminus \{\infty\}$ and let $(x, y)$ be analytic coordinates on $S$ at $q$ as in Lemma 4.7. Using Lemma 4.8, choose a family of maps $\tilde{f} : \mathbb{P}^1_{F[q]} \to S$ such that $f|_{q=0} = f$ and near $p$,
\[
f(\epsilon, t) = a_{10} \epsilon + a_{02} t^2 + a_{11} \epsilon t + a_{03} t^3 \mod (\epsilon^2, \epsilon t^2, t^4)
\]
where $a_{ij} \in F^2$ are given by
\[
a_{02} = (1, 0), \quad a_{11} = (0, 3a), \quad a_{03} = (0, 1).
\]
Let
\[
\tilde{\mathcal{F}} : \mathbb{P}^1_{F[q]} \times_{\text{Spec } (F[q])} \mathbb{P}^1_{F[q]} \to \bar{X}_{0,n}^{\text{good}} \times_{M_{0,n}(S,D)^{\text{good}}} \bar{X}_{0,n}^{\text{good}}
\]
de term the induced map. We have local coordinates on $\mathbb{P}^1_{F[q]} \times_{\text{Spec } (F[q])} \mathbb{P}^1_{F[q]}$ at $(p, p, 0)$ given by $\epsilon$ and two copies $t_1, t_2$, of the parameter $t$. Let $I$ be the ideal
sheaf of $\tilde{\mathcal{O}}$ on $\tilde{X}_{0,n} \times \tilde{\mathcal{M}}_{0,n}(S,D)_{\text{good}} \times \tilde{X}_{0,n}$. So, analytically locally $\mathcal{F}^* \mathcal{I} = (t_1, t_2, \epsilon)$. By Remark 5.6, the pull-back $\tilde{\mathcal{F}}^* \mathcal{I}_D$ is generated analytically locally by
\begin{equation}
\mathcal{Y} = \frac{f(t_1) - f(t_2)}{t_1 - t_2} = a_{02}(t_1 + t_2) + a_{11}\epsilon + a_{03}(t_1^2 + t_2^2 + t_1 t_2) \mod (\tilde{\mathcal{F}}^* \mathcal{I})^3 + (\epsilon^2, t_1, t_2, \epsilon).
\end{equation}
Since $a_{02}$ and $a_{11}$ are linearly independent, it follows that the subscheme determined by the ideal sheaf $\tilde{\mathcal{F}}^* \mathcal{I}_D$ is smooth of codimension 2 at $(p, p, 0)$. Apply Lemma 5.10 with $l = 2$, $z = \tilde{p}$,
\begin{equation}
X = \mathbb{P}^1_{F[[\epsilon]]} \times_{\text{Spec}(F[[\epsilon]])} \mathbb{P}^1_{F[[\epsilon]]}, \quad Z = \mathcal{D}, \quad Y = \tilde{X}_{0,n}^{\text{good}} \times \tilde{\mathcal{M}}_{0,n}(S,D)_{\text{good}} \times \tilde{X}_{0,n}^{\text{good}},
\end{equation}
and $X \times_Y Z$ the subscheme of $X$ determined by $\tilde{\mathcal{F}}^* \mathcal{I}_D$. Since $\mathcal{D}$ is given by two equations, we can take $Z_0 = Z$. It follows that $\mathcal{D}$ is smooth at $\tilde{p}$ of dimension $d - 1 + n$.

Now assume that $\tilde{p}$ is a geometric generic point of $\mathcal{D}_{\text{cusp}}$. Let $\tilde{\mathcal{F}}$ be as in the proof of Lemma 5.7. That is, we set the parameter $\epsilon$ to zero in the $\tilde{\mathcal{F}}$ above. Let $q \in \mathcal{O}_{(p,p), \mathbb{P}^1_{\mathbb{P}^1}}$ be the ideal sheaf of $\tilde{\mathcal{F}}^{-1}(\pi^{-1}(f))$. By equation (5.1), the quotient $\mathcal{O}_{(p,p), \mathbb{P}^1_{\mathbb{P}^1}}/q$ is given by $F[t_1, t_2] / (t_1 + t_2, t_1^2)$, which has length two and induces an isomorphism to $\text{Spec} F$ after taking the reduced subscheme. Therefore, the ramification index of $\pi$ at $\tilde{p}$ is 2. By Proposition 2.40(2), the map $\pi : \mathcal{D}_{\text{cusp}} \to \mathcal{D}_{\text{cusp}} \cap \mathcal{M}_{0,n}(S,D)_{\text{good}}$ is generically a bijection; since we are in characteristic zero, this implies that $\pi : \mathcal{D}_{\text{cusp}} \to \mathcal{D}_{\text{cusp}} \cap \mathcal{M}_{0,n}(S,D)_{\text{good}}$ is birational. □

Lemma 5.12. Let $f : \mathbb{P}^1 \to S$ represent a point of $\mathcal{D}_{\text{tac}}$. Let $p \neq p' \in \mathbb{P}^1$ such that $q = f(p) = f(p')$ is the tacnode.

(1) There exist local coordinates $t,t'$ at $p,p'$, respectively, and local analytic coordinates $x,y$, on $S$ at $q$ such that near $p$,
\begin{equation}
(x, y) \circ f = (t, t^2)
\end{equation}
and near $p'$,
\begin{equation}
(x, y) \circ f = (t', 0).
\end{equation}

(2) There exists a family $\tilde{f} : \mathbb{P}^1_{F[[\epsilon]]} \to S$ such that $\tilde{f}|_{(\epsilon = 0)} = f$, and near $p$,
\begin{equation}
(x, y) \circ \tilde{f}(t, \epsilon) = (t, t^2 + \epsilon) \mod \epsilon^2
\end{equation}
and near $p'$,
\begin{equation}
(x, y) \circ \tilde{f}(t, \epsilon) = (t', 0) \mod \epsilon^2.
\end{equation}

Lemma 5.13. The double point locus $\mathcal{D}$ is smooth of dimension $d - 1 + n$ at the geometric points of the tacnodal locus $\mathcal{D}_{\text{tac}}$, the map $\pi : \mathcal{D} \to \mathcal{M}_{0,n}(S,D)_{\text{good}}$ has ramification index 2 at $\mathcal{D}_{\text{tac}}$, and the map $\pi|_{\mathcal{D}_{\text{tac}}} : \mathcal{D}_{\text{tac}} \to \mathcal{D}_{\text{tac}}$ is two to one.

Proof. Let $\tilde{p} = ((p,f),(p',f'))$ represent a geometric point
\begin{equation}
\text{Spec} F \to \mathcal{D}_{\text{tac}} \subset \tilde{X}_{0,n} \times \tilde{\mathcal{M}}_{0,n}(S,D)_{\text{good}} \tilde{X}_{0,n}.
\end{equation}
Let $q = f(p)$. Let $t, t'$, be local coordinates at $p, p'$, respectively, and let $x, y$, be analytic coordinates at $q$ as in Lemma 5.12(1). Let $\tilde{f} : \mathbb{P}^1_{F[[\epsilon]]} \to S$ be the family of Lemma 5.12(2). Let $\tilde{\mathcal{F}} : \mathbb{P}^1_{F[[\epsilon]]} \times_{\text{Spec}(F[[\epsilon]])} \mathbb{P}^1_{F[[\epsilon]]} \to \tilde{X}_{0,n}^{\text{good}} \times \tilde{\mathcal{M}}_{0,n}(S,D)_{\text{good}} \tilde{X}_{0,n}$
denote the induced map. We have local coordinates on $\mathbb{P}^1_{F[[\epsilon]]}$ at $(p, p', 0)$ given by $t, t', \epsilon$. The pull-back $\tilde{\mathcal{F}}^* \mathcal{I}_D$ is generated analytically locally by \begin{equation}
abla \left( x, y \right) \circ f(\epsilon, t) - \left( x, y \right) \circ f(\epsilon, t') = (t - t', t^2 + \epsilon) \mod \epsilon^2. \end{equation} It follows that the subscheme determined by $\tilde{\mathcal{F}}^* \mathcal{I}_D$ is smooth of codimension 2 at $(p, p', 0)$. Apply Lemma 5.10 with $l = 2$, $z = \tilde{p}$, \begin{align*}
abla \frac{\mathbb{P}^1_{F[[\epsilon]]} \times \text{Spec}(F[[\epsilon]])}{\mathbb{P}^1_{F[[\epsilon]]}} \times \mathbb{P}^1_{F[[\epsilon]]} = \mathcal{D}, \quad \mathbb{P}^1_{F[[\epsilon]]} = \mathcal{D} \mathcal{D}, \quad Y = X^{\text{good}}_0 \times \mathcal{M}_0, (S, D)^{\text{good}}, X^{\text{good}}_0,
abla \quad \text{and} \quad X \times_Y \mathcal{D} \text{ the subscheme of } \mathcal{D} \text{ determined by } \tilde{\mathcal{F}}^* \mathcal{I}_D. \text{ Since } \mathcal{D} \text{ is given by two equations, we can take } Z_0 = Z. \text{ It follows that } \mathcal{D} \text{ is smooth at } \tilde{p} \text{ of dimension } \dim(d - 1 + n).

Now, assume that $\tilde{p}$ is a geometric generic point of $\mathcal{D}_{\text{cusp}}$. Let \[ \tilde{\mathcal{F}}|_0 : \mathbb{P}^1 \to S \] be the restriction to $\epsilon = 0$. Let $q \subset \mathcal{O}_{(p, p'), \mathbb{P}^1 \times \mathbb{P}^1}$ be the ideal sheaf of $(\tilde{\mathcal{F}}|_0)^{-1}(f)$. By equation (5.4), the quotient $\mathcal{O}_{(p, p'), \mathbb{P}^1 \times \mathbb{P}^1}/q$ is given by $F(t, t')/(t - t', t^2)$, which has length two. Therefore, the ramification index of $\pi$ at $\tilde{p}$ is 2.

Finally, the map $\pi|_{\mathcal{D}_{\text{cusp}}} : \mathcal{D}_{\text{cusp}} \to \mathcal{D}_{\text{cusp}}$ is two to one because $\pi(f, p, p') = \pi(f, p', p) = f$. \hfill \qed

**Corollary 5.14.** The double point locus $\mathcal{D}$ is smooth of dimension $\dim(d - 1 + n)$. The map $\pi : \mathcal{D} \to \mathcal{M}_{0, n}(S, D)^{\text{good}}$ is finite, flat and generically étale. The ramification of $\pi$ is supported on $\mathcal{D}_{\text{cusp}}$ and $\mathcal{D}_{\text{cusp}}$ where it is simply ramified.

**Proof.** The smoothness and dimension of $\mathcal{D}$ follow from Theorem 4.5 (1) and Lemmas 5.4, 5.9, 5.11 and 5.13. The map $\pi$ is quasi-finite because the fiber over a point of $\mathcal{M}_{0, n}(S, D)^{\text{good}}$ is represented by a map $f : \mathbb{P} \to S$ contained in the union of the ramification locus of $f$ and the locus where $f$ is not $1 - 1$. Since $\pi$ is proper, it follows that $\pi$ is finite. Since the domain and range of $\pi$ are smooth of the same dimension and $\pi$ is quasi-finite, it follows that $\pi$ is flat. Lemmas 5.4 and 5.9 show that $\pi$ is étale over $\mathcal{D}_{\text{cusp}}$ and $\mathcal{D}_{\text{trip}}$. In particular, it is generically étale. The ramification over $\mathcal{D}_{\text{cusp}}$ and $\mathcal{D}_{\text{cusp}}$ was computed in Lemmas 5.11 and 5.13. \hfill \qed

### 6. Orienting the evaluation map

In this section, we continue to assume that $\mathcal{M}_{0, n}(S, D)^{\text{good}}$ is non-empty.

Let $A$ be a Noetherian ring. Let $f : Y \to Z$ be a finite flat morphism of smooth $A$-schemes. It follows that $f_* \mathcal{O}_Y$ is a locally free $\mathcal{O}_Z$-module. The multiplication map on $\mathcal{O}_Y$ gives the morphism of $\mathcal{O}_Z$-modules \[ m : f_* \mathcal{O}_Y \otimes_{\mathcal{O}_Z} f_* \mathcal{O}_Y \to f_* \mathcal{O}_Y, \] and since $f_* \mathcal{O}_Y$ is a finite locally free $\mathcal{O}_Z$-module, we have the trace map \[ \text{Tr}_f : f_* \mathcal{O}_Y \to \mathcal{O}_Z \] defined by sending $s \in f_* \mathcal{O}_Y(U)$ to the trace of the multiplication map $\times s : f_* \mathcal{O}_Y(U) \to f_* \mathcal{O}_Y(U)$. Rewriting the composition $\text{Tr} \circ m$ as \[ \delta : f_* \mathcal{O}_Y \to f_* \mathcal{O}_Y^\vee \]
we have the discriminant $\text{disc}_f : \det f_* \mathcal{O}_Y \to \det f_* \mathcal{O}_Y$, given by
\begin{equation}
(6.1) \quad \text{disc}_f = \det \delta.
\end{equation}
Equivalently,
\[ \text{disc}_f : \mathcal{O}_Z \to (\det f_* \mathcal{O}_Y)^{-2} \]
Now suppose that $f$ is étale over each generic point of $Z$, and that $Z$ is reduced. Since $\text{Tr}_f$ is a surjection if $f$ is étale, we see that $\text{disc}_f$ is generically injective, hence injective since $Z$ is reduced. This gives us the effective Cartier divisor $\text{div} (\text{disc}_f)$, supported on the branch locus of $f$.

For $V$ a locally free sheaf of rank $r$ on $Z$, we write $\det^n V$ for the $n$-tensor power over $\mathcal{O}_Z$ of the invertible sheaf $\det V = \Lambda^r V$. Recall Definition 5.3.

Let $S$ be a del Pezzo surface of degree $d_S$ over a field $k$ of characteristic 0, and let $D$ be an effective Cartier divisor. Suppose that Basic Assumptions 4.1 (2) (3) holds. We may then apply Theorem 4.5 and obtain $\text{ev} : \bar{M}_{0,n}^\text{good} \to S^n$. By Definition 5.3, we have the map $\pi : D \to \bar{M}_{0,n}^\text{good} (S, D)$ from the double point locus, which is finite, flat and generically étale by Corollary 5.14. We therefore have $\text{disc}_\pi : \mathcal{O}_{\bar{M}_{0,n}^\text{good} (S, D)} \to (\det \pi_* \mathcal{O}_D)^{-2}$ by the above construction. The results of Section 5 compute the divisor of this section.

**Proposition 6.1.** We have
\[ \text{div} (\text{disc}_\pi) = 1 \cdot D_{\text{cusp}} + 2 \cdot D_{\text{tac}} \]
and thus $\text{disc}_\pi$ defines an isomorphism
\[ \text{disc}_\pi : \mathcal{O}_{\bar{M}_{0,n}^\text{good} (S, D)} (D_{\text{cusp}}) \to (\det \pi_* \mathcal{O}_D) (-D_{\text{tac}})^{-2} \]

**Proof.** A finite, flat map $f : X \to Y$ between smooth schemes over a field has a different $D_f$, which is an ideal sheaf on an effective Cartier divisor [49, Tag 0BTC]. In characteristic 0, the different is computed [55, Chapter 5 Theorem 28] to be the product of ideal sheaves $p^{e_p-1}$ where $p$ runs over the codimension 1 ideal sheaves of $X$ where $f$ is ramified and $e_p$ is the ramification index. By Corollary 5.14 it follows that $\text{div} D_\pi = D_{\text{cusp}} + D_{\text{tac}}$. By Propositions 14 of Chapter 3 in [31], $\text{div} (\text{disc}_\pi) = \pi_* (\text{div} D_\pi)$. Thus $\text{div} (\text{disc}_\pi) = \pi_* D_{\text{cusp}} + \pi_* D_{\text{tac}}$. By Lemmas 5.11 and 5.13 respectively,
\[ \pi_* D_{\text{cusp}} = D_{\text{cusp}} \text{ and } \pi_* D_{\text{tac}} = 2 \cdot D_{\text{tac}}. \]

We are now in a position to orient the evaluation map in characteristic 0. We first recall the definition of an orientation of a map.

**Definition 6.2.** Let $f : X \to Y$ be a local complete intersection morphism [49 Tag 068E]. Let $L_f$ denote the perfect cotangent complex of $f$ [49 08SH] and $\det L_f$ its determinant. An orientation of $f$ is a line bundle $\mathcal{L}$ on $X$ and an isomorphism $\mathcal{L}^\otimes 2 \cong \det L_f$.

$ev : \bar{M}_{0,n}^\text{good} \to S^n$ is a map between smooth schemes by Theorem 4.5 (1) and is therefore a local complete intersection morphism [49 Tag 068E]. By Proposition 4.9 $ev$ defines an isomorphism
\[ \det d(ev) : \mathcal{O}_{\bar{M}_{0,n}^\text{good} (S, D)} (D_{\text{cusp}}) \to \omega_{ev} \]
Theorem 6.3. Let $k$ be a field of characteristic 0 and let $S$ and $D$ be as in Theorem 1.1. Let $L$ be the invertible sheaf on $M_{0,n}^\mathrm{good}(S,D)$ given by

$$L = \det^{-1} \pi_* \mathcal{O}_D(-D_{\text{disc}})$$

Then the composition $\det \circ \text{disc}_\pi^{-1} : L^\otimes 2 \to \omega_{ev}$ is an isomorphism on $\overline{M}_{0,n}^\mathrm{good}(S,D)$.

Proof. This is a direct consequence of Proposition 4.9 and Proposition 6.1. □

7. The Symmetrized Moduli Space

Let $k$ be a field of characteristic 0. Let $S$ be a del Pezzo surface equipped with an effective Cartier divisor $D$ satisfying Basic Assumptions 4.1[1][2][3] and such that $\deg K_S \cdot D \geq 1$. We continue to assume that $M_{0,n}(S,D)^{\text{sup}}$ is non-empty. See Remark 5.2.

The symmetric group $\mathfrak{S}_n$ acts freely on $\overline{M}_{0,n}(S,D)$ by permuting the marked points, and acting trivially on the underlying curve and the morphism to $S$. This extends to an action on the universal curve $X_{0,n} \to \overline{M}_{0,n}(S,D)$ and the usual permutation action on $S^n$, giving us the following $\mathfrak{S}_n$-equivariant diagram.

$$\overline{X}_{0,n} \to \overline{M}_{0,n}(S,D) \xrightarrow{ev} S^n$$

Let $S_0^n$ denote the complement of the pairwise diagonals in $S^n$, so the restriction of the $\mathfrak{S}_n$ action to $S_0^n$ is free. Let $n = d - 1$. Moreover, $ev(M_{0,n}(S,D)^{\text{good}}) \subset S_0^n$ because by Theorem 4.5[4] there are no contracted components in the stable maps of $M_{0,n}(S,D)^{\text{good}}$. As above, let $\overline{X}_{0,n}^\text{good}$ denote the inverse image of $M_{0,n}(S,D)^{\text{good}}$ under $\overline{X}_{0,n} \to \overline{M}_{0,n}(S,D)$. Thus, we obtain the following $\mathfrak{S}_n$-equivariant diagram in which all actions are free.

$$\overline{X}_{0,n}^\text{good} \times \overline{M}_{0,n}(S,D)^{\text{good}} \xrightarrow{\pi} \overline{M}_{0,n}^\text{good}(S,D) \xrightarrow{ev} S_0^n$$

Since $S^n$ is projective over $k$ and $ev$ and $\pi$ are quasi-finite (Theorem 4.3[1] and Corollary 5.14), the schemes $\overline{M}_{0,n}^{\text{good}}(S,D)$, $S_0^n$, and $D$ are quasi-projective. So, one may take their quotients by $\mathfrak{S}_n$ in the category of quasi-projective $k$-schemes. Since the actions are free, these quotients are smooth over $k$. We denote the respective quotients by $D_{\emptyset}$, $M_{0,n,\emptyset}^{\text{good}}(S,D)$, and $\text{Sym}^n S$. We denote the induced maps by $\pi_{\emptyset}$ and $ev_{\emptyset}$. Note that $\text{Sym}^n S$ is an open subscheme of the standard $n$th symmetric product $\text{Sym}^n S$. Thus we obtain the following diagram of smooth quasi-projective $k$-schemes.

$$D_{\emptyset} \xrightarrow{\pi_{\emptyset}} M_{0,n,\emptyset}^{\text{good}}(S,D) \xrightarrow{ev_{\emptyset}} \text{Sym}^n S$$

Observe that all squares in the following diagram are Cartesian, where the vertical maps are the quotient maps.

$$\overline{X}_{0,n} \text{good} \xrightarrow{\pi} \overline{M}_{0,n}^\text{good}(S,D) \xrightarrow{ev} S^n$$
For $\square \in \{\text{cusp}, \text{tac}, \text{trip}\}$, we let $D^\square_\square_\square$ denote the reduced image of $D_\square_\square_\square$ in $\tilde{M}^\text{good}_{0,n,S}(S,D)$.

**Theorem 7.1.** Let $k,S,D$ be as in Theorem 1.1.

1. The canonical section $\det \ev_\square_\square_\square : \mathcal{O}_{\tilde{M}^\text{good}_{0,n,S}(S,D)} \to \omega_{\ev_\square_\square_\square}$ has divisor $1 \cdot D^\square_\text{cusp}$ and induces an isomorphism
   
   $$\det \ev_\square_\square_\square : \mathcal{O}_{\tilde{M}^\text{good}_{0,n,S}(S,D)}(D^\square_\text{cusp}) \to \omega_{\ev_\square_\square_\square}.$$  

2. The divisor of $\text{disc}_\square_\square_\square : \mathcal{O}_{\tilde{M}^\text{good}_{0,n,S}(S,D)} \to \det^{-2} \pi^* \mathcal{O}_{D^\square_\square}$ is $D^\square_\text{cusp} + 2 \cdot D^\square_\text{tac}$ and induces an isomorphism
   
   $$\text{disc}_\square_\square : \mathcal{O}_{\tilde{M}^\text{good}_{0,n,S}(S,D)}(D^\square_\text{cusp}) \to [\det \pi^* \mathcal{O}_{D^\square_\square}(-D^\square_\text{tac})]^{-2}.$$  

3. Letting $\mathcal{L}^\square : [\det \pi^* \mathcal{O}_{D^\square_\square}(-D^\square_\text{tac})]^{-1}$, we have the isomorphism
   
   $$\det \ev_\square_\square_\square \circ \text{disc}_\square^{-1} : (\mathcal{L}^\square)^{\otimes 2} \to \omega_{\ev_\square_\square_\square}.$$  

**Proof.** This follows from Proposition 4.9 and Proposition 6.1 noting that the relative dualizing sheaf $\omega_f$ of a morphism $f$ is compatible with étale base-change, as is the divisor of the discriminant of a morphism, and the divisor of a section of an invertible sheaf is detectible after finite étale base-change. Specifically, the fact that $\text{div} \ev = 1 \cdot D^\text{cusp}$ and $\text{div} \text{disc}_\pi = 1 \cdot D^\text{cusp} + 2 \cdot D^\text{tac}$ implies that $\text{div} \ev^\square = 1 \cdot D^\square$ and $\text{div} \text{disc}_\square = 1 \cdot D^\square_\text{cusp} + 2 \cdot D^\square_\text{tac}$; the remaining assertions are direct consequences of these two identities. \(\square\)

**8. Twisting the degree map**

As before, we let $k$ be a field. Let $S$ be a smooth del Pezzo surface over $k$ equipped with an effective Cartier divisor $D$. Let $k \subseteq k^{\text{sep}}$ denote a separable closure of $k$. For a $k$-scheme $Y$ and field extension $k \subset L$, we write $Y_L$ for $Y \times_k L$. Let

$$\sigma = (L_1, \ldots, L_r)$$

be an $r$-tuple of subfields $L_i \subset k^{\text{sep}}$ containing $k$ for $i = 1, \ldots, r$ subject to the requirement that $\sum_{i=1}^r [L_i : k] = n$. We think of $\sigma$ as the fields of definition of a list of points of $S$ that our curves will be required to pass through.

The list $\sigma$ is used to define twists $\ev_{\sigma}$ of the evaluation map $\ev : \tilde{M}_{0,n}(S,D) \to S^n$ in the following manner. The Galois group $\text{Gal}(k^{\text{sep}}/k)$ acts on the $k^{\text{sep}}$-points of $k$-schemes. Thus $\sigma$ gives rise to a canonical homomorphism $\rho(\sigma) : \text{Gal}(k^{\text{sep}}/k) \to \mathfrak{S}_n(\sigma)$, where $\mathfrak{S}_n(\sigma)$ denotes the $k^{\text{sep}}$-points of the $k$-scheme $\prod_{i=1}^n \text{Spec} L_i$ and $\mathfrak{S}_n(\sigma) \cong \mathfrak{S}^n$ denotes the symmetric group. For convenience, we fix an identification $\mathfrak{S}_n(\sigma) = \{1, 2, \ldots, n\}$ and thus a canonical isomorphism $\mathfrak{S}_n(\sigma) = \mathfrak{S}_n$.

There is a canonical inclusion of $\mathfrak{S}_n$ into $\text{Aut}(S^n)$. We include $\mathfrak{S}_n$ into $\text{Aut}(\tilde{M}_{0,n}(S,D))$ by permutation of the marked points, and acting trivially on the underlying curve and the morphism to $S$; for $\tau \in \mathfrak{S}_n$, set

$$\tau(u : C \to S, p_1, \ldots, p_n) = (u : C \to S, p_{\tau^{-1}(1)}, \ldots, p_{\tau^{-1}(n)}).$$

Let $X = S^n, X = \tilde{M}_{0,n}(S,D)$ or the double point locus $D$. The 1-cocycles

$$g \mapsto \rho(\sigma)(g) \times g$$

$$\text{Gal}(k^{\text{sep}}/k) \to \text{Aut}(X_{k^{\text{sep}}}$$

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determine twists $X_\sigma$ of $X$. Since $ev_{k^\text{sep}}$ and $\pi_{k^\text{sep}} : D_{k^\text{sep}} \to \bar{M}_{0,n}(S,D)_{k^\text{sep}}^{\text{good}}$ are Galois equivariant for the twisted action, they descend to $k$-maps denoted $ev_\sigma$ and $\pi_\sigma$ respectively

$$
\begin{align*}
\pi_\sigma & : D_\sigma \to \bar{M}_{0,n}(S,D)_\sigma, \\
\end{align*}
$$

The twist $(S^n)_\sigma$ of $S^n$ by $\sigma$ can be expressed as the restriction of scalars

(8.2) $$
(S^n)_\sigma \cong \prod_{i=1}^r \text{Res}_{L_i/k} S_i,
$$

allowing us to view $ev_\sigma$ as a map with this codomain.

In this section, we orient an appropriate restriction of $ev_\sigma$ in characteristic 0. Assume that $k, S$, and $D$ are as in the hypotheses of Theorem 4.5. We continue to assume that $\bar{M}_{0,n}(S,D)^{\text{good}}$ is non-empty. (See Remark 5.2.) We may assume the set $A \subset S^n$ used to construct $\bar{M}_{0,n}(S,D)$ in Theorem 4.5 is stable under the action of $\mathfrak{S}_n$. This action then restricts to an action on $S^n_A$ defining an open $k$-subscheme $(S^n - A)_\sigma \subset S^n$ whose closed complement $A_\sigma$ has codimension $\geq 2$.

Forgetting the marked points determines a $k$-map $\bar{M}_{0,n}(S,D)_{k^\text{good}}^{\text{good}} \to \bar{M}_0(S,D)$ from the twisted good moduli space to the untwisted moduli space of stable curves because $\mathfrak{S}_n$ acts trivially on the underlying curve. For $\square \in \{\text{cusp, tac, trip}\}$, we let $D_{\square, \sigma}$ denote the preimage of $D_{\square}$ under this map.

**Theorem 8.1.** Let $k, S, D$ be as in Theorem 1.1.

1. $ev_\sigma : \bar{M}_{0,n}(S,D)_\sigma \to (S^n)_\sigma$ is a map between smooth $k$-schemes.
2. The canonical section $\det ev_\sigma : \mathcal{O}_{\bar{M}_{0,n}(S,D)_\sigma} \to \omega_{ev_\sigma}$ has divisor $1 : D_{\text{cusp}, \sigma}$ and induces an isomorphism

$$
\det ev_\sigma : \mathcal{O}_{\bar{M}_{0,n}(S,D)_\sigma}(D_{\text{cusp}, \sigma}) \to \omega_{ev_\sigma}.
$$

3. The divisor of $\text{disc}_{\pi_\sigma} : \mathcal{O}_{\bar{M}_{0,n}(S,D)_\sigma} \to [\det(\pi_\sigma)_* \mathcal{O}_{\bar{D}_\sigma}]^{\otimes -2}$ is

$$
D_{\text{cusp}, \sigma} + 2 : D_{\text{tac}, \sigma}
$$

and induces an isomorphism

$$
\text{disc}_{\pi_\sigma} : \mathcal{O}_{\bar{M}_{0,n}(S,D)_\sigma}(D_{\text{cusp}, \sigma}) \to [\det(\pi_\sigma)_* \mathcal{O}_{\bar{D}_\sigma}(-D_{\text{tac}, \sigma})]^{\otimes -2}.
$$

4. Letting $\mathcal{L}_\sigma := [\det(\pi_\sigma)_* \mathcal{O}_{\bar{D}_\sigma}(-D_{\text{tac}})]^{-1}$, we have the isomorphism

$$
\det ev_\sigma \circ \text{disc}_{\pi_\sigma}^{-1} : (\mathcal{L}_\sigma)^{\otimes 2} \to \omega_{ev_\sigma}.
$$

**Proof.** Let $L$ be a finite normal extension of $k$ containing $L_i$ for $i = 1, \ldots, r$. Then the cocycle $\{[5.1]\}$ factors through $\text{Gal}(L/k)$, and there is a canonical isomorphism $X_L \cong X_{\sigma,L}$ for $X = \bar{M}_{0,n}(S,D)$, $S^n - A$ or $S^n$. Similarly the base-change $ev_{\sigma,L}$ of $ev_\sigma$ is identified with $ev_L$ via these canonical isomorphisms.

Theorem 4.5 then follows from Theorem 4.5 and the smoothness of $S$ because smoothness is fpqc local and may therefore be checked after base-change to $L$.

Note that $k \subset L$ is étale as $k$ is characteristic 0. The claims $[2]$ and $[3]$ follow from Proposition 4.9 and Proposition 6.1 respectively, noting that the relative dualizing sheaf $\omega_f$ of a morphism $f$ is compatible with étale base-change, as is the divisor of the discriminant of a morphism, and the divisor of a section of an invertible sheaf is detectable after finite étale base-change.

For the comparison of the $\mathbb{A}^1$-degrees corresponding to the orientations of Theorem 8.1 (4) and Theorem 7.1 (3) in [22], we note that there is a pullback diagram

$$
\begin{array}{ccc}
M_{0,n}^{\text{good}} (S,D)_\sigma & \longrightarrow & M_{0,n,i}^{\text{good}} (S,D) \\
\downarrow \text{ev}_\sigma & & \downarrow \text{ev}_i \\
S^\sigma & \longrightarrow & \text{Sym}^n S
\end{array}
$$

where the horizontal maps are determined by the quotient maps over the algebraic closure or $L$ (e.g. $S^\sigma \to \text{Sym}^n S$), where $L$ is a finite normal extension of $k$ containing the $L_i$. Since the bottom horizontal map is étale over $\text{Sym}^n S$, the upper horizontal map is étale.

9. Positive characteristic

In this section we will extend many of our constructions that up to now have been restricted to characteristic zero to del Pezzo surfaces in positive characteristic. The method is to lift to characteristic zero.

9.1. Lifting to characteristic zero. We first recall some basic facts from deformation theory.

**Proposition 9.1.** Let $\Lambda$ be a complete discrete valuation ring with residue field $k$ and quotient field $K$.

1. Let $X,Y$ be smooth, proper $\Lambda$-schemes and let $f_0 : Y_k \to X_k$ be a morphism. Suppose that $H^1(Y_k,f_0^*T_{X_k/k}) = 0$. Then there is a $\Lambda$-morphism $f : Y \to X$ with $f_k = f_0$. If $f_0$ is a closed immersion, then so is $f$.

2. Let $X_0$ be a smooth proper $k$-scheme. Suppose $H^2(X_0,T_{X_0/k}) = 0$. There is a smooth proper $\Lambda$-scheme $X$ with an isomorphism $\phi : X_k \iso X_0$ over $k$. If in addition $H^1(X_0,T_{X_0/k}) = 0$, then $(X,\phi)$ is unique up to isomorphism over $\Lambda$.

3. Let $X$ be a proper $\Lambda$-scheme and let $\mathcal{L}_0$ be an invertible sheaf on $X_k$. If $H^2(X_k,\mathcal{O}_{X_k}) = 0$, there is an invertible sheaf $\mathcal{L}$ on $X$ and an isomorphism $\psi : \mathcal{L}_k \iso \mathcal{L}_0$ of coherent sheaves on $X_k$. If in addition $H^1(X_k,\mathcal{O}_{X_k}) = 0$, then $(\mathcal{L},\psi)$ is unique up to isomorphism of invertible sheaves on $X$.

4. Let $\mathcal{L}$ be an invertible sheaf on a proper $\Lambda$-scheme $p : X \to \text{Spec} \, \Lambda$. If $H^1(X_k,\mathcal{L}_k) = 0$, then $p_*\mathcal{L}$ is a free $\Lambda$-module and the natural map $\pi_*\mathcal{L} \otimes_{\Lambda} k \to H^0(X_k,\mathcal{L}_k)$ is an isomorphism. In particular, each section $s_0 \in H^0(X_k,\mathcal{L}_k)$ lifts to a section $s \in H^0(X,\mathcal{L})$.

**Proof.** (1) Let $\hat{X}, \hat{Y}$ denote the formal schemes associated to the $\Lambda$-schemes $X,Y$. By [10] Exp. III, Corollaire 5.6], $f_0$ extends to a morphism of formal schemes $\hat{f} : \hat{Y} \to \hat{X}$. By [9] Chap. III, Théorème 5.4.1], there is a unique $\Lambda$-morphism $f : Y \to X$ inducing $\hat{f}$ on the formal schemes. In particular, $f_k = f_0$.

If moreover $f_0$ is a closed immersion, then it follows that $f : \hat{Y} \to \hat{X}$ is a (formal) closed immersion. Then [9] Chap. III, Corollaire 5.1.8, Théorème 5.4.1] implies that $f : Y \to X$ is a closed immersion.

(2) This can be found in [18] Theorem 8.5.9(b)].

(3) Use [18] Theorem 8.5.5].

(4) Apply [18] Theorem 8.5.3(a)] with $E = \Lambda X$, $F = \mathcal{L}$.
Lemma 9.2. Let $S$ be a del Pezzo surface over a field $k$, with effective Cartier divisor $D$. As above, let $d_S = \deg_k(K_S \cdot K_S)$ and $d = \deg_k(-K_S \cdot D)$. Then

1. $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$.

2. Let $T_{S/k}$ denote the tangent sheaf. Then $H^2(S, T_{S/k}) = 0$ and

$$\dim_k H^1(S, T_{S/k}) = \begin{cases} 
0 & \text{if } d_S \geq 5 \\
5 - d_S & \text{if } 1 \leq d_S < 5 
\end{cases}$$

3. $H^1(S, \mathcal{O}_S(D)) = 0$

Proof. Since cohomology commutes with flat base-change, we may extend from $k$ to its algebraic closure, and assume from the start that $k$ is algebraically closed. Then $S$ is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow-up of $\mathbb{P}^2$ at $r := 9 - d_S \geq 0$ points.

For [1], if $S = \mathbb{P}^1 \times \mathbb{P}^1$, then the K"unneth formula gives $H^1(S, \mathcal{O}_S) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^2 = 0$, $H^2(S, \mathcal{O}_S) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^2 = 0$.

For $S = \mathbb{P}^2$, the vanishing of $H^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})$ for $i > 0$ may be found in [47, Chap. III, §3, Proposition 8].

If $\pi : S \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ at $\{p_1, \ldots, p_r\}$, $r \geq 1$, let $E_i = \pi^{-1}(p_i)$. We compute $H^i(S, \mathcal{O}_S)$ via the Leray spectral sequence

$$E_2^{p,q} = H^p(\mathbb{P}^2, R^q\pi_* \mathcal{O}_S) \Rightarrow H^{p+q}(S, \mathcal{O}_S)$$

Since $\pi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2}$, we need only show that $R^q\pi_* \mathcal{O}_S = 0$ for $q > 0$. We use the formal functions thereon

$$(R^q\pi_* \mathcal{O}_S)_{p_i} = \lim_{n \to 0} H^q(E_i, \mathcal{O}_{S/k}/T_{E_i}^{n+1})$$

As $E_i \cong \mathbb{P}^1$ and $T_{E_i}/T_{E_i}^{n+1} \cong \mathcal{O}_{\mathbb{P}^1}(n)$, we find that $(R^q\pi_* \mathcal{O}_S)_{p_i} = 0$; clearly $(R^q\pi_* \mathcal{O}_S)_{p_i} = 0$ for $p$ not among the $p_i$, completing the proof of [1].

For [2], in case $S = \mathbb{P}^1 \times \mathbb{P}^1$, we have $T_{S/k} = p_1^* \mathcal{O}_{\mathbb{P}^1}(2) \oplus p_2^* \mathcal{O}_{\mathbb{P}^1}$, from which [2] easily follows. If $\pi : S \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ at $\{p_1, \ldots, p_r\}$, let $E_i = \pi^{-1}(p_i)$. If $r = 0$, we have the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^3 \to T_{\mathbb{P}^2/k} \to 0$$

and $H^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = 0$ for $i > 0, d \geq -1$, giving $H^i(\mathbb{P}^2, T_{\mathbb{P}^2/k}) = 0$ for $i > 0$. This also shows that $\dim_k H^0(\mathbb{P}^2, T_{\mathbb{P}^2/k}) = 8$.

For $0 < r$, we have the exact sequence

$$0 \to T_{S/k} \xrightarrow{d\pi} \pi^* T_{\mathbb{P}^2/k} \to \bigoplus_{j=1}^r i_{j*} \mathcal{O}_{E_j}(-E_j^{(2)}) \to 0$$

with $i_j : E_j \to S$ the inclusion. Identifying $E_j$ with $\mathbb{P}^1$, we have $\mathcal{O}_{E_j}(-E_j^{(2)}) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, so $H^i(\mathcal{O}_{E_j}(-E_j^{(2)})) = 0$ for $i > 0$. Using the Leray spectral sequence again, we see that $H^i(S, \pi^* T_{\mathbb{P}^2/k}) \cong H^i(\mathbb{P}^2, T_{\mathbb{P}^2/k}) = 0$ for $i > 0$. Thus $H^2(S, T_{S/k}) = 0$ and we have the exact sequence

$$H^0(\mathbb{P}^2, T_{\mathbb{P}^2/k}) \xrightarrow{\sum i_{j*}^j} \bigoplus_{j=1}^r H^0(E_j, \mathcal{O}_{E_j}(-E_j^{(2)})) \to H^1(S, T_{S/k}) \to 0$$

Taking parameters $(x, y)$ at $p_j$, we identify $E_i$ with $\mathbb{P}^1$, $\mathcal{O}_{E_i}(-E_i^{(2)})$ with $\mathcal{O}_{\mathbb{P}^1}(1)$, $T_{\mathbb{P}^2, p_j}$ with $k \cdot \partial / \partial x \oplus k \cdot \partial / \partial y$, and we have $i_{j*}^j(\partial / \partial x) = -X_1$, $i_{j*}^j(\partial / \partial y) = X_0$. This identifies $\pi_* (\mathcal{O}_{E_i}(-E_i^{(2)}))$ with $T_{\mathbb{P}^2, p_j}$, giving the exact sequence

$$H^0(\mathbb{P}^2, T_{\mathbb{P}^2/k}) \xrightarrow{\sum i_{j*}^j} \bigoplus_{j=1}^r T_{\mathbb{P}^2, p_j} \to H^1(S, T_{S/k}) \to 0.$$
The automorphism group \( \text{PGL}_3 \) of \( \mathbb{P}^2 \) acts 4-transitively on 4-tuples of points, no three of which lie on a line. Since \( S \) is a del Pezzo surface, \( S \) has no \(-a\) curves for \( a > 0 \), so this condition is satisfied for the set \( \{p_1, \ldots, p_r\} \), and thus the map \( \sum_i i_{i,p_j}^* \) is surjective for \( r \leq 4 \). For \( r = 4 \), counting dimensions shows \( \sum_i i_{i,p_j}^* \) is an isomorphism, and for \( r > 4, \sum_i i_{i,p_j}^* \) is injective, giving

\[
\dim_k H^1(S, T_{S/k}) = r - 4 = 5 - d_s
\]
as claimed.

For (3) we have the exact sequence

\[
0 \to \mathcal{O}_S \to \mathcal{O}_S(D) \to i_{D,*} \mathcal{O}_D(D^{(2)}) \to 0
\]
so we reduce to showing \( H^1(D, \mathcal{O}_D(D^{(2)})) = 0 \). Letting \( \omega_D \) denote the dualizing sheaf on \( D \), Serre duality gives \( H^1(D, \mathcal{O}_D(D^{(2)})) \cong H^0(D, \omega_D \otimes \mathcal{O}_D(-D^{(2)})) \). But the adjunction formula says \( \omega_D = K_S(D) \otimes \mathcal{O}_S \), so \( \omega_D \otimes \mathcal{O}_D(-D^{(2)}) \cong \mathcal{O}_D(D \cdot S) \). Since \( -K_S \) is ample, \( \deg_{D} \mathcal{O}_D(K_S \cdot D) < 0 \), so \( H^0(D, \mathcal{O}_D(K_S \cdot D)) = 0 \).

**Lemma 9.3.** Let \( S \) be a del Pezzo surface over a field \( k \), with effective Cartier divisor \( D \). Let \( d_S = \deg_k(K_S \cdot K_S) \) and \( d = \deg_k(-K_S \cdot D) \). Let \( \Lambda \) be a complete discrete valuation ring with residue field \( k \) and quotient field \( K \). Then

1. There is a smooth proper \( \Lambda \)-scheme \( \tilde{S} \to \text{Spec} \, \Lambda \) with an isomorphism \( \phi : \tilde{S} \to S \).
2. For each lifting \( (\tilde{S}, \phi) \) of \( S \) over \( \Lambda \) as in (2), letting \( i : S \to \tilde{S} \) be the closed immersion induced by \( \phi \), the restriction map \( i^* : \text{Pic}(\tilde{S}) \to \text{Pic}(S) \) is an isomorphism.
3. For each lifting \( (\tilde{S}, \phi) \) of \( S \) over \( \Lambda \) as in (2), \( \tilde{S} \) is a del Pezzo surface over \( \Lambda \) and the generic fiber \( \tilde{S}_K \) is a del Pezzo surface over \( K \). Moreover, we have \( d_{\tilde{S}_K} = d_s \).
4. For each lifting \( (\tilde{S}, \phi) \) of \( S \) over \( \Lambda \) as in (4), there is an effective Cartier divisor \( \tilde{D} \) on \( \tilde{S} \) with \( \phi(\tilde{D}) = D \). Moreover, we have \( \deg_k(-K_{\tilde{S}_K} \cdot \tilde{D}_K) = d \).

**Proof.**

(1) follows from Proposition 9.1(2) and Lemma 9.2(2).

For (2) we have \( H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0 \) by Lemma 9.2(1). Applying Proposition 9.1(3) shows that \( i^* \) is an isomorphism.

For (3) \(-K_S \) is ample, and \( K_S \) lifts canonically to the relative dualizing sheaf \( \omega_{\tilde{S}/\Lambda} \) which restricted to \( \tilde{S}_K \) is the canonical sheaf \( K_{\tilde{S}_K} \). By 9.1 Théorème 5.4.5, there is an ample invertible sheaf \( \mathcal{L} \) on \( \tilde{S} \) with \( \phi(\mathcal{L}_k) = -K_S \). But then by (2) \( \mathcal{L} \) is isomorphic to \( \omega_{\tilde{S}/\Lambda} \), so \( -K_{\tilde{S}_K} \) is ample on \( \tilde{S}_K \). Thus \( \tilde{S} \) is a del Pezzo surface over \( \Lambda \) and \( \tilde{S}_K \) is a del Pezzo surface over \( K \). The assertion that \( d_{\tilde{S}_K} = d_s \) follows from the conservation of intersection numbers (see e.g. 25, B18).

Finally, to prove (4), we have \( H^1(\tilde{S}, \mathcal{O}_S(D)) = 0 \) by Lemma 9.2(3) and by (2) there is an invertible sheaf \( \mathcal{L} \) on \( \tilde{S} \) lifting \( \mathcal{O}_S(D) \). Let \( s_0 \in H^0(\tilde{S}, \mathcal{O}_S(D)) \) be the canonical section, so \( \divisor(s_0) = D \). By Proposition 9.4(4) we may lift \( s_0 \) to a section \( s \in H^0(\tilde{S}, \mathcal{L}) \); letting \( \tilde{D} = \divisor(s) \), we see that \( \phi(\tilde{D}_k) = D \). The identity \( \deg(-K_{\tilde{S}_K} \cdot \tilde{D}_K) = d \) follows as above by conservation of intersection numbers.

**Lemma 9.4.** Let \( S \) be a del Pezzo surface over a field \( k \), with effective Cartier divisor \( D \), and let \( \tilde{S}, \tilde{D} \) be a lifting of \((S, D)\) over \( \Lambda \) as in Lemma 9.3. Suppose that \( S, D \)
Then \( S_K, D_K \) satisfy Basic Assumptions 4.1(2)(3). Then \( S_K, D_K \) satisfy Basic Assumptions 4.1(2)(3) and Assumption 2.30.

**Proof.** By Lemma 9.3, we have \( d_{S_K} = d_S \) and \( d_K = d \). Thus Basic Assumption 4.1(3) for \( S, D \) implies this assumption for \( S_K, D_K \). Suppose \( E \subset S \) is a -1 curve. Then by Lemma 9.3, there is a lifting \( \tilde{E} \) of \( E \) to a relative Cartier divisor on \( \tilde{S} \), and \( \tilde{E}_K \) is a -1 curve on \( \tilde{S}_K \). Moreover, by Lemma 9.3(2) if \( D = m \cdot E \), then \( D_K = m \cdot \tilde{E}_K \), so Basic Assumptions 4.1(2) for \( S, D \) implies this assumption for \( S_K, D_K \). Basic Assumption 4.1(1) is trivially satisfied for \( S_K, D_K \) since \( K \) has characteristic zero. Similarly, Lemma 2.31 implies that \( S_K, D_K \) satisfy Assumption 2.30. □

### 9.2. The moduli space \( \mathcal{M}_{0,n}(S, D)_{\text{good}} \) and its first properties.

Let \( k \) be a perfect field of characteristic \( p > 3 \). Let \( S \) be a del Pezzo surface over \( k \) with effective Cartier divisor \( D \). We assume that \( D \) is not the zero divisor; let \( d = \deg(-K_S \cdot D) \geq 1 \).

Let \( \Lambda \) be a complete discrete valuation ring with residue field \( k \) and quotient field \( K \) of characteristic 0. We fix a lifting \( \tilde{S} \rightarrow \Spec \Lambda, \tilde{D} \) of \((S, D)\), which exists by Lemma 9.3 also by that result, the generic fiber \( S_K \) is a del Pezzo surface with \( d_{S_K} = d_S \), and the effective Cartier divisor \( D_K \) on \( S_K \) has degree \( d_K := \deg_K(-K_{S_K} \cdot \tilde{D}_K) = d \).

The following elementary lemma will be used below.

**Lemma 9.5.** Let \( Y \rightarrow \Spec(\Lambda) \) be of finite type and let \( Z' \subset Y_K \) be a closed subscheme. Let \( Z \) be the closure of \( Z' \) in \( Y \). Then

1. \( Z \) is flat over \( \Lambda \). In particular, no irreducible component of \( Z \) is contained in the fiber of \( Y \) over the closed point of \( \Spec \Lambda \).
2. Suppose that \( Z' \) is reduced. Then \( Z \) is reduced.
3. Let \( W \) be a reduced closed subscheme of \( Y \) such that no irreducible component of \( W \) is contained in the fiber of \( Y \) over the closed point of \( \Spec \Lambda \). Then \( W \) is flat over \( \Lambda \).
4. Let \( W \subset Y \) be a closed subscheme of \( Y \) containing \( Z' \), with support of \( W \) equal to the support of \( Z \) and with \( W_K = Z' \). Suppose that the special fiber \( W_K \) is reduced. Then \( W = Z \).

**Proof.** We claim that the sheaf \( \mathcal{O}_Z \) is t-torsion free, where \( t \in \Lambda \) is a generator of the maximal ideal. This implies the result, since a \( \Lambda \)-module \( M \) is flat if and only if \( M \) is t-torsion free.

To see that \( \mathcal{O}_Z \) is t-torsion free, we may assume that \( Y \) is affine and finite type over \( \Lambda \), \( Y = \Spec \Lambda \). Then \( Y \) is a closed subscheme of an affine space over \( \Lambda \), \( \mathbb{A}^n_\Lambda \), and the closure of \( Z' \) in \( Y \) is the same as the closure in \( \mathbb{A}^n_\Lambda \), so we may assume that \( Y = \mathbb{A}^n_\Lambda \). Let \( K \) be the quotient field of \( \Lambda \) and let \( I' \subset K[x_1, \ldots, x_n] \) be the ideal of \( Z' \). Then the ideal \( I \) of \( Z \) in \( \Lambda[x_1, \ldots, x_n] \) is the maximal ideal \( J \) such that \( JK[x_1, \ldots, x_n] = I' \).

Take \( \bar{x} \in \Lambda[x_1, \ldots, x_n]/I \) such that \( t\bar{x} = 0 \). Lifting \( \bar{x} \) to \( x \in \Lambda[x_1, \ldots, x_n] \), we have \( tx \in I \). Lifting \( x \) to \( \bar{x} \in \Lambda[x_1, \ldots, x_n] \) we have \( x \in I \) and \( \bar{x} = 0 \). This proves the first assertion of (1).

For (2), suppose that \( Z' \) is reduced. Let \( J \subset \mathcal{O}_Z \) be the ideal sheaf of \( Z_{\text{red}} \) in \( Z \), then since \( Z' \) is reduced and \( Z_K = Z' \), we have \( J_K = 0 \). Again by the maximality of \( \mathcal{I}_Z \), we must have \( J = 0 \), so \( Z \) is reduced.

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Let \( W \subset Y \) be as in (3) and let \( W' \subset W \) be the closure of \( W_K \). By (1) \( W' \) is flat over \( \Lambda \). But as \( W' \) and \( W \) have the same support and \( W \) is reduced, we must have \( W' = W \), so \( W \) is flat over \( \Lambda \), proving (3).

For (4) let \( \mathcal{I}_W \subset \mathcal{O}_W \) be the ideal sheaf of \( W \). Again by maximality of \( \mathcal{I}_Z \), we have \( \mathcal{I}_W \subset \mathcal{I}_Z \); let \( \mathcal{J} \subset \mathcal{O}_W \) be the image of \( \mathcal{I}_Z \). Since \( W_K = Z' = Z_K \), we have \( \mathcal{J}_K = 0 \), that is, \( \mathcal{J} \) is supported on \( Y_k \). Applying \(- \otimes_{\Lambda} k\) to the exact sequence
\[
0 \to \mathcal{J} \to \mathcal{O}_W \to \mathcal{O}_Z \to 0
\]
and recalling that \( Z \) is flat over \( \Lambda \), we have the exact sequence
\[
0 \to \mathcal{J}/t\mathcal{J} \to \mathcal{O}_{W_k} \to \mathcal{O}_{Z_k} \to 0
\]
But \( Z_k \) is a closed subscheme of \( W_k \) with the same irreducible components, and \( W_k \) is reduced, so \( Z_k = W_k \) and thus \( \mathcal{J}/t\mathcal{J} = 0 \). Since \( \mathcal{J} \) is supported on \( Y_k \), it follows from Nakayama’s lemma that \( \mathcal{J} = 0 \), so \( W = Z \). \( \square \)

We recall the moduli stack \( \bar{M}_{0,n}(\tilde{S}, \tilde{D}) \). We have the evaluation map
\[
ev_k : \bar{M}_{0,n}(\tilde{S}, \tilde{D}) \to \tilde{S}^n
\]
lifting \( \ev_k : \bar{M}_{0,n}(S, D) \to S^n \); here we write \( \tilde{S}^n \) for the \( n \)-fold fiber product of \( \tilde{S} \) over \( \text{Spec} \Lambda \).

We now extend the construction of the open subset \( \bar{M}_{0,n}(S', D')_{\text{good}} \) as outlined in Theorem 4.5 to the mixed characteristic case

**Construction 9.6.** Suppose that \((S, D)\) satisfies Basic Assumptions 4.1 (2) (3) and Assumption 2.30. Then by Lemma 9.4 \( \tilde{S}, \tilde{D} \) satisfy all the Basic Assumptions 4.1. Thus, we may apply Theorem 4.5 to take a closed subset \( A_K \subset \tilde{S}^n_K \) as in Theorem 4.5 and let \( \bar{A}_K \subset \tilde{S}^n \) be its closure. By Lemma 9.5 \( \bar{A}_K \) has codimension \( \geq 2 \) in \( \tilde{S}^n \).

Recalling that \( M_{0,n}^{\text{odp}}(S, D) \) is open in \( \bar{M}_{0,n}(S, D) \) by Lemma 2.14 we let \( \mathcal{A} \) be the closed subset \( \ev_k (\bar{M}_{0,n}(S, D) \setminus M_{0,n}^{\text{odp}}(S, D)) \) of \( S^n \). By Corollary 3.15 \( \mathcal{A} \) has positive codimension in \( S^n \), since we are assuming \( S, D \) satisfies Assumption 2.30. Let
\[
(9.1) \quad \mathcal{A} := \bar{A}_K \cup A_k
\]
and define
\[
\bar{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} := \bar{M}_{0,n}(\tilde{S}, \tilde{D}) - \ev^{-1}(\mathcal{A}).
\]

We may freely enlarge \( \mathcal{A} \), as long as we ensure that \( \mathcal{A} \) remains closed in \( \tilde{S}^n \), \( \bar{A}_K \) satisfies the conditions of Theorem 4.5 for \( K, \tilde{S}_K, \tilde{D}_K \), and \( \mathcal{A}_K \) has positive codimension in \( S^n \).

For the remainder of \$\S 9\$ we will assume that \( S, D \) satisfies the conditions of Construction 9.6, that is Basic Assumptions 4.1 (2) (3) and Assumption 2.30 all hold for \( S, D \).

**Remark 9.7.** We show in the Appendix (Corollary 4.15) that del Pezzo surfaces in characteristic \( \neq 2 \) with \( d_S \geq 6 \) satisfy Assumption 2.30.

Our next task is to show that \( \bar{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \) is smooth over \( \Lambda \); we first need a lemma.

**Lemma 9.8.** Let \( f_0 : \mathbb{P}^1 \to S \) be a morphism in \( M_0(S, D) \). If \( f_0 \) is in \( M_0^{\text{unr}}(S, D) \), then \( f_0 \) lifts to a morphism \( f \in M_0^{\text{unr}}(\tilde{S}, \tilde{D}) \).
Proof. Since $f_0$ is unramified, we have $\mathcal{N}_{f_0} \cong O_{\mathbb{P}^1}(d - 2)$. Since $d \geq 1$, we have $H^1(\mathbb{P}^1, \mathcal{N}_f) = 0$. From the exact sequence

$$0 \to T_{\mathbb{P}^1} \to f_0^* T_S \to \mathcal{N}_f \to 0$$

and the fact that $T_{\mathbb{P}^1} \cong O_{\mathbb{P}^1}(2)$, we see that $H^1(\mathbb{P}^1, f_0^* T_S) = 0$. Applying Proposition 9.9, we see that $f_0$ lifts to a morphism $f : \mathbb{P}_\Lambda^1 \to \tilde{S}$. By Lemma 9.3(2), we see that $f$ is in $\tilde{M}_0(\tilde{S}, \tilde{D})$. Since the support of the cokernel of $df : f^* \Omega_{\tilde{S}/\Lambda} \to \Omega_{\mathbb{P}^1/\Lambda}$ is closed and has empty intersection with the special fiber $\mathbb{P}^1_\Lambda$. the fact that $\mathbb{P}^1_\Lambda$ is proper over $\Lambda$ implies that this cokernel is zero, hence $f$ is unramified.

Let $M_{0,n}(\tilde{S}, \tilde{D})^{\good} := M_{0,n}(\tilde{S}, \tilde{D}) \cap \tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$.

**Proposition 9.9.** $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ is smooth over $\Lambda$. Moreover, $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ is non-empty if and only if $M_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\good}$ is non-empty.

**Proof.** After replacing $\Lambda$ with an unramified extension $\Lambda \to \Lambda'$, with $\Lambda'$ a complete discrete valuation ring with residue field the algebraic closure of $k$, applying the base-change to $\Lambda'$ and changing notation, we may assume that $k$ is algebraically closed.

We first consider the case in which $\tilde{M}_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\good}$ is empty. We claim that in this case, $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ is itself empty. Indeed, by the construction of $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$, this is the same as asserting that $M_{0,\odp}^{\odp}(S, D)$ is empty. If not, take a $k$-point $f_0 : \mathbb{P}^1 \to S$ of $M_{0,\odp}(S, D)$. By Lemma 9.8, $f_0$ lifts to a morphism $f \in M_{0,\odp}^{\odp}(\tilde{S}, \tilde{D})$, and thus restricts to $f_K \in M_{0,\odp}^{\odp}(\tilde{S}_K, \tilde{D}_K)$. But then $M_{0,\odp}^{\odp}(\tilde{S}_K, \tilde{D}_K) \cap M_{0,\odp}^{\odp}(\tilde{S}_K, \tilde{D}_K) \neq \emptyset$, hence by Theorem 4.1, $M_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\good} \neq \emptyset$, contrary to our assumption. This proves the second assertion in the statement of the proposition.

Thus, if $\tilde{M}_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\good}$ is empty, then $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ is empty and hence is smooth over $\Lambda$, as desired.

We now assume that $\tilde{M}_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\good}$ is non-empty. Since $\tilde{M}_{0,n}$ commutes with base-change, the generic fiber of $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ is smooth by construction and Theorem 4.3, and is non-empty by assumption. Similarly, the special fiber of $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ is contained in $M_{0,\odp}^{\odp}(S, D)$ by construction, which is smooth by Lemma 4.2. Thus the structure map $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good} \to \text{Spec } \Lambda$ has smooth fibers.

By [49 01V8], it is then enough to show that $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ is flat over $\Lambda$.

Let $Z$ be the closure of the generic fiber $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ in $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$. By Lemma 9.5(1), it suffices to show that $Z = \tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$.

Clearly $Z$ is a closed subscheme of $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$. We first show that $Z$ and $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ have the same support.

To show this, it suffices to show that for each point $x_0$ in the special fiber $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$, there is a point $x \in \tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ that specializes to $x$. In particular, if $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ is empty, there is nothing to prove, so assume that $\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good}$ is non-empty.

Choose a point $x_0 := (f_0, p_\ast) \in \tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\good} \subset M_{0,\odp}^{\odp}(S, D)$.

By Lemma 9.8, $f_0$ lifts to a morphism $f \in M_{0,\odp}^{\odp}(\tilde{S}, \tilde{D})$.

Since $\Lambda$ is a complete discrete valuation ring, each of the $k$-points $p_1, \ldots, p_n$ of $\mathbb{P}^1_k$ lift to $\Lambda$-points $p_1, \ldots, p_n$ of $\mathbb{P}^1_\Lambda$.
of \([\mathbb{P}^1]_{\Lambda}\), giving us the lifting of \((f_0, p_\ast)\) to a point \((f, p_\ast)\) of \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})\). Because the closure in \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})\) of the complement of \(M_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\text{good}}\) in \(\tilde{M}_{0,n}(\tilde{S}_K, \tilde{D}_K)\) is disjoint from \(M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\) by construction, it follows that \(x := (f, p_\ast)\) is a point of \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\). Thus \(x_0\) is a specialization of \(x\), as desired.

Finally, since \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\tilde{k}}^{\text{good}}\) is smooth over \(k\), the special fiber \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\tilde{k}}^{\text{good}}\) of \(M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\) is reduced. We have \(Z_K = M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\) be construction. Thus by Lemma \([9.3][4]\), we have \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} = Z\), completing the proof. \(\square\)

For the remainder of § 9, we assume that \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\) is non-empty; equivalently (Proposition \[9.9\]), \(\tilde{M}_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\text{good}}\) is non-empty.

### 9.3. Divisors and the double point locus for \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\)

The morphism \(\tilde{ev} : \tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \to S^n - \tilde{A}\) is proper because it is the pullback of a proper morphism, and quasi-finite by Theorem \([4.5][1]\) and Lemma \([2.27]\). Thus \(\tilde{ev} : \tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \to S^n - \tilde{A}\) is finite.

Define \(D_{\text{tac}}\) to be the closure of \((D_{\text{tac}})_K\) in \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\). By Lemma \([9.5]\), the intersection \(D_{\text{tac}} \cap \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\tilde{k}}^{\text{good}}\) has codimension 1 in \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\tilde{k}}^{\text{good}}\) and is flat over \(\Lambda\). Since \(\tilde{ev}\) is finite, \(\tilde{ev}(D_{\text{tac}} \cap \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\tilde{k}}^{\text{good}})\) is at least codimension 1 in \(S^n_k\), whence codimension 2 in \(\tilde{S}^n\). Adding \(\tilde{ev}(D_{\text{tac}} \cap \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\tilde{k}}^{\text{good}})\) to \(\tilde{A}\), we may assume that \(D_{\text{tac}}\) has empty intersection with \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\tilde{k}}^{\text{good}}\). Since \(D_{\text{tac}}\) is closed and codimension 1 in a smooth scheme, \(D_{\text{tac}}\) is a relative Cartier divisor. We may similarly define \(D_{\text{cusp}}\) to be the closure of \((D_{\text{cusp}})_K\) and assume that \(D_{\text{cusp}}\) is a Cartier divisor on \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\) which has empty intersection with \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\tilde{k}}^{\text{good}}\).

#### Lemma 9.10

The divisor of the section \(\det d(\tilde{ev}) : \mathcal{O}_{\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}} \to \omega_{\tilde{ev}}\) is \(D_{\text{cusp}}\), and thus \(\det d(\tilde{ev})\) defines an isomorphism

\[
\det d(\tilde{ev}) : \mathcal{O}_{\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}}(D_{\text{cusp}}) \to \omega_{\tilde{ev}}
\]

on \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\).

**Proof.** The evaluation map is compatible with base change. Suppose that \(M_{0,n}^{\text{odp}}(\tilde{S}_K, \tilde{D}_K)\) is non-empty. On the special fiber, \(\tilde{M}_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\text{good}}\) is contained in \(M_{0,n}^{\text{odp}}(\tilde{S}_K, \tilde{D}_K)\). By Lemma \([2.27]\), \(\tilde{ev} \equiv \tilde{ev}_k\) is étale on \(M_{0,n}^{\text{odp}}(\tilde{S}_k, \tilde{D}_k)\), so \(\tilde{ev}_k : \tilde{M}_{0,n}(\tilde{S}_k, \tilde{D}_k)^{\text{good}} \to S^n\) is flat and unramified. Since ramification can be checked on fibers of a smooth \(\Lambda\)-scheme, \(\tilde{ev}\) is unramified at points of \(\tilde{M}_{0,n}(\tilde{S}_k, \tilde{D}_k)^{\text{good}}\). Since \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\) is flat over \(\Lambda\) (Proposition \([9.9]\)), it follows from \([49]\) tag \(02G8)\) that \(\tilde{ev}\) is flat on \(\tilde{M}_{0,n}(\tilde{S}_k, \tilde{D}_k)^{\text{good}}\). Thus \(\tilde{ev}\) is étale on \(\tilde{M}_{0,n}(\tilde{S}_k, \tilde{D}_k)^{\text{good}}\), whence \(\tilde{ev}\) is étale over an open neighborhood \(U\) of the special fiber \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\tilde{k}}^{\text{good}} \subset \tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\). Thus \(\det d(\tilde{ev}) \cap U = 0\).

We recall that \(\tilde{S}_K, \tilde{D}_K\) satisfy Basic Assumptions \([4.1]\) by Lemma \([9.4]\). Over the open set of \(\tilde{M}_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\text{good}}\) given by the generic fiber of \(M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}\), the proposition follows from Proposition \([4.9]\).

If \(M_{0,n}^{\text{odp}}(\tilde{S}_k, \tilde{D}_k)\) is empty, then we need only check on \(\tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\tilde{k}}^{\text{good}} \subset \tilde{M}_{0,n}(\tilde{S}_K, \tilde{D}_K)^{\text{good}}\), which follows as above from Proposition \([4.9]\). \(\square\)
Forgetting the last marked point defines a map \( \tilde{M}_{0,n+1}(\tilde{S}, \tilde{D}) \to \tilde{M}_{0,n}(\tilde{S}, \tilde{D}) \) from the universal curve to the moduli space. Define \( \tilde{X}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \subset \tilde{M}_{0,n+1}(\tilde{S}, \tilde{D}) \) to be the inverse image of \( \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \).

Define the double point locus \( \tilde{\pi} : \tilde{D} \to \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \) using the natural analogue of Definition 5.3.

**Lemma 9.11.** Let \( \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} := \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \cap \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D}) \). The double point locus \( \tilde{D} \) satisfies the following.

1. By possibly enlarging \( \Lambda \), we may take \( \tilde{D} \) to be smooth over \( \Lambda \).
2. The map \( \tilde{\pi} \) is finite and flat.
3. The map \( \tilde{\pi} \) is étale over \( \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \) and \( \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \) is an open neighborhood of \( \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \) in \( \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \).

**Proof.** At points of the generic fiber \( \tilde{D}_K \), it follows from Corollary 5.14 that, after enlargeing \( \Lambda \) if necessary, \( \tilde{D} \) is smooth over \( \Lambda \); similarly, (2) holds for

\[ \tilde{\pi} : \tilde{D}_K \to \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}}. \]

The fact that \( \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \) is an open neighborhood of \( \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \) in \( \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \)

follows from the construction of \( \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \).

Let \( \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} := \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \cap \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D}) \) and let \( \tilde{D}^{\text{odp}} \) be the restriction of \( \tilde{D} \) over the open subscheme \( \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \). Define \( \tilde{X}_{0,n}^{\text{odp}, \text{good}} \to \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \) similarly. Since \( \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \) is an open neighborhood of the special fiber \( \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \) in \( \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \), to complete the proof, it suffices to prove (1) and (2) for the restriction

\[ \tilde{\pi}^{\text{odp}} : \tilde{D}^{\text{odp}} \to \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \]

of \( \tilde{\pi} \).

Since \( \tilde{D}^{\text{odp}} \) is closed in \( \tilde{X}_{0,n}^{\text{odp}, \text{good}} \times \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \) and \( \tilde{X}_{0,n}^{\text{odp}, \text{good}} \times \tilde{M}_{0,n}(\tilde{S}, \tilde{D})_{\text{good}} \),

\[ \tilde{X}_{0,n}^{\text{odp}, \text{good}} \to \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \]

is proper, this shows that \( \tilde{D}^{\text{odp}} \to \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \) is proper.

By Lemma 2.11

\[ \tilde{D}^{\text{odp}} \cap \Delta \tilde{X}_{0,n}^{\text{odp}, \text{good}} = \emptyset \]

the intersection taking place in \( \tilde{X}_{0,n}^{\text{odp}, \text{good}} \times \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \). Thus

\[ \tilde{D}^{\text{odp}} = (\tilde{X}_{0,n}^{\text{odp}, \text{good}} \times \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \tilde{X}_{0,n}^{\text{odp}, \text{good}}) \setminus (\Delta \tilde{X}_{0,n}^{\text{odp}, \text{good}}) \]

where \( \tilde{X}_{0,n}^{\text{odp}, \text{good}} \to \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \times \Lambda \tilde{S} \) is the universal map and \( \Delta \tilde{X}_{0,n}^{\text{odp}, \text{good}} \) is the relative diagonal.

Next we recall that \( \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D}) \) is smooth over \( \Lambda \) and \( \tilde{X}_{0,n}^{\text{odp}} \) is smooth over \( \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D}) \), hence \( \tilde{X}_{0,n}^{\text{odp}, \text{good}} \times \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \tilde{X}_{0,n}^{\text{odp}, \text{good}} \) is smooth over \( \Lambda \). To prove (1) it thus suffices to show that \( \tilde{X}_{0,n}^{\text{odp}, \text{good}} \times \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \tilde{X}_{0,n}^{\text{odp}, \text{good}} \) is transverse to the inclusion \( \Delta \tilde{X}_{0,n}^{\text{odp}, \text{good}} \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D})_{\text{good}} \times \Lambda \tilde{S} \times \Lambda \tilde{S} \), at points away from \( \Delta \tilde{X}_{0,n}^{\text{odp}, \text{good}} \). This transversality follows immediately from the definition of \( \tilde{M}_{0,n}^{\text{odp}}(\tilde{S}, \tilde{D}) \).

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Similarly, this transversality implies that \( \tilde{D}^{\text{odp}} \to M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \) is a smooth morphism and that \( D^{\text{odp}} \) has codimension two in \( \tilde{X}_{0,n}^{\text{odp}, \text{good}} \times_{M_{0,n}(\tilde{S}, \tilde{D})} \tilde{X}_{0,n}^{\text{odp}, \text{good}} \).

Since \( \tilde{D}^{\text{odp}} \to M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \) is thus smooth, proper and of relative dimension zero, it follows that \( D^{\text{odp}} \to M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \) is finite and étale. This completes the proof.

\[ \square \]

As before, we have \( \text{disc}_{\tilde{\pi}} : \mathcal{O}_{\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}} \to (\det \tilde{\pi}_* \mathcal{O}_{\tilde{D}})^{\otimes -2} \). (See Section 6.6.

By Lemma 9.11 (2) (3) \( \tilde{\pi} \) admits the claimed discriminant, and \( \det \tilde{\pi}_* \mathcal{O}_{\tilde{D}} \) is a line bundle.

**Lemma 9.12.** The divisor of \( \text{disc}_{\tilde{\pi}} \) is computed as

\[ \text{div} \text{disc}_{\tilde{\pi}} = 1 \cdot D_{\text{cusp}} + 2 \cdot D_{\text{tac}} \]

and thus \( \text{disc}_{\tilde{\pi}} \) defines an isomorphism

\[ \text{disc}_{\tilde{\pi}} : \mathcal{O}_{\tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}}} \to (\det \tilde{\pi}_* \mathcal{O}_{\tilde{D}})(-D_{\text{tac}})^{\otimes -2} \]

**Proof.** By Lemma 9.11 (3) \( \tilde{\pi} \) is étale over an open neighborhood \( U \) of the special fiber \( M_{0,n}(\tilde{S}, \tilde{D})_k^{\text{good}} \subset \tilde{M}_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \). Thus \( \text{div} \text{disc}_{\tilde{\pi}} \cap U = 0 \). Over the open set of \( M_{0,n}(\tilde{S}_K, D_K)^{\text{good}} \) given by the generic fiber, we may apply Proposition 6.1 which proves the claim.

**Theorem 9.13.** Let \( \mathcal{L} \) be the invertible sheaf on \( M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \) given by

\[ \mathcal{L} = (\det \tilde{\pi}_* \mathcal{O}_{\tilde{D}}(-D_{\text{tac}}))^{\otimes -1} \]

Then the composition \( \det d\tilde{\nu} \circ \text{disc}_{\tilde{\pi}}^{-1} : \mathcal{L}^{\otimes 2} \to \omega_{\tilde{\pi}_*} \) is an isomorphism on \( M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \).

**Proof.** Follows immediately from Lemmas 9.10 and 9.12 \( \square \)

9.4. The symmetrized evaluation map in positive characteristic. We continue to assume that \( M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \) is non-empty.

Just as in Section 7, let \( \tilde{S}_n \) denote the complement of the relative diagonals in \( \tilde{S}^n \), where the product is taken over \( \Lambda \). The symmetric group \( \mathcal{S}_n \) acts freely on \( \tilde{S}_n \), \( M_{0,n}(\tilde{S}, \tilde{D}) \), and the universal curve \( \tilde{M}_{0,n+1}(\tilde{S}, \tilde{D}) \). By enlarging the closed subset \( A \) of \( [9.1] \) to be invariant under \( \mathcal{S}_n \), we likewise obtain a free action on \( M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \), \( X_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \) and the double point locus \( \tilde{\pi} : \tilde{D} \to M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \).

Proceedings as in Section 7, we take quotients and form the following commutative diagram with Cartesian squares and vertical maps finite étale quotient maps:

\[ (9.2) \]

\[ \tilde{D} \overset{\tilde{\pi}}{\longrightarrow} M_{0,n}(\tilde{S}, \tilde{D})^{\text{good}} \overset{\tilde{\nu}}{\longrightarrow} \tilde{S}_n \]

\[ \overset{\tilde{\pi}_e}{\longrightarrow} M_{0,n,\mathcal{E}}(\tilde{S}, \tilde{D})^{\text{good}} \overset{\tilde{\nu}_e}{\longrightarrow} \text{Sym}_n \tilde{S} \]

For \( \square \in \{\text{cusp}, \text{tac}, \text{trip}\} \), we let \( D_{\square}^{\mathcal{E}} \) denote the reduced image of \( D_{\square} \) in \( M_{0,n,\mathcal{E}}^{\text{good}}(\tilde{S}, \tilde{D}) \).

**Theorem 9.14.** Let \( k \) be a perfect field of characteristic \( p > 3 \). Let \( S \) be a del Pezzo surface over \( k \) with effective Cartier divisor \( D \), satisfying Basic Assumptions 4.1 (3) and Assumption 2.3.\( \square \)
(1) The canonical section $\det d\tilde{e}v_{\bar{\sigma}} : \mathcal{O}_{\bar{M}_{0,n},\bar{\sigma}}(\bar{S},\bar{D}) \rightarrow \omega_{\bar{e}v_{\bar{\sigma}}}$ has divisor $1 \cdot D_{\text{cusp},\bar{\sigma}}$ and induces an isomorphism
$$\det d\tilde{e}v_{\bar{\sigma}} : \mathcal{O}_{\bar{M}_{0,n},\bar{\sigma}}(\bar{S},\bar{D})(D_{\text{cusp},\bar{\sigma}}) \rightarrow \omega_{\bar{e}v_{\bar{\sigma}}}. $$

(2) The divisor of $\text{disc}_{\bar{\sigma}} : \mathcal{O}_{\bar{M}_{0,n},\bar{\sigma}}(\bar{S},\bar{D}) \rightarrow [\det \tilde{\pi}_{\bar{\sigma}} \mathcal{O}_{\bar{D}_{\bar{\sigma}}}]^{\otimes -2}$ is $D_{\text{cusp},\bar{\sigma}} + 2 \cdot D_{\text{lac},\bar{\sigma}}$ and induces an isomorphism
$$\text{disc}_{\bar{\sigma}} : \mathcal{O}_{\bar{M}_{0,n},\bar{\sigma}}(\bar{S},\bar{D})(D_{\text{cusp},\bar{\sigma}}) \rightarrow [\det (\tilde{\pi}_{\bar{\sigma}})_{\ast} \mathcal{O}_{\bar{D}_{\bar{\sigma}}}]^{\otimes -2}.$$

(3) Letting $\mathcal{L}_{\bar{\sigma}} := \det^{-1} \tilde{\pi}_{\bar{\sigma}} \mathcal{O}_{\bar{D}_{\bar{\sigma}}}(\mathcal{O}_{\bar{D}_{\bar{\sigma}}}(\mathcal{O}_{\bar{D}_{\bar{\sigma}}}(-D_{\text{lac},\bar{\sigma}})))$ have the isomorphism
$$\det d\tilde{e}v_{\bar{\sigma}} \circ \text{disc}_{\bar{\sigma}}^{-1} : (\mathcal{L}_{\bar{\sigma}})^{\otimes 2} \rightarrow \omega_{\bar{e}v_{\bar{\sigma}}}. $$

Proof. The proof is essentially the same as the proof of Theorem 7.1. Replace the uses Proposition 4.9 and Proposition 6.1 with Lemmas 9.10 and 9.12.

9.5. Twists of the evaluation map in positive characteristic. We continue to assume that $M_{0,n}(\bar{S},\bar{D})_{\text{good}}$ is non-empty.

As in Section 8, let $\sigma = (L_1, \ldots, L_r)$ be an $r$-tuple of subfields $L_i \subset k_{\text{sep}}$ containing $k$ for $i = 1, \ldots, r$ subject to the requirement that $\sum_{i=1}^{k} [L_i : k] = n$. The reduction map defines an equivalence between the category of finite étale extensions of $\Lambda$ and the analogous category over $k$. [Exposé IX 1.10]. Thus the twisting construction from Section 8 lifts over $\Lambda$.

Let $\Lambda \subset \Lambda_{\text{unr}}$ be the extension corresponding to the separable closure $k_{\text{sep}}$ of $k$. ($k_{\text{sep}}$ is ind-finite.) Let $\bar{\Lambda} \rightarrow \bar{S}^n$ be a closed set as constructed in (9.1). By potentially enlarging $\bar{\Lambda}$ we may assume that $\Lambda$ is invariant under the action of symmetric group $S_n$. Proceeding as in Section 8 we obtain a $\Lambda$-map
$$\bar{e}v_{\bar{\sigma}} : \bar{M}_{0,n}(\bar{S},\bar{D})_{\text{good}} \rightarrow (\bar{S}^n \setminus \bar{\Lambda})_{\bar{\sigma}}$$
with special fiber $e\bar{v}_{\bar{\sigma}}$ and which is canonically identified with $\bar{e}v$ after base change to $\Lambda_{\text{unr}}$. We similarly twist the double point locus $\tilde{\sigma} : \tilde{D} \rightarrow \bar{M}_{0,n}(\bar{S},\bar{D})_{\text{good}}$ producing a $\Lambda$-map
$$\tilde{\sigma} : \tilde{D} \rightarrow \bar{M}_{0,n}(\bar{S},\bar{D})_{\text{good}}$$
We again have a forgetful map $\bar{M}_{0,n}(\bar{S},\bar{D})_{\text{good}} \rightarrow \bar{M}_{0,n}(\bar{S},\bar{D})$. For $\square \in \{\text{cusp, tac}\}$, we let $D_{\square,\bar{\sigma}}$ denote the preimage of $D_{\square}$ under this map.

Theorem 9.15. Let $k$ be a perfect field of characteristic $p > 3$. Let $S$ be a del Pezzo surface over $k$ with effective Cartier divisor $D$, satisfying Assumptions 4.7 (3) and Assumption 2.30. Let $\sigma$ be a map between smooth $\Lambda$-schemes.

(1) $e\bar{v}_{\sigma} : \bar{M}_{0,n}(\bar{S},\bar{D})_{\sigma} \rightarrow (\bar{S}^n)_{\sigma}$ is a map between smooth $\Lambda$-schemes.

(2) The canonical section $\det d\bar{e}v_{\sigma} : \mathcal{O}_{\bar{M}_{0,n},\sigma}(\bar{S},\bar{D})_{\sigma} \rightarrow \omega_{\bar{e}v_{\sigma}}$ has divisor $1 \cdot D_{\text{cusp},\sigma}$ and induces an isomorphism
$$\det d\bar{e}v_{\sigma} : \mathcal{O}_{\bar{M}_{0,n},\sigma}(\bar{S},\bar{D})_{\sigma}(D_{\text{cusp},\sigma}) \rightarrow \omega_{\bar{e}v_{\sigma}}. $$

(3) The divisor of $\text{disc}_{\bar{\sigma}} : \mathcal{O}_{\bar{M}_{0,n},\bar{\sigma}}(\bar{S},\bar{D})_{\bar{\sigma}} \rightarrow [\det (\tilde{\pi}_{\bar{\sigma}})_{\ast} \mathcal{O}_{\bar{D}_{\bar{\sigma}}}]^{\otimes -2}$ is
$$D_{\text{cusp},\bar{\sigma}} + 2 \cdot D_{\text{lac},\bar{\sigma}}, $$
and induces an isomorphism
$$\text{disc}_{\bar{\sigma}} : \mathcal{O}_{\bar{M}_{0,n},\bar{\sigma}}(\bar{S},\bar{D})_{\bar{\sigma}}(D_{\text{cusp},\bar{\sigma}}) \rightarrow [\det (\tilde{\pi}_{\bar{\sigma}})_{\ast} \mathcal{O}_{\bar{D}_{\bar{\sigma}}}(-D_{\text{lac},\bar{\sigma}})]^{\otimes -2}.$$
(4) Letting $L_{\sigma} := [\det(\tilde{\pi}_\sigma)_* \mathcal{O}_{\tilde{\mathcal{P}}_{\sigma}}(-D_{\text{tac}})]^{-1}$, we have the isomorphism
\[
\text{det } d\tilde{\psi}_\sigma \circ \text{disc}_{\tilde{\mathcal{P}}_{\sigma}}^{-1} : (L_{\sigma})^\otimes 2 \to \omega_{\tilde{\mathcal{P}}_{\sigma}}.
\]

Proof. The proof is parallel to the proof of Theorem 8.1. \hfill \Box

APPENDIX A. Unramified maps in positive characteristic

Fix a field $k$ and let $S$ be a smooth 2-dimensional toric variety over a field $k$. Such an $S$ is determined by a fan $\Delta$ in $\mathbb{R}^2$. Let $\Delta(1)$ denote the 1-dimensional cones of $\Delta$. Each $\rho$ in $\Delta(1)$ determines a divisor $D_\rho$ of $S$. Let $\Delta(2)$ denote the 2-dimensional cones. See [5] and [12] for further information.

Theorem A.1. Let $S$ be a smooth toric del Pezzo surface over a field $k$ and let $D$ be a divisor. Then the stack $M_0(S,D)$ and the scheme $\text{Maps}(\mathbb{P}^1, S; D)$ are both irreducible.

Proof. The stack $M_0(S,D)$ is the quotient stack of the space $\text{Maps}(\mathbb{P}^1, S; D)$ by the action of $\text{Aut}(\mathbb{P}^1)$. So, it suffices to prove that $\text{Maps}(\mathbb{P}^1, S; D)$ is irreducible. Let $d_\rho = D \cdot D_\rho$. Let $\mathcal{P}(D)$ denote the set of tuples of homogeneous polynomials $P_\rho \in k[t_0, t_1]$ of degree $d_\rho$ indexed by $\rho \in \Delta(1)$. We also think of $P_\rho$ as a section of $\mathcal{O}_P(d_\rho)$. Define $Z \subset \text{Spec } k[x_\rho : \rho \in \Delta(1)]$ by the vanishing for all $\sigma$ in $\Delta(2)$ of $\prod_{\rho \in \sigma} x_\rho$. Let $\mathcal{P}(D)$ denote the set of $(P_\rho)_\rho \in \mathcal{P}(D)$ such that $(P_\rho(t_0, t_1))_\rho \in Z$ only if $t_0 = t_1 = 0$. Let $G = \text{Maps}(\text{Pic}(S), \mathbb{G}_m)$. The group $G$ acts on $\mathcal{P}(D)$ by
\[
g \cdot (P_\rho)_\rho = (g([D_\rho])P_\rho)_\rho.
\]
By (A.1), we have
\[
\text{Maps}(\mathbb{P}^1, S; D) = \mathcal{P}(D)/G.
\]
Thus, it suffices to show that $\mathcal{P}(D)$ is an open subset of the affine space $\mathcal{P}(D)$. We do this now. For $\sigma \in \Delta(2)$ define $U_\sigma \subset \mathbb{P}^1 \times \mathcal{P}(D)$ by
\[
U_\sigma = \{(y, (P_\rho)_\rho)|P_\rho(y) \neq 0 \forall \rho \notin \sigma\}.
\]
Let $X \subset \mathbb{P}^1 \times \mathcal{P}(D)$ denote the complement of $\bigcup_\sigma U_\sigma$. $X$ is the locus of points $[t_0, t_1] \times (P_\rho)_\rho$ such that $(P_\rho(t_0, t_1))_\rho \in Z$ for some and therefore any choice of affine coordinates $(t_0, t_1)$ of the point $[t_0, t_1]$ in $\mathbb{P}^1$. The subset $X$ is closed because each $U_\sigma$ is open. Let $\pi : \mathbb{P}^1 \times \mathcal{P}(D) \to \mathcal{P}(D)$ denote the projection, which is proper because $\mathbb{P}^1$ is proper. Thus $\pi(X)$ is closed. By definition, $\mathcal{P}(D)$ is the complement of $\pi(X)$. Thus $\mathcal{P}(D)$ is an open subset of the affine space $\mathcal{P}(D)$ and $M_0(S,D)$ is irreducible as claimed. \hfill \Box

We now concentrate on the case of the blow-up of $\mathbb{P}^2$ at three non-collinear points; up to projective equivalence this is the blowup $\pi : S \to \mathbb{P}^2$ at the three points $p_0 := [1 : 0 : 0]$, $p_1 := [0 : 1 : 0]$, $p_2 := [0 : 0 : 1]$. Let $L_i \subset \mathbb{P}^2$ be the line $X_i = 0$, let $D_i \subset S$ be the proper transform of $L_i$, and let $D'_i = \pi^{-1}(p_i)$ be the exceptional divisors. We have
\[
\begin{align*}
D_i^{(2)} &= D_i' = -1, \quad i = 0, 1, 2 \\
D_i \cdot D'_i &= 0 = D_i' \cdot D'_j, \quad i \neq j \\
D_i \cdot D'_j &= 1, \quad i \neq j
\end{align*}
\]

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Moreover, $\text{CH}^1(S)$ is generated by the classes $\{[D_i], [D'_i] \mid i = 0, 1, 2\}$, with the relations

$$[D_i] + [D'_j] = [D_j] + [D'_i], \quad i \neq j$$

In particular, $\text{Pic}(S) \cong \mathbb{Z}^4$, with basis $[D'_0], [D'_1], [D'_2], [D_0]$ and $\pi^*(c_1(\mathcal{O}_{\mathbb{P}^2}(1))) = [D_i] + [D'_j] + [D'_k]$ for each choice of $i, j, k$ with $\{i, j, k\} = \{0, 1, 2\}$. The canonical class $K_S$ is $-\sum [D_0] - [D_1] - [D_2] - [D'_0] - [D'_1] - [D'_2]$.

From the above, we see that the curve class $D$ in $\text{Pic}(S)$ is uniquely determined by the intersection numbers $d_i := D \cdot D_i$ and $d'_i := D \cdot D'_i$ for all $i$. Then either $d_i \geq 0$, $d'_i \geq 0$ for $0, 1, 2$, or $D_0$ is the class of a positive multiple one of the -1 curves $D_i, D'_i$; in case $D_0 = mD_i$ with $m > 0$, then $d_i = -m$, $d'_i = 0$, $d_j = 0$ for $j \neq i$ and $d'_j = m$ for $j \neq i$. If $D_0 = mD'_i$ with $m > 0$, then $d_i' = -m$, $d_j = 0$, $d_j' = m$ for $j \neq i$.

$\textbf{Proof.}$ $\text{Maps}(\mathbb{P}^1, S, D_{D_0}) \neq \emptyset$ implies that the complete linear system $|D_{D_0}|$ contains a positive multiple of an integral curve $C$, namely $f_*(\mathbb{P}^1)$. If we have $mC \in |D_{D_0}|$ with $C$ integral and $m > 0$, and $C$ is not one of the $D_i$ or $D'_i$, then $C$ intersects each $D_i$ and $D'_i$ in a finite set of points, so $d_i = mC \cdot D_i \geq 0$ and $d'_i = mC \cdot D'_i \geq 0$ for all $i$. If $f_*(\mathbb{P}^1) = mD_i$, then $m > 0$ and we have $d_i = -m$, $d_i' = 0$, $d_j = 0$ for $j \neq i$ and $d_j' = m$ for $j \neq i$; the case $f_*(\mathbb{P}^1) = mD'_i$ is similar. \hfill $\Box$

We may blow down the disjoint -1 curves $D_0, D_1, D_2$, yielding a morphism $\pi' : S \to \mathbb{P}^2$ with $\pi'(D_i) = p_i$ and $\pi'(D'_i) = L_i$. The automorphism $\tau := \pi^{-1} \circ \pi'$ of $S$ exchanges $D_i$ with $D'_i$ for $i = 0, 1, 2$.

We recall a choice of representation of $\mathbb{P}^2$ and $S$ as smooth projective toric varieties. Let $T$ be the torus $\mathbb{G}_m^2$, $N = \mathbb{Z}^2$ the co-character lattice and $M = N^\vee \cong \mathbb{Z}^2$ the character lattice. $\mathbb{P}^2$ is represented as the toric variety for the fan $\Delta_{\mathbb{P}^2}$ in $\mathbb{R}^2 = N \otimes \mathbb{R}$ with $\Delta_{\mathbb{P}^2}(1)$ the set of rays $\{(1, 0) \cdot \mathbb{R}, (0, 1) \cdot \mathbb{R}, (-1, -1) \cdot \mathbb{R} \}$. This represents the blow-up $S$ as the toric variety for the fan $\Delta_S$ having rays $\Delta_S(1) = \{(1, 0) \cdot \mathbb{R}, (0, 1) \cdot \mathbb{R}, (-1, -1) \cdot \mathbb{R}, (1, 1) \cdot \mathbb{R}, (0, -1) \cdot \mathbb{R}, ((-1, 0) \cdot \mathbb{R}\}$. We have the bijection of $\Delta_S(1) \cap N$ with the monoid of effective $T$-invariant divisors in $S$ [12] bottom of pg. 60]; explicitly this is given by

$$\begin{align*}
(1, 0) & \leftrightarrow D_1 \\
(0, 1) & \leftrightarrow D_2 \\
(-1, -1) & \leftrightarrow D_0 \\
(1, 1) & \leftrightarrow D'_0 \\
(-1, 0) & \leftrightarrow D'_1 \\
(0, -1) & \leftrightarrow D'_2 
\end{align*}$$

and extending by $\mathbb{N}$-linearity; we will identify the set $\{D_0, D_1, D_2, D'_0, D'_1, D'_2\}$ with $\Delta_S(1)$ via this bijection. The automorphism $\tau : S \to S$ is induced by the automorphism of $N$ sending $n$ to $-n$. 

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Let \( k[\Delta_S(1)] \) be the polynomial ring on the set of variables \( x_\rho, \rho \in \Delta_S(1) \), which via the above identification we write as
\[
k[\Delta_S(1)] = k[x_{D_0}, x_{D_1}, x_{D_2}, x_{D'_1}, x_{D'_2}]
\]
and we write \( A_k^\Delta(1) \) for \( \operatorname{Spec} k[\Delta_S(1)] \).

For \( \sigma \in \Delta_S(2) \), let \( \sigma(1) \) denote the rays in \( \sigma \). For each \( \sigma \in \Delta_S(2) \), we have the monomial \( x_\sigma = \prod_{\rho \in \sigma(1)} x_\rho \) in \( k[\Delta_S(1)] \) as defined above, and the closed subset \( Z \subset A_k^\Delta(1) \) defined by \( \{ x_\sigma \mid \sigma \in \Delta_S(2) \} \).

Let \( G \cong \mathbb{G}_m^4 \) be the torus \( \operatorname{Hom}(\operatorname{Pic}(S), \mathbb{G}_m) \). The map sending the divisor \( D = \sum_i a_i D_i + a'_i D'_i \) to its class \([D] \in \operatorname{Pic}(S)\) defines a surjection \( \oplus_i \mathbb{Z} D_i \oplus \mathbb{Z} D'_i \to \operatorname{Pic}(S) \) and dually describes \( G \) as the subtorus in \( \mathbb{G}_m^6 = \mathbb{G}_m^{\Delta_S(1)} \) defined as
\[
G = \{ (z_0, z_1, z_2, z'_0, z'_1, z'_2) \mid z_iz'_i = z_jz'_j, i \neq j \},
\]
where \( z_i = z_{D_i}, z'_i = z_{D'_i} \) are the coordinates corresponding to \( D_i, D'_i \), respectively.

As above, \( G \) acts on \( A_k^\Delta(1) \) via the canonical action of \( \mathbb{G}_m^{\Delta_S(1)} \) on \( A_k^{\Delta_S(1)} \) composed the \( \mathbb{G}_m^6 \), and this gives a canonical isomorphism of \( S \) with the geometric quotient \( A_k^{\Delta_S(1)} \setminus Z/G \). We let \( \pi : A_k^{\Delta_S(1)} \setminus Z \to S \) be the corresponding quotient map.

For each \( i \in \{ 0, 1, 2 \} \) let \( \sigma_{D_i}, \sigma_{D'_i} : G \to \mathbb{G}_m \) be the character given by the projection on the factor \( D_i, D'_i \), respectively. The coordinates \( x_{D_i}, x_{D'_i} \) are respective \( \sigma_{D_i}, \sigma_{D'_i} \)-weight vectors. There is an isomorphism of \( \mathcal{O}_S(D_i) \) with the invertible sheaf on \( S \) defined by the character \( \sigma_{D_i} \), uniquely determined by requiring that the canonical section \( \mathcal{O}_S \to \mathcal{O}_S(D_i) \) corresponds to the section \( x_{D_i} \) of \( \mathcal{O}_S(D_i) \). We have a corresponding canonical isomorphism of \( \mathcal{O}_S(D'_i) \) with the invertible sheaf on \( S \) defined by the character \( \sigma_{D'_i} \). Via these isomorphisms, we consider \( x_{D_i}, x_{D'_i} \) as the canonical sections of \( \mathcal{O}_S(D_i), \mathcal{O}_S(D'_i) \). Looking at our presentation of \( \operatorname{Pic}(S) \), this gives us for each \( i \neq j \) a unique isomorphism
\[
\alpha_{ij} : \mathcal{O}_S(D_i) \otimes \mathcal{O}_S(D'_j) \cong \mathcal{O}_S(D'_i) \otimes \mathcal{O}_S(D_j)
\]
with \( \alpha_{ij}(x_{D_i} \otimes x_{D'_j}) = x_{D_j} \otimes x_{D'_i} \).

For a morphism \( f : \mathbb{P}^1 \to S \), there are uniquely determined integers \( d_i, d'_j \) with
\[
f^* \mathcal{O}_S(D_i) \cong \mathcal{O}_{\mathbb{P}^1}(d_i), \quad f^* \mathcal{O}_S(D'_i) \cong \mathcal{O}_{\mathbb{P}^1}(d'_j).
\]
Let
\[
\phi_{ij} : \mathcal{O}_{\mathbb{P}^1}(d_i) \otimes \mathcal{O}_{\mathbb{P}^1}(d'_j) \cong \mathcal{O}_{\mathbb{P}^1}(d'_i) \otimes \mathcal{O}_{\mathbb{P}^1}(d_j)
\]
be the canonical isomorphism given by the identity \( d_i + d'_j = d'_i + d_j \) and the canonical isomorphisms
\[
\mathcal{O}_{\mathbb{P}^1}(d_i) \otimes \mathcal{O}_{\mathbb{P}^1}(d'_j) \cong \mathcal{O}_{\mathbb{P}^1}(d_i + d'_j) = \mathcal{O}_{\mathbb{P}^1}(d'_i + d_j) \cong \mathcal{O}_{\mathbb{P}^1}(d'_i) \otimes \mathcal{O}_{\mathbb{P}^1}(d_j).
\]

Let \( d(f) \) be the corresponding tuple \( d_* := (d_0, d_1, d_2, d'_1, d'_2) \in L \subset \mathbb{Z}^6 \); clearly \( f_*([\mathbb{P}^1]) \) is in the curve class \( D_{d_*} \). Thus, for \( d_* \in L \), we have the scheme of morphisms \( f : \mathbb{P}^1 \to S \) such that \( d(f) = d_* \). Maps\( (\mathbb{P}^1, S; D_{d_*}) \), mapping onto the moduli stack \( \mathcal{M}_0(S, D_{d_*}) \).

Given \( d_* \in L \) and a field \( F \supset k \), define an equivalence relation \( \sim \) on tuples \((f_0, f_1, f_2, f'_0, f'_1, f'_2) \) with \( f_i \in F[T_0, T_1]_{d_i} \), by \((f_0, f_1, f_2, f'_0, f'_1, f'_2) \sim (g_0, g_1, g_2, g'_0, g'_1, g'_2) \) if there is a \( z := (z_1, \ldots, z_6) \in G(F) \) such that
\[
(f_0, f_1, f_2, f'_0, f'_1, f'_2) = z \cdot (g_0, g_1, g_2, g'_0, g'_1, g'_2).
\]
Proposition A.3. Given \( d_* \in L \), and a field \( F \supset k \), there is a bijection of Maps(\( \mathbb{P}^1; S; D_{d_*} \))(\( F \)) with the set of equivalence classes of tuples \((f_0, f_1, f_2, f'_0, f'_1, f'_2)\) with \( f_i \in F[T_0, T_1]_{d_i} \), \( f'_i \in F[T_0, T_1]_{d'_i} \), such that

\[
\text{(A.3)}
\]

for each \((t_0, t_1) \in F^2 \setminus \{0\}\), there is a triple \((i, j, k)\) with \(\{i, j, k\} = \{0, 1, 2\}\) and with \((f, f'_i, f'_j, f'_k)(t_0, t_1) \neq 0\).

Moreover, if \( F \) is an infinite field, then

\[
\text{Maps}(\mathbb{P}^1, S; D_{d_*})(F) \neq \emptyset
\]

Proof. The 2-dimensional cones \( \sigma \in \Delta_S(2) \) are spanned by rays of the form \( \rho_{D_i}, \rho_{D'_j} \) with \( i \neq j \). The corresponding monomial \( x_\sigma \) is thus \( x_{D_i} x_{D'_j} \) with \(\{i, j, k\} = \{0, 1, 2\}\). In addition, since \( G \) is a torus, we have, by Hilbert’s theorem 90, a bijection

\[
\text{Maps}(\mathbb{P}^1, S; D_{d_*})(F) = (\mathbb{A}^{\Delta_S(1)} \setminus Z)(F)/G(F)
\]

Our explicit description of Maps(\( \mathbb{P}^1, S; D_{d_*} \))(\( F \)) as a set of equivalence classes then follows from the description of the scheme Maps(\( \mathbb{P}^1, S; D_{d_*} \)) furnished by [3 Theorem 3.1]; more precisely, there is a choice of isomorphisms

\[
\phi_i : f^* \mathcal{O}_S(D_i) \sim \mathcal{O}_{P^1_p}(d_i), \quad \phi'_i : f^* \mathcal{O}_S(D'_i) \sim \mathcal{O}_{P^1_p}(d'_i),
\]

\( i = 0, 1, 2 \), such that the diagram

\[
f^* \mathcal{O}_S(D_i) \otimes f^* \mathcal{O}_S(D'_j) \xrightarrow{f^* \alpha_{ij}} f^* \mathcal{O}_S(D'_i) \otimes f^* \mathcal{O}_S(D_j)
\]

\[
\phi_i \otimes \phi'_j
\]

\[
\phi'_{ij} \sim \mathcal{O}_{P^1_p}(d_i) \otimes \mathcal{O}_{P^1_p}(d'_j)
\]

commutes for all \( i, j \), and the choice of \( \{\phi_i, \phi'_i\} \) is unique modulo the action of \( G(F) \). Given the family \( \{\phi_i, \phi'_i\} \), the family \( \{f_i, f'_i\} \) is defined by taking

\[
f_i := \phi_i(f^*(x_{D_i})) \in H^0(\mathbb{P}^1, \mathcal{O}_{P^1_p}(d_i)) = F[T_0, T_1]_{d_i},
\]

\[
f'_i := \phi'_i(f^*(x_{D'_i})) \in H^0(\mathbb{P}^1, \mathcal{O}_{P^1_p}(d'_i)) = F[T_0, T_1]_{d'_i}.
\]

Now suppose that \( F \) is an infinite field. We may find \( f_i \in F[T_0, T_1]_{d_i} \) and \( f'_i \in F[T_0, T_1]_{d'_i} \), all non-zero and with pairwise disjoint zero-loci. Then the set

\[
\{p \in \mathbb{P}^1 | f'_0 f_0 f_1 f'_2(p) = f_1 f'_0 f'_2 f_2(p) = f_2 f'_2 f_0 f'_1(p) = 0\}
\]

is empty, so \((f_0, f_1, f_2, f'_0, f'_1, f'_2)\) defines a point of \((\mathbb{A}^{\Delta_S(1)} \setminus Z)(F)\).

Following this result, we write \([f_0, f_1, f_2, f'_0, f'_1, f'_2]\) for the equivalence class of a tuple \((f_0 : f_1 : f_2, f'_0, f'_1, f'_2)\) and write the corresponding morphism as

\[
f = [f_0, f_1, f_2, f'_0, f'_1, f'_2] : \mathbb{P}^1 \to S.
\]

The following is a partial converse to Lemma A.2.

Corollary A.4. For each tuple \( d_* \in L \setminus \{0\} \) with \( d_i \geq 0, d'_i \geq 0 \) for \( 0, 1, 2, \) the scheme Maps(\( \mathbb{P}^1, S; D_{d_*} \)) and the stack \( M_0(S, D_{d_*}) \) are both non-empty and irreducible.

Proof. This follows from Theorem A.1 and Proposition A.3. □
We concentrate on \( f \) of the form

\[(A.4) \quad f(T_0, T_1) = (T_0^{d_1}, f_1, f_2, T_1^{d_1}, (T_1 + T_0)^{d_2}, (T_1 - T_0)^{d_2})\]

with \( f_i = f_i(T_0, T_1) \in k[T_0, T_1] \) of degree \( d_i \), \( i = 1, 2 \), with

\[f_1(0, 1) f_2(0, 1) f_1(1, -1) f_2(1, 1) \neq 0,\]

and with \( f_1, f_2 \) having no zeros in common on \( \mathbb{P}^1 \setminus \{[0, 1], [1, 0], [1, 1], [1, -1]\}. As usual, we also require that \( d_i + d'_i = d_j + d'_j \) for all \( i, j \). We note that this does indeed satisfy the condition \( (A.3) \), indeed, we have the monomials

\[f_0 f'_0 f_1 f'_2 = T_0^{d_0} T_1^{d'_0} f_1(T_1 - T_0)^{d_2},\]

\[f_0 f'_0 f_2 f'_1 = T_0^{d_0} T_1^{d'_0} f_2(T_1 + T_0)^{d_2},\]

\[f_1 f'_1 f_0 f'_2 = T_0^{d_0} f_1(T_1 + T_0)^{d_2}(T_1 - T_0)^{d_2},\]

\[f_1 f'_1 f_2 f'_0 = f_1 f_2 T_1^{d_0} (T_1 + T_0)^{d_2},\]

\[f_2 f'_2 f_0 f'_1 = T_0^{d_0} f_2(T_1 + T_0)^{d_2}(T_1 - T_0)^{d_2},\]

\[f_2 f'_2 f_1 f'_0 = f_1 f_2 T_1^{d_0} (T_1 - T_0)^{d_2}\]

If \( p = (p_0, p_1) \neq (0, 0) \) is a common zero for all these monomials, then for \([p] \notin \{[0, 1], [1, 0], [1, 1], [1, -1]\}, none of \( f_0, f'_0, f'_1, f'_2 \) are zero and either \( f_1(p) \neq 0 \) or \( f_2(p) \neq 0 \), so at least one of the above monomials is non-zero. For \( p = (0, 1) \), \( f_1 f'_1 f_0 f'_2(p) \neq 0 \), for \( p = (1, 0) \), either \( f_1 f'_1 f_0 f'_2(p) \neq 0 \) or \( f_2 f'_2 f_0 f'_1(p) \neq 0 \). For \( p = (1, 1) \), \( f_0 f'_0 f_2 f'_1(p) \neq 0 \) and for \( p = (1, -1) \), \( f_0 f'_0 f_2 f'_1(p) \neq 0 \).

We write \( f_1(T_0, T_1) = \sum_{i=0}^{d_1} a_i T_1^i T_0^{d_1-i} \) and \( f_2(T_0, T_1) = \sum_{i=0}^{d_2} b_i T_1^i T_0^{d_2-i} \). If we want to explicitly include the \( a_i, b_j \) in the notation, we write \( f_{a_i, b_j} \) for the corresponding map \( f : \mathbb{P}^1 \to S \).

**Lemma A.5.** Let \( f : \mathbb{P}^1 \to S \) be of the form \((A.4)\). Then \( f(\mathbb{P}^1) \) is not contained in \( \cup_i D_i \cup D'_i \) and

\[f^*(D_0) = d_0 \cdot [0, 1]\]

\[f^*(D'_0) = d'_0 \cdot [1, 0]\]

\[f^*(D_1) = d_1 \cdot [1, -1]\]

\[f^*(D'_2) = d'_2 \cdot [1, 1].\]

In particular, \( f(\mathbb{P}^1 \setminus \{[0, 1], [1, 0], [1, 1], [1, -1]\}) \subset \mathbb{P}^2 \setminus L_0 \cup p_0. \) Finally, suppose that \( d_1 \cdot d_2 \neq 0 \) and that \( \text{char } k \neq 2 \). Then for \( f_i \) in some non-empty open subset of the affine space \( k[T_0, T_1]_{d_i}, i = 1, 2 \), the map \( f : \mathbb{P}^1 \to S \) is unramified over an open subset of \( D'_0 \cup D_0 \cup D'_1 \cup D'_2 \).

**Proof.** \( D_i \) and \( D'_i \) are defined by the respective coordinates \( x_{D_i}, x_{D'_i} \), and

\[f^*(x_{D_0}) = T_0^{d_0}, f^*(x_{D'_0}) = T_1^{d'_0}, f^*(x_{D'_1}) = (T_1 + T_0)^{d_1}, f^*(x_{D_2}) = (T_1 - T_0)^{d_2}\]

which shows that \( f(\mathbb{P}^1) \not\subseteq \cup_i D_i \cup D'_i \) and gives the computations of \( f^*(D_0) \) and the \( f^*(D'_i) \). In addition, for \( p \in \mathbb{P}^1 \setminus \{[0, 1], [1, 0], [1, 1], [1, -1]\} \), we have

\[f^*(x_{D_0}) f^*(x_{D'_0}) f^*(x_{D'_1}) f^*(x_{D_2})(p) \neq 0,\]

so \( f(p) \) is in \( S \setminus D_0 \cup D'_0 \cup D'_1 \cup D'_2 \).
For the final assertion, we first check in a neighborhood of $[0,1]$. A system of local coordinates near $f(0,1) \in D_0$ is given by $(x_{D_0}/x_{D_1}, x_{D_2}/x_{D_D}, x_{D_D}/x_{D_D}, x_{D_D}/x_{D_D})$, for which $f$ is locally described by

$$f(t,1) = (t^{d_0}(1-t)^{d_2}/f_2(t,1), f_2(t,1) \cdot (1 + t)\delta / f_1(t,1) \cdot (1 - t)^{d_2})$$

with $t = T_0/T_1$. Then

$$\frac{d}{dt} \bigg|_{t=0} \left( \frac{f_2(t,1) \cdot (1 + t)\delta}{f_1(t,1) \cdot (1 - t)^{d_2}} \right) = \frac{1}{f_1(0,1)^2} (f_1(0,1)f_2'(0,1) - f_2(0,1)f_1'(0,1) + (d_1' + d_2')f_1(0,1)f_2(0,1))$$

where we take the derivatives of $f_1(t,1), f_2(t,1)$ as functions of $t$. This will be non-zero for general $f_1, f_2$ if $d_1 > 0$ and $d_2 > 0$. Since $f^{-1}(D_0) = \{0,1\}$ this shows that $f$ is unramified over a neighborhood of $D_0$, under these conditions.

Next, check in a neighborhood of $[1,0]$. Here a local coordinate system near $f(1,0) \in D_0'$ is given by $(x_{D_0}/x_{D_1}, x_{D_2}/x_{D_D}, x_{D_D}/x_{D_D})$, and $f$ is locally described as

$$f(1,t) = (t^{d_0}f_1(1,t)/(1 + t)^{d_2}, f_2(1,t)(1 + t)^{d_1}/f_1(1,t)\cdot (1 - t)^{d_2})$$

with $t = T_1/T_0$. We similarly compute that $d(f_2(1,t)/(1 + t)^{d_1}/f_1(1,t)\cdot (1 - t)^{d_2})/dt$ is non-zero at $t = 0$ for general choice of $f_1, f_2$ if $d_1 > 0$ and $d_2 > 0$.

For $[1,1]$ we use $(x_{D_2}/x_{D_0}, x_{D_2}/x_{D_D}, x_{D_D}/x_{D_D})$ as local coordinate system near $f(1,1) \in D_2'$. We let $t = T_1/T_0$ and compute the derivative at $t = 1$, giving

$$\frac{d}{dt} \bigg|_{t=1} \left( \frac{f_1(t,1) \cdot t^{d_0}}{(t + 1)^{d_1}} \right) = \frac{1}{2^{d_1+1}} \cdot (2f_1'(1,1) - (2d_0' - d_1')f_1(1,1))$$

Since we are assuming that the characteristic is not 2, this will be non-zero for general $f_1$ if $d_l > 0$.

Finally, at $[1,-1]$, we take the local coordinate system near $f(1,-1) \in D_1'$ to be $(x_{D_2}/x_{D_0}, x_{D_2}/x_{D_D}, x_{D_D}/x_{D_D})$. We let $t = T_1/T_0$, and compute

$$\frac{d}{dt} \bigg|_{t=-1} \left( \frac{f_2(t,1)\cdot t^{d_0} \cdot (t - 1)^{d_2}}{(t + 1)^{d_1}} \right) = \frac{(-1)^{d_0}}{(-2)^{d_2+1} \cdot (2f_2'(1,-1) + (2d_0' - d_2')f_2(1,-1))}$$

which is non-zero for general $f_2$ if $d_2 > 0$.

\[ \square \]

**Lemma A.6.** Let $p \geq 0$ be the characteristic of $k$; we assume that $p \neq 2$. Suppose that the index $d(f) = (d_0, d_1, d_2, d_0', d_1', d_2')$ has $d_i, d_i' \geq 0$ for all $i$, $d_1d_2 > 0$, and moreover, at least one of the following conditions hold

\[(A.5)\]

a. $d_0' \neq 0$ \mod $p$

b. $d_1d_2' \equiv 0$ \mod $p$.

c1. $d_0 \equiv d_0' \equiv 0$ \mod $p$ and $d_1' \neq 0$ \mod $p$.

c2. $d_0 \equiv d_0' \equiv 0$ \mod $p$ and $d_2' \neq 0$ \mod $p$.

Then there is an open subset $U$ of $\mathbb{A}^{d_1+1} \times \mathbb{A}^{d_2+1}$ such that the map $f_{a, b} : \mathbb{P}_{k(a, b)}^1 \rightarrow S_{k(a, b)}$ is everywhere unramified for each $(a, b) \in U$.  

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Lemma A.5, we reduce to showing that the restriction of $f := f_{a_*, b_*}$ to $f : \mathbb{P}^1 \setminus \{0, 1, [1, 0], [1, 1], [1, -1]\} \to S \setminus D_0 \cup D'_0 \cup D'_1 \cup D'_2$ is unramified for all $(a_*, b_*)$ in some non-empty open subset $U$ of $\mathbb{A}^{d_1+1} \times \mathbb{A}^{d_2+1}$. We note that the blow-down map $\pi : S \to \mathbb{P}^2$ identifies $S \setminus D_0 \cup D'_0 \cup D'_1 \cup D'_2$ with $\mathbb{P}^2 \setminus p_0 \cup L_0$, and the corresponding map

$$\pi \circ f : \mathbb{P}^1 \setminus \{0, 1, [1, 0], [1, 1], [1, -1]\} \to \mathbb{P}^2 \setminus L_0 \cup p_0 \subset \mathbb{A}^2$$

is given by

$$f(t) := \pi \circ f(1, t) = \left( \begin{array}{c} f_1(1, t) t^{d'_0} \\ f_2(1, t) t^{d'_0} \end{array} \right) \left( \begin{array}{c} \frac{1}{(t + 1)^{d_1}} \\ \frac{1}{(t - 1)^{d_2}} \end{array} \right)$$

with $t = T_1/T_0$. We consider the coefficients $(a_*, b_*)$ as generic, that is, we work over the polynomial ring $k[a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2}]$.

Computing $df(t)/dt$, we find

$$df(t)/dt = \left( \begin{array}{c} t^{d'_0-1} g_1(t) \\ t^{d'_0-1} g_2(t) \end{array} \right) \left( \begin{array}{c} \frac{1}{(t + 1)^{d_1}} \\ \frac{1}{(t - 1)^{d_2}} \end{array} \right)$$

with $g_1(t) = \sum_{i=0}^{d_1} a'_i t^i$, $g_2(t) = \sum_{i=0}^{d_2} b'_i t^i$, with $a'_i$, $b'_i$ given by the matrix equations (for $m := d'_0 - d'_1$, $n = d'_0 - d'_2$)

$$(A.6) \quad \begin{pmatrix} a'_0 \\ a'_1 \\ \vdots \\ a'_{d_1} \\ a'_{d_1+1} \end{pmatrix} = \begin{pmatrix} a_0 \\ m \\ d'_0 + 1 \\ 0 \\ m + 1 \end{pmatrix} \begin{pmatrix} d'_0 + 1 \\ m + 1 \end{pmatrix} \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$$

$$(A.7) \quad \begin{pmatrix} b'_0 \\ b'_1 \\ \vdots \\ b'_{d_2} \\ b'_{d_2+1} \end{pmatrix} = \begin{pmatrix} b_0 \\ n \\ d'_0 + 1 \\ 0 \\ n + 1 \end{pmatrix} \begin{pmatrix} d'_0 + 1 \\ n + 1 \end{pmatrix} \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$$

In what follows we will be computing determinants of matrices of the form

$$(A.8) \quad M := \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$
with $A$, $B$, $C$, $\alpha$ and $\gamma$ all square matrices, and the missing entries all zero. It follows from Sublemma \[A.8\] below that

\[(A.9) \quad \det(M) = \det(A) \det(B) \det(C)\]

We first discuss Case (a), $d'_0 \not\equiv 0 \pmod{p}$, which we divide into the following subcases:

i. $d_0d'_0 \not\equiv 0 \pmod{p}$

i'. $d_0d'_1 \not\equiv 0 \pmod{p}$

ii. $d_0 \equiv 0 \pmod{p}$ and $d'_1 \not\equiv 0 \pmod{p}$

ii'. $d_0 \equiv 0 \pmod{p}$ and $d'_2 \not\equiv 0 \pmod{p}$

We first handle cases (i) and (i')

Case A: $d_0d'_0 \not\equiv 0 \pmod{p}$.

Case A1: $d_0d'_0 - (d'_0 + d_1)(d'_0 - d'_2) \not\equiv 0 \pmod{p}$. In this case we will show that the coefficient of $\alpha d_{d_2+1}b_{d_0+1}$ in the resultant is non-zero. For this, we may set $a_i = 0$ for $i < d_1$, and $b_j = 0$ for $j > 0$ in computing the resultant. Noting that $m + d_1 = d_0$, this is the determinant of the matrix $M$ given by \[A.8\] with $A$ the $d_1 \times d_1$ matrix

$$A = \begin{pmatrix}
  d'_0b_0 & nb_0 & \ldots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \ldots & d'_0b_0 & nb_0 \\
  0 & \ldots & 0 & d'_0b_0
\end{pmatrix}$$

$B$ the $2 \times 2$ matrix

$$B = \begin{pmatrix}
  d'_0b_0 & nb_0 \\
  (d'_0 + d_1)a_{d_1} & d_0a_{d_1}
\end{pmatrix}$$

$C$ the $d_2 \times d_2$ matrix

$$C = \begin{pmatrix}
  d_0a_{d_1} & \ldots & \ldots & 0 \\
  (d'_0 + d_1)a_{d_1} & d_0a_{d_1} & \ldots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \ldots & (d'_0 + d_1)a_{d_1} & d_0a_{d_1}
\end{pmatrix}$$

with the missing entries all zero, and $\alpha$ and $\gamma$ the $1 \times 1$ matrices $\alpha = (nb_0)$, $\gamma = ((d'_0 + d_1)a_{d_1})$. By \[A.9\], we have

$$\det(M) = \det(A) \det(B) \det(C) = d'_0d_{d_2}d_0d'_0 - (d'_0 + d_1)(d'_0 - d'_2) \cdot \alpha d_{d_2+1}b_{d_0+1} \not\equiv 0$$

Case A2: $d_0d'_0 - (d'_0 + d_1)(d'_0 - d'_2) \equiv 0 \pmod{p}$, $d'_2 \not\equiv 0 \pmod{p}$ and $d_1 \geq 1$. Note that in this case $n := d'_0 - d'_2 \not\equiv 0 \pmod{p}$, since we are assuming that $d_0d'_0 \not\equiv 0 \pmod{p}$.

Here we set $a_i = 0$ for $i < d_1 - 1$, $b_j = 0$ for $j > 0$. The resultant is then the determinant of the matrix $M$ given by \[A.8\] with $A$ the $d_1 - 1 \times d_1 - 1$ matrix

$$A = \begin{pmatrix}
  d'_0b_0 & nb_0 & \ldots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \ldots & d'_0b_0 & nb_0 \\
  0 & \ldots & 0 & d'_0b_0
\end{pmatrix}$$
$B$ the $3 \times 3$ matrix

$$B = \begin{pmatrix}
  d'_0b_0 & nb_0 & 0 \\
  0 & d'_0b_0 & nb_0 \\
  (d_0 + d'_1 - 1)a_{d_1-1} & (d'_0 + d_1)a_{d_1} + (d_0 - 1)a_{d_1-1} & d_0a_{d_1}
\end{pmatrix}$$

$C$ the $d_2 \times d_2$ matrix

$$C = \begin{pmatrix}
  d_0a_{d_1} & \ldots & \ldots & 0 \\
  (d'_0 + d_1)a_{d_1} + (d_0 - 1)a_{d_1-1} & d_0a_{d_1} & \ldots & 0 \\
  (d_0 + d'_1 - 1)a_{d_1-1} & (d'_0 + d_1)a_{d_1} + (d_0 - 1)a_{d_1-1} & d_0a_{d_1} & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \ldots & \ldots \\
  0 & \ldots & (d_0 + d'_1 - 1)a_{d_1-1} & (d'_0 + d_1)a_{d_1} + (d_0 - 1)a_{d_1-1} & d_0a_{d_1}
\end{pmatrix}$$

with the missing entries all zero, $\gamma$ the $2 \times 2$ matrix

$$\gamma = \begin{pmatrix}
  (d_0 + d'_1 - 1)a_{d_1-1} & (d'_0 + d_1)a_{d_1} + (d_0 - 1)a_{d_1-1} \\
  0 & (d_0 + d'_1 - 1)a_{d_1-1}
\end{pmatrix}$$

$\alpha$ the $1 \times 1$ matrix ($nb_0$). Under our assumption that $d_0d'_0 - (d'_0 + d_1)(d'_0 - d'_2) \equiv 0 \mod p$, we have

$$\det(B) = [(d'_0 - d'_2)(d_0 + d'_1 - 1) - d'_0(d_0 - 1)] nb_0^2a_{d_1-1} = nd_2^2b_0^2a_{d_1-1}.$$ 

By [A.9], we have

$$\det(M) = \det(A)\det(B)\det(C) = d_0^{d_1-1}nd_2^2b_0^{d_1+1}a_{d_1-1}d_1^{d_2}.$$ 

Our assumptions imply $d'_0d_0nd_2^2 \neq 0 \mod p$, so the result is non-zero.

Case A3: $d_0d'_0 - (d'_0 + d_1)(d'_0 - d'_2) \equiv 0 \mod p$, $d'_1 \neq 0 \mod p$, and $d_2 \geq 1$. In this case $d'_0 + d_1 \neq 0 \mod p$, since we are assuming that $d_0d'_0 \neq 0 \mod p$. 

[83]
We set \( a_i = 0 \) for \( i < d \) and \( b_j = 0 \) for \( j > 1 \). The resultant is then the determinant of the matrix \( M \) given by (A.8) with \( A \) the \( d_1 \times d_1 \) matrix

\[
A = \begin{pmatrix}
    d'_0 b_0 & nb_0 + (d'_0 + 1)b_1 & (n + 1)b_1 & \ldots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \ldots & d'_0 b_0 & nb_0 + (d'_0 + 1)b_1 & (n + 1)b_1 \\
    0 & \ldots & \ldots & d'_0 b_0 & nb_0 + (d'_0 + 1)b_1 \\
    0 & \ldots & \ldots & \ldots & d'_0 b_0 \\
\end{pmatrix}
\]

\( B \) the \( 3 \times 3 \) matrix

\[
B = \begin{pmatrix}
    d'_0 b_0 & nb_0 + (d'_0 + 1)b_1 & (n + 1)b_1 \\
    (d'_0 + d_1) a_{d_1} & d_0 a_{d_1} & 0 \\
    0 & (d'_0 + d_1) a_{d_1} & d_0 a_{d_1} \\
\end{pmatrix}
\]

\( C \) the \( d_2 - 1 \times d_2 - 1 \) matrix

\[
C = \begin{pmatrix}
    d_0 a_{d_1} & \ldots & \ldots & 0 \\
    (d'_0 + d_1) a_{d_1} & d_0 a_{d_1} & \ldots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \ldots & (d'_0 + d_1) a_{d_1} & d_0 a_{d_1} \\
\end{pmatrix}
\]

with the missing entries all zero, \( \alpha \) the \( 2 \times 2 \) matrix

\[
\alpha = \begin{pmatrix}
    (n + 1)b_1 & 0 \\
    nb_0 + (d'_0 + 1)b_1 & (n + 1)b_1 \\
\end{pmatrix}
\]

and \( \gamma \) the \( 1 \times 1 \) matrix \( ((d'_0 + d_1) a_{d_1}) \). Under our assumption that

\[
d_0 a'_0 - (d'_0 + d_1)(d'_0 - d_2) \equiv 0 \mod p,
\]

and noting that \( d'_0 + d_1 = d_0 + d'_1 \), we find

\[
\det(B) = [(d'_0 - d'_2 + 1)(d'_0 + d_1) - d_0(d'_0 + 1) \cdot (d'_0 + d_1)b_1a_{d_1}^2] = d'_1(d'_0 + d_1)b_1a_{d_1}^2,
\]

so by (A.9) we have

\[
\det(M) = \det(A) \det(B) \det(C) = a'_0 d'_1 d'_1(d'_0 + d_1)d_0^{d_2 - 1}b_0 b_1 a_{d_1}^{d_2 + 1} \neq 0.
\]

This takes care of cases (i) and (i'), we now handle cases (ii) and (ii')

Case B \( d_0 \equiv 0 \mod p \). In that case, we have \( a'_{d_1+1} = b'_{d_2+1} = 0 \) for all \( a_*, b_* \), and we need to compute the resultant of polynomials of degrees \( d_1 \) and \( d_2 \), respectively.
Case B1 $d_0'd_1' \neq 0 \mod p$. Note that

$$d_0' + d_1 = d_0 + d_1' \equiv d_1' \neq 0.$$ 

We show that the coefficient of $b_0'd_1d_2$ is non-zero, by setting $a_i = 0$ for $i < d_1$, $b_j = 0$ for $j > 0$. The resultant is the determinant of

$$
\begin{pmatrix}
  d_0'b_0 & nb_0 & 0 & \ldots & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & \ldots & d_0'b_0 & nb_0 & 0 & \ldots \\
  0 & \ldots & 0 & d_1'a_{d_1} & \ldots & 0 \\
  0 & \ldots & \ldots & \ldots & \ldots & d_1'a_{d_1}
\end{pmatrix}
$$

giving us $d_0'd_1d_2b_0'a_{d_1}d_2$.

Case B2 $d_0'd_2' \neq 0 \mod p$. This is just like Case B1, after switching the roles of $a_*$ and $b_*$. The resultant has coefficient of $a_0'd_0'd_2$ equal to $a_0'd_0'd_2d_1' \neq 0$.

This takes care of cases (ii) and (ii'), and finishes Case (a). We now discuss Case (b): $d_1'd_2' \equiv 0 \mod p$.

If $d_1' \equiv 0 \mod p$, then

$$df(t)/dt = \left( \frac{t^{d_0'-1}g_1(t)}{(t+1)^{d_1'}} + \frac{t^{d_0'-1}g_2(t)}{(t-1)^{d_2}+1} \right)$$

with $g_2(t)$ as in [A.7] and with

$$g_1(t) = \sum_{i=0}^{d_1}(d_0' + i)a_it^i.$$ 

Similarly, if $d_2' \equiv 0 \mod p$, then

$$df(t)/dt = \left( \frac{t^{d_0'-1}g_1(t)}{(t+1)^{d_1}+1} + \frac{t^{d_0'-1}g_2(t)}{(t-1)^{d_2}} \right)$$

with $g_1(t)$ as in [A.6] and with

$$g_2(t) = \sum_{i=0}^{d_2}(d_0' + i)b_it^i.$$ 

If both $d_1' \equiv 0 \mod p$ and $d_2' \equiv 0 \mod p$, then

$$df(t)/dt = \left( \frac{t^{d_0'-1}g_1(t)}{(t+1)^{d_1}+1} + \frac{t^{d_0'-1}g_2(t)}{(t-1)^{d_2}} \right)$$

with

$$g_1(t) = \sum_{i=0}^{d_1}(d_0' + i)a_it^i, \quad g_2(t) = \sum_{i=0}^{d_2}(d_0' + i)b_it^i.$$ 

Recall that we are in the generic case, with $k[a_0, \ldots, a_{d_1}]$ and $k[b_0, \ldots, b_{d_2}]$ polynomial rings over $k$. Since $d_1d_2 > 0$, $g_1(t)$ and $g_2(t)$ are both non-zero polynomials in all three cases: $d_1' \equiv 0 \mod p$ but $d_2' \neq 0 \mod p$, $d_1' \equiv 0 \mod p$ but $d_2' \neq 0 \mod p$ and $d_1' \equiv d_2' \equiv 0 \mod p$. 85
Suppose $d'_1 \equiv 0 \mod p$. Let $d'_1$, $d'_2$ be the degrees of $g_1, g_2$, respectively, so $0 \leq d'_i \leq d_i$, $i = 1, 2$. Let $d''_1, d''_2$ be the respective degrees of the lowest degree terms in $g_1, g_2$, and let $h_i(t) = g_i(t)/t^{d''_i}$, so the roots of $h_i$ are the non-zero roots of $g_i$, and we need only show that $h_1, h_2$ have no roots in common.

If $d''_1 = d''_2$, then $h_1$ is a non-zero constant polynomial, hence has no roots. If $d''_1 > d''_2$, let $r = d''_1 - d''_2$. Let $h_1$ be the polynomial formed from $h_1$ by setting $a_i = 0$ for $i \neq d''_1, d''_2$. Then $h_1$ has roots $\xi$ with

$$
\xi^r = \frac{d''_0 + d''_1}{d''_0 + d''_1} \cdot \frac{a_{d''_1}}{a_{d''_2}} \neq 0
$$

In particular, $\xi$ is in the algebraic closure $\overline{k(a_0, a_1, \ldots, a_{d_1})}$ of $k(a_0, a_1, \ldots, a_{d_1})$ but is transcendental over $k$. By Sublemma A.7 below,

$$
\overline{k(a_0, a_1, \ldots, a_{d_1})} \cap \overline{k(b_0, b_1, \ldots, b_{d_2})} = \overline{k},
$$

the intersection taking place in the algebraic closure of $k(a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2})$, hence $h_1(t)$ and $h_2(t)$ have no roots in common. Thus their resultant is not zero (as an element of $k(a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2})$), and hence the resultant of $h_1(t)$ and $h_2(t)$ is also non-zero, as desired. Thus $g_1(t)$ and $g_2(t)$ have no non-zero roots in common.

A similar argument shows that $g_1(t)$ and $g_2(t)$ have no non-zero roots in common if $d'_2 \equiv 0 \mod p$.

We conclude with case (c).

Case c1 $d_0 \equiv d'_0 \equiv 0 \mod p$ and $d'_1 \neq 0 \mod p$. In this case we consider $g_1(t)$ and $g_2(t)$ as in Case A, but note that now $g_i$ has degree $d_i$ and $t$ divides $g_i$, for $i = 1, 2$. We replace $g_i$ with the degree $d_i - 1$ polynomial $h_i(t) = g_i(t)/t$, and compute the coefficient of $t_1^{d_i - 1}a_{d_i - 1}^{d_i - 1}$ in the resultant of $h_1, h_2$. We set $a_i = 0$ for $i < d_1$ and $b_j = 0$ for $j \neq 1$, so the resultant is the determinant of

$$
\begin{pmatrix}
    b_1 & (n + 1)b_1 & 0 & \ldots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    0 & \ldots & b_1 & (n + 1)b_1 & 0 \\
    0 & \ldots & d'_1a_{d_1} & 0 & \ddots \\
    0 & \ldots & \ldots & \ldots & d'_1a_{d_1}
\end{pmatrix}
$$

which is $d_1^{nd_2 - 1}b_1^{d_1 - 1}a_{d_1 - 1}^{d_1 - 1} \neq 0$.

Case c2 $d_0 \equiv d'_0 \equiv 0 \mod p$ and $d'_2 \neq 0 \mod p$. This is just as for (c1), where we switch the roles of $g_1$ and $g_2$.

\[\square\]

Here are the two sublemmas we used in the proof of Lemma A.6.

**Sublemma A.7.** Let $F$ be a field and let $F[a_0, \ldots, a_{d_1}], F[b_0, \ldots, b_{d_2}]$ be polynomial rings, considered as subrings of the polynomial ring $F[a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2}]$. Let $F(a_\ast), F(b_\ast)$ and $F(a_\ast, b_\ast)$ be the respective fraction fields.

Let $\overline{F(a_\ast, b_\ast)}$ be an algebraic closure of $F(a_\ast, b_\ast)$, and let $F, \overline{F(a_\ast)}$ and $\overline{F(b_\ast)}$ be the respective algebraic closures of $F$, $F(a_\ast)$, $F(b_\ast)$ in $\overline{F(a_\ast, b_\ast)}$. Then

$$
F = \overline{F(a_\ast)} \cap \overline{F(b_\ast)}
$$

the intersection taking place in $\overline{F(a_\ast, b_\ast)}$. 

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Proof. Take
\[ x \in F(a_u) \cap F(b_u) \]
and let
\[ f(t) \in F(a_0, \ldots, a_{d_1})[t], \ g(t) \in F(b_0, \ldots, b_{d_2})[t] \]
be the respective minimal polynomials of \( x \) over \( F(a_0, \ldots, a_{d_1}) \) and \( F(b_0, \ldots, b_{d_2}) \).

Since \( F(a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2}) \) is a pure transcendental extension of \( F(a_0, \ldots, a_{d_1}) \),
a simple specialization argument shows that \( f(t) \) remains irreducible in the extension \( F(a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2})[t] \), and similarly, \( g(t) \) remains irreducible in the extension \( F(a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2})[t] \). Indeed, if
\[ f = f_1 \cdot f_2 \in F(a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2})[t], \]
with both \( f_i \) non-constant polynomials, then since \( F \) is an infinite field, there is a \( b'_i := (b'_0, \ldots, b'_{d'_2}) \in F^{d'_2+1} \) such that the common denominator of the coefficients of \( f_1 \) and \( f_2 \) is non-zero at the point \( (a_u, b'_u) := (a_0, \ldots, a_{d_1}, b'_0, \ldots, b'_{d_2}) \). Evaluating the coefficients of \( f_1 \) and \( f_2 \) at \( (a_u, b'_u) \) yields non-constant monic polynomials \( f_1, f_2 \in F(a_0, \ldots, a_{d_1})[t] \) with \( f = f_1 f_2 \), contrary to the irreducibility of \( f \).

Since \( f \) and \( g \) are irreducible monic polynomials in \( F(a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2})[t] \) with \( x \) as a root, we must have \( f(t) = g(t) \), hence \( f \) is in
\[ (F(a_0, \ldots, a_{d_1}) \cap F(b_0, \ldots, b_{d_2}))[t] = F[t] \]
and thus \( x \) is in \( F \). \( \square \)

Sublemma A.8. Let
\[
A = \begin{pmatrix}
A_{11} & \ldots & A_{1m} \\
\vdots & \ddots & \vdots \\
A_{m1} & \ldots & A_{mm}
\end{pmatrix}
\]
be an \( n \times n \) matrix divided into \( n_i \times n_j \) blocks \( A_{ij} \), \( 1 \leq i, j \leq m \), \( \sum n_i = n \). Suppose there is a total order \( \prec \) on the set \( \{1, \ldots, m\} \) such that \( A_{ij} = 0 \) unless \( i \prec j \). Then
\[
\det A = \prod_{i=1}^{m} \det A_{ii}
\]
Proof. By conjugating by a suitable permutation matrix, we may assume that \( \prec \)
is the standard order on \( \{1, \ldots, m\} \), so \( A \) is block upper triangular with diagonal blocks \( A_{ii} \). Write \( A = (a_{ij})_{1 \leq i, j \leq n} \) and take \( \sigma \) in the permutation group \( S_n \). If \( \sigma \) does not preserve each of the blocks \( \{n_1 + \ldots + n_r + 1, \ldots, n_1 + \ldots + n_{r+1}\} \), it is easy to see that there is an \( i \in \{1, \ldots, n\} \) with \( \sigma(i) < i \), so the term sign(\( \sigma \)) \( \prod_{j=1}^{m} a_{j\sigma(j)} \)
is zero. This shows that \( \det A = \prod_{i=1}^{m} \det A_{ii} \), as claimed. \( \square \)

We now take into account the available symmetries:
\[
(d_0, d_1, d_2, d'_0, d'_1, d'_2) \mapsto (d_1, d_2, d_0, d'_1, d'_2, d'_0)
\]
\[
(d_0, d_1, d_2, d'_0, d'_1, d'_2) \mapsto (d'_0, d'_1, d'_2, d_0, d_1, d_2)
\]

Proposition A.9. Suppose that \( d_* = (d_0, d_1, d_2, d'_0, d'_1, d'_2) \in L \) has \( d_i, d'_i \geq 0 \) for all \( i \), and there are a pair of indices \( i \neq j \) such that either \( d_i d_j \neq 0 \) or \( d'_i d'_j \neq 0 \). Let \( D_{d_*} \) denote the corresponding curve class on \( S \). Suppose that \( \text{char } k \neq 2 \). Then \( M_{0,nr}(S, D_{d_*}) \) is non-empty.
Proof. Using the symmetries, it suffices to show that $M_{0,0}^\text{unr}(S,D_d)$ is non-empty if $d_1d_2 > 0$. Let $p \geq 0$ be the characteristic of $k$. Looking at Lemma A.6, we see that the only case not covered is if $d'_0 \equiv 0 \mod p$ but $d_0 \not\equiv 0 \mod p$. In this case, we have $d_0 \neq 0$, so $d_0d_2 > 0$. We now apply Lemma A.6 after exchanging the indices 0 and 1, and use the case (b): $d'_1 \cdot d'_2 \equiv 0 \mod p$.

Remark A.10. We complete the picture by examining the cases not covered by Proposition A.9. If $d_i \geq 0$, $d'_i > 0$ for all $i$, then up to symmetry there are two cases:

1. $d_1 = d_2 = d'_0 = d'_1 = 0$
2. $d_1 = d_2 = d'_1 = d'_2 = 0$

In case (1), the relations $d_i + d'_j = d'_i + d_j$ implies that $d_i = d'_j = 0$ for all $i, j$, so each map $\mathbb{P}^1 \to S$ in the class $D_d$ maps $\mathbb{P}^1$ to a point, and $M_{0,0}^\text{unr}(S,D_d) = \emptyset$. In case (2), the relations imply that $d_0 = d'_0$. We can blow down the -1 curves $D_1, D'_1$, $\pi : S \to S'$, with $S' \cong \mathbb{P}^1 \times \mathbb{P}^1$, $D_d$ being the pullback of $d_0 \cdot \mathbb{P}^1 \times pt$ and composing with $\pi$ induces an isomorphism $M_0(S,D_d) \cong M_0(\mathbb{P}^1 \times \mathbb{P}^1, d_0 \cdot \mathbb{P}^1 \times pt)$. Fixing a particular model, we may assume that the curves $D_1, D'_1$ blow down to the points $([0 : 1], [1 : 0])$ and $([1 : 0], [0 : 1])$. It is not difficult to show that in this case, each map $f : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ in the curve class $d_0 \cdot \mathbb{P}^1 \times pt$ is an $d_0$-fold cover of its image $f(\mathbb{P}^1)$, and $f(\mathbb{P}^1) = \mathbb{P}^1 \times t$ for some $t \in \mathbb{P}^1$. As each multiple cover $\mathbb{P}^1 \to \mathbb{P}^1$ is ramified, we see that $M_{0,0}^\text{unr}(S,D_d) = \emptyset$ if $d_0 > 1$, and consists of the pull back of maps $f : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ with $f : \mathbb{P}^1 \to f(\mathbb{P}^1)$ an isomorphism to the image curve $f(\mathbb{P}^1) = \mathbb{P}^1 \times t$, for some $t \in \mathbb{P}^1 \setminus \{[1 : 0], [0 : 1]\}$, if $d_0 = 1$.

By Lemma A.2 if some $d_i < 0$ or some $d'_i < 0$, then the only cases with Maps($\mathbb{P}^1, S, D_d$) $\neq \emptyset$ have $f : \mathbb{P}^1 \to D_i$ or $f : \mathbb{P}^1 \to D'_i$ an $m$-fold cover; in the first case $d_i$ has $d_i = -m$, $d_j = d_k = 0$, $d'_j = d'_k = m$ for $\{i, j, k\} = \{0, 1, 2\}$ and the second case is covered by the symmetry $d_i \leftrightarrow d'_i$.

As above, $M_{0,0}^\text{unr}(S,D_d) = \emptyset$ unless $m = 1$, and we have the case of an isomorphism $f : \mathbb{P}^1 \to D_i$, or $f : \mathbb{P}^1 \to D'_i$. We summarize with the following result.

Theorem A.11. Let $k$ be a field of characteristic $\neq 2$. Let $S$ be the blowup of $\mathbb{P}^2$ at three non-collinear points, considered as a smooth toric surface over $k$. Let $d_* = (d_0, d_1, d_2, d'_0, d'_1, d'_2) \in L$ with corresponding curve class $D_d$ on $S$. Suppose that $D_d$ is represented by an effective Cartier divisor on $S$, or is zero.

1. $M_{0,0}^\text{unr}(S,D_d)$ is non-empty precisely the following cases
   a. $d_i, d'_i \geq 0$ for all $i$ and there exist $i \neq j$ such that either $d_id_j \neq 0$ or $d'_id'_j \neq 0$
   b. There exists a triple $\{i, j, k\} = \{0, 1, 2\}$ such that $d_i = d'_i = 1$, $d_j = d_k = d'_j = d'_k = 0$
   c. There exists a triple $\{i, j, k\} = \{0, 1, 2\}$ such that $d_i = 1$, $d_j = d_k = 0$, $d'_j = d'_k = 1$ or $d'_i = -1$, $d'_j = d'_k = 0$, $d_i = d_k = 1$

2. In all the above cases, $1(a,b,c)$, $M_{0,0}^\text{unr}(S,D_d)$ is irreducible, and is dense and open in $M_0(S,D_d)$ (in case (1c)), $M_{0,0}^\text{unr}(S,d_d) = M_0(S,D_d)$ is a single point). Letting $d = \sum_i d_i + d'_i$, $M_{0,n}(S,D_d)$ is irreducible of dimension
   \[ \dim_k M_{0,n}(S,D_d) = d + n - 1. \]

3. $M_0(S,D_d) \neq \emptyset$ but $M_{0,0}^\text{unr}(S,D_d) = \emptyset$ if and only if one of the following holds:
   a. $D_d = 0$
b. there is a triple \(i, j, k\) with \(\{i, j, k\} = \{0, 1, 2\}\) and an integer \(m > 1\) such that 
\[d_i = d_i' = m, \quad d_j = d_j' = d_k = d_k' = 0\]
c. there is an integer \(m > 1\) with \(D_{d_i} = mD_i\) or \(D_{d_k} = mD_k'\).

4. \(M^\text{bir}_1(S, D_{d_*})\) is non-empty if and only if \(M^\text{unr}_1(S, D_{d_*})\) is non-empty.

**Proof.** Statements (1) and (2) follow from Proposition [A.9] and Remark [A.10]. For (3), we have \(M^\text{unr}_1(S, D_{d_*}) \subset M^\text{bir}_1(S, D_{d_*})\), and all the cases covered by Proposition [A.9] have \(M^\text{unr}_1(S, D_{d_*}) \neq \emptyset\), so if \(M_0(S, D_{d_*})\) is non-empty but \(M^\text{bir}_1(S, D_{d_*}) = \emptyset\), then we are in the situation discussed in Remark [A.10]. (3a) is case (1) of the Remark, (3b) is case (2) with \(m = d_0 > 1\), and (3c) is the remaining case, with 
\[-m = d_i < 1 \text{ or } -m = d_i' < 1 \text{ for some } i.\]
(4) follows from (1)-(3) together with Lemma [A.2].

**Remark.** We note that in the cases (1a, b) of Theorem [A.11] we have \(d \geq 2\), which implies that for \(f : P^1 \to S\) in \(M^\text{unr}_1(S, D_{d_*})\), \(f\) is free and birational (see Lemma [2.8] and Remark [2.7]), and thus we have the sequence of inclusions of non-empty, irreducible subschemes of the stack \(M_0(S, D_{d_*})\):
\[M^\text{unr}_1(S, D_{d_*}) \subset M^\text{bir}_1(S, D_{d_*}) \subset M^\text{bir}_1(S, D_{d_*}) \subset M_0(S, D_{d_*})\]

**Remark.** Theorem [A.11] and Remark [A.12] extend to all smooth toric del Pezzo surfaces \(S'\) by blowing down the appropriate -1 curves according to the following table:

<table>
<thead>
<tr>
<th>Surface (S')</th>
<th>Curves to blow down</th>
<th>Condition on the degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bl_{p_0, p_1}(P^2)</td>
<td>(D_2)</td>
<td>(d_2 = 0)</td>
</tr>
<tr>
<td>Bl_{p_0, p_1}(P^2)</td>
<td>(D_1, D_2)</td>
<td>(d_1 = d_2 = 0)</td>
</tr>
<tr>
<td>P^2</td>
<td>(D_0, D_1, D_2)</td>
<td>(d_0 = d_1 = d_2 = 0)</td>
</tr>
<tr>
<td>P^1 \times P^1</td>
<td>(D_0, D_0')</td>
<td>(d_0 = d_0' = 0)</td>
</tr>
</tbody>
</table>

and imposing the corresponding condition on \(d_* \in L\) to describe the curve classes on \(S'\).

Indeed, fix one of the above blowdown maps \(\pi : S \to S'\), and consider the set of \(d_* \in L\) for which the appropriate components are zero. Then each curve class on \(S'\) arises as \(\pi_*(D_{d_*})\) for a unique such \(d_*\).

Let \(p \subset S'\) be the set of points of the form \(p = \pi(E)\) for \(E \subset S\) an exceptional curve of \(\pi\), and let \(V = S' \setminus p\). Then \(\pi\) admits a well-defined inverse over \(V\), \(\pi^{-1} : V \to S\). We claim that the map
\[\pi_* : \text{Maps}(P^1, S, D_{d_*}) \to \text{Maps}(P^1, S', \pi_*(D_{d_*}))\]
is an open immersion with dense image.

Indeed, \(\pi_*\) has image \(U\) those \(f : P^1 \to S'\) with \(f(P^1) \subset V\). Since the condition \(f(P^1) \cap p \neq \emptyset\) defines a closed subset of Maps\(\(P^1, S', \pi_*(D_{d_*})\)), we see that \(U\) is an open subscheme of Maps\(\(P^1, S', \pi_*(D_{d_*})\)). Moreover, we have the morphism \(\pi^{-1} : U \to \text{Maps}(P^1, S, D_{d_*})\) with \(\pi^{-1} \circ \pi_* = \Id_{\text{Maps}(P^1, S, D_{d_*})}\). Thus \(\pi_*\) is an open immersion. Since Maps\(\(P^1, S, D_{d_*}\)) and Maps\(\(P^1, S', \pi_*(D_{d_*})\)) are irreducible (Theorem [A.1]), \(U\) is dense except in case Maps\(\(P^1, S, D_{d_*} = \emptyset\) and Maps\(\(P^1, S', \pi_*(D_{d_*})\)) \neq \emptyset\).

But if this were the case, this says that each \(f \in \text{Maps}(P^1, S', \pi_*(D_{d_*}))\) has \(f(P^1) \cap p \neq \emptyset\). Let \(\text{Aut}(S')^0\) denote the connected component of the identity in the automorphism group \(\text{Aut}(S')\). It is easy to see by explicit computation that for
for \(M\) the set of components of \(\text{Aut}(S)\) that are disjoint from the exceptional curves of \(\pi\), and hence \(p \in p\) does not lie on any -1 curve in \(S'\). Since \(\text{Aut}(S')\) acts trivially on \(\text{Pic}(S')\), we have the action of \(\text{Aut}(S')\) on \(\text{Maps}(\mathbb{P}^1, S', \pi_*(D_{d_*}))\) by post-composition, from which we see that the set of \(f \in \text{Maps}(\mathbb{P}^1, S', \pi_*(D_{d_*}))\) with \(f(\mathbb{P}^1) \cap p \neq \emptyset\) is a proper (closed) subset of \(\text{Maps}(\mathbb{P}^1, S', \pi_*(D_{d_*}))\). To conclude, this shows that \(\pi_*\) has dense image.

Thus, one can deduce results for \(M_0^{\text{unirr}}(S', D)\), etc., from the corresponding results for \(M_0^{\text{unirr}}(S, D_{d_*})\) with \(D = \pi_*(D_{d_*})\), and for which the appropriate components of \(d_*\) are zero.

**Remark A.14.** The above results also extend, suitably interpreted, to the case of a twisted form of a smooth toric del Pezzo surface, since the question of whether \(M_0(S, D_{d_*})\) is irreducible in the non-twisted case, the twisted form will have \(M_0(S, D)\) being geometrically irreducible. The only real difference is that the Picard group of a twisted form will in general be a proper subgroup of that of the split version, and even a proper subgroup of the Galois invariants of \(\text{Pic}(S_{k_{sep}})\), for example, if \(S\) is a Severi-Brauer surface (a form of \(\mathbb{P}^2\)), or a smooth quadric surface in \(\mathbb{P}^3\) (a form of \(\mathbb{P}^1 \times \mathbb{P}^1\)). Thus, the results detailed above will not hold for all \(d_*\), but only for those \(d_*\) for which the Cartier divisor \(D_{d_*}\) on \(S_{k}\) is in the image of \(\text{Pic}(S)\) under the (injective) base-change map \(\text{Pic}(S) \to \text{Pic}(S_{k})\) for passing to the algebraic closure \(k\) of \(k\).

**Corollary A.15.** Let \(S\) be a form of a toric del Pezzo surface over a field \(k\) with \(\text{char } k \neq 2\), that is \(S\) is a del Pezzo surface over \(k\) with \(d_S \geq 6\). Then \(S\) satisfies Assumption 2.30.

**Proof.** Let \(D\) be an effective Cartier divisor on \(S\). Since the statement of Assumption 2.30 concerns geometric points of \(M_0^{\text{unirr}}(S,D)\), we may assume that \(S\) is a toric del Pezzo surface over \(k\). By Remark A.13, we reduce to the case of \(S\) the blow-up of \(\mathbb{P}^2\) at three non-collinear points. In this case, this follows from Theorem A.11 and Theorem A.1. 

**References**


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