2-NILPOTENT REAL SECTION CONJECTURE

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ABSTRACT. We show a 2-nilpotent section conjecture over $\mathbb{R}$: for a geometrically connected curve X over $\mathbb{R}$ such that each irreducible component of its normalization has $\mathbb{R}$-points, $\pi_0(X(\mathbb{R}))$ is determined by the maximal 2-nilpotent quotient of the fundamental group with its Galois action, as the kernel of an obstruction of Jordan Ellenberg. This implies that for X smooth and proper, $X(\mathbb{R})^{\pm}$ is determined by the maximal 2-nilpotent quotient of $\mathrm{Gal}(\mathbb{C}(X))$ with its $\mathrm{Gal}(\mathbb{R})$ action, where $X(\mathbb{R})^{\pm}$ denotes the set of real points equipped with a real tangent direction, showing a 2-nilpotent birational real section conjecture.

1. INTRODUCTION

Grothendieck's section conjecture predicts that the rational points of hyperbolic curves over finitely generated fields are determined by their étale fundamental groups. Let X denote a geometrically connected, finite type scheme over a characteristic 0 field k, equipped with a geometric point $b : \operatorname{Spec}\Omega \to X$. Let $\overline{k}$ denote the algebraic closure of k in Ω. There is a canonical lift of b to $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec}\overline{k}$ and an exact sequence of étale fundamental groups

$$1 \to \pi_1(X_{\overline{k}}, b) \to \pi_1(X, b) \to G_k \to 1, \tag{1}$$

where $G_k = \mathrm{Gal}(\overline{k}/k)$ is the absolute Galois group of k, and all fundamental groups are based at the geometric points naturally associated to b [SGAI, IX Thm 6.1]. A rational point $x : \operatorname{Spec} k \to X$ induces a map $\pi_1(x) : G_k \to \pi_1(X, x)$, where $\pi_1(X, x)$ denotes the étale fundamental group of X based at the geometric point $\operatorname{Spec}\overline{k} \to \operatorname{Spec} k \to X$ associated to x. View x and b as geometric points of $X_{\overline{k}}$ and choose a path between them, where path means a natural transformation between the associated fiber functors, giving a path between x and b in X and an isomorphism $\pi_1(X, x) \cong \pi_1(X, b)$ respecting the projections to G_k. Composing $\pi_1(x)$ with this isomorphism $\pi_1(X, x) \cong \pi_1(X, b)$ produces a section $s : G_k \to \pi_1(X, b)$ of (1), and a different choice of path will change the section to $g \mapsto \lambda s(g)\lambda^{-1}$ for some λ in $\pi_1(X_{\overline{k}}, b)$. Sections obtained from s in this way are said to be conjugate. Let $\mathcal{S}_{\pi_1(X/k)}$ denote the conjugacy classes of sections of (1), and let $X(k)$ denote the set of k-points of X. Given X and b as above, let κ be the map

$$\kappa : X(k) \to \mathcal{S}_{\pi_1(X/k)}$$

just constructed. For X a smooth, proper curve of genus > 1 over a finitely generated field, Grothendieck's section conjecture, which is unknown, is that κ is a bijection.

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For $k = \mathbb{R}$, the map κ factors through $\pi_0(X(\mathbb{R}))$, and the real section conjecture, saying that

$$\kappa : \pi_0(X(\mathbb{R})) \to \mathcal{S}_{\pi_1(X/\mathbb{R})}$$

is a bijection is proven, but non-trivial [Moc03] [Sul05] [Mil84] [Car91] [Pál11]. This paper proves a 2-nilpotent real section conjecture, determining $\pi_0(X(\mathbb{R}))$ from the maximal 2-nilpotent quotient of $\pi_1(X_{\mathbb{C}})$ with its $G_{\mathbb{R}}$-action.

For a (profinite) group π, let $\pi = [\pi]_1 > [\pi]_2 > [\pi]_3 > \ldots$ denote the lower central series of π, i.e. $[\pi]_{n+1} = [[\pi]_n, \pi]$ (respectively $[\pi]_{n+1} = \overline{[[\pi]_n, \pi]}$) is (the closure of) the subgroup generated by commutators of elements of $[\pi]_n$ and π. Pushing out (1) by the quotient $\pi_1(X_{\overline{k}}, b) \to \pi_1(X_{\overline{k}}, b)/[\pi_1(X_{\overline{k}}, b)]_n$ yields an exact sequence

$$1 \to \pi_1(X_{\overline{k}}, b)/[\pi_1(X_{\overline{k}}, b)]_n \to \pi_1(X, b)/[\pi_1(X_{\overline{k}}, b)]_n \to G_k \to 1. \tag{2}$$

Let κ^{ab} be the map taking a k-point x of X to the section of (2) for $n = 2$ determined by $\kappa(x)$.

Define *curve* to mean a pure dimension 1, finite type scheme over a field. A curve X over k will be said to be *based* if it is equipped with a choice of a geometric point whose image is a k-point of X or which is associated to a k-tangent vector based at a smooth point of a compactification of X as described in [Del89, §15] [Nak99] [Wic12, 12.2.1]. The complex analytic space $X(\mathbb{C})$ associated to a based curve X over $\mathbb{R}$ has a distinguished point or tangent vector based at a smooth point of a compactification, allowing us to apply the topological or orbifold fundamental group functors to $X(\mathbb{C})$ or $X(\mathbb{C})/G_{\mathbb{R}}$, giving maps κ and κ^{ab} as above.

In the following theorem, X is a based curve over $\mathbb{R}$, π denotes either the étale or topological fundamental group of $X_{\mathbb{C}}$ or $X(\mathbb{C})$ respectively, and $\pi_1(X)$ denotes either the étale or orbifold fundamental group of X or $X(\mathbb{C})/G_{\mathbb{R}}$ respectively.

1.1. *Theorem. — Let X be a geometrically connected, based curve over $\mathbb{R}$, such that each irreducible component of its normalization has $\mathbb{R}$-points. Then κ^{ab} is a natural bijection from $\pi_0(X(\mathbb{R}))$ to conjugacy classes of sections of*

$$1 \to \pi/[\pi]_2 \to \pi_1(X)/[\pi]_2 \to G_{\mathbb{R}} \to 1$$

which lift to sections of

$$1 \to \pi/[\pi]_3 \to \pi_1(X)/[\pi]_3 \to G_{\mathbb{R}} \to 1.$$

Note that the assumption that X is based gives (1) a splitting, and that this implies that Theorem 1.1 says that the 2-nilpotent quotient $\pi/[\pi]_3$ of π with its $G_{\mathbb{R}}$-action determines the connected components of $X(\mathbb{R})$. The real section conjecture shows that π with its $G_{\mathbb{R}}$-action determines the connected components of $X(\mathbb{R})$ when the topological space $X(\mathbb{C})$ is a $K(\pi, 1)$, which is the case precisely when no component of the normalization of $X_{\mathbb{C}}$ is $\mathbb{P}^1$ – see Remark 2.8. The proof of Theorem 1.1 given below is independent of the real section conjecture, although assuming it, one would be saved the trouble of proving Proposition 2.2.

For X smooth and proper, Theorem 1.1 applied to smaller and smaller Zariski opens of X shows that $X(\mathbb{R})$ is determined by the maximal 2-nilpotent quotient of the absolute Galois group of the function field $\mathbb{C}(X)$ of $X_{\mathbb{C}}$ with its $G_{\mathbb{R}}$-action. Let $X(\mathbb{R})^{\pm}$ denote the set of real points of X equipped with a real tangent direction, i.e. a vector in the tangent space of the smooth 1-manifold $X(\mathbb{R})$ up to scaling by elements of $\mathbb{R}_{>0}$. The notation $X(\mathbb{R})^{\pm}$ is meant to indicate that after orienting $X(\mathbb{R})$, the two tangent directions associated to each element of $X(\mathbb{R})$ consist of the direction distinguished by the orientation and its negative. For any Zariski open U of X, there is a map $X(\mathbb{R})^{\pm} \to \pi_0(U(\mathbb{R}))$ given by taking a tangent direction to the connected component it is pointing towards. Note that the resulting map $X(\mathbb{R})^{\pm} \to \varprojlim_U \pi_0(U(\mathbb{R}))$ is a bijection. It follows that a corollary of the 2-nilpotent real section conjecture is that $X(\mathbb{R})^{\pm}$ is determined by $G_{\mathbb{R}(X)}/[G_{\mathbb{C}(X)}]_3 \to G_{\mathbb{R}}$ (§4 Corollary 4.1).

1.2. *Corollary. — Let X be a smooth, proper, connected curve over $\mathbb{R}$ equipped with a chosen element of $X(\mathbb{R})^{\pm} \neq \emptyset$. There is a natural bijection between $X(\mathbb{R})^{\pm}$ and the conjugacy classes of sections of*

$$1 \to G_{\mathbb{C}(X)}/[G_{\mathbb{C}(X)}]_2 \to G_{\mathbb{R}(X)}/[G_{\mathbb{C}(X)}]_2 \to G_{\mathbb{R}} \to 1$$

which lift to sections of

$$1 \to G_{\mathbb{C}(X)}/[G_{\mathbb{C}(X)}]_3 \to G_{\mathbb{R}(X)}/[G_{\mathbb{C}(X)}]_3 \to G_{\mathbb{R}} \to 1.$$

The real section conjecture and its 2-nilpotent version are closely related to Sullivan's conjecture, as we now discuss, first introducing some notation. This also helps summarize the proof of Theorem 1.1, which we do below.

Let $G = \mathbb{Z}/2$ and EG denote a contractible topological space with a free action of G. For a sufficiently well-behaved topological space Y with a G-action, e.g. Y a G-CW complex, let $\mathrm{Map}(EG, Y)$ denote the function space of continuous maps $EG \to Y$ equipped with the G action given by $gf = gfg^{-1}$. The homotopy fixed points of G on Y are defined $Y^{hG} = \mathrm{Map}(EG, Y)^G$ and there is a canonical map $Y^G \to Y^{hG}$ from the fixed points to the homotopy fixed points induced by the G-equivariant map from EG to the point.

Let $\mathcal{S}_{\pi_1(Y/G)}$ denote the conjugacy classes of sections of

$$1 \to \pi \to \pi_1(Y) \to G \to 1 \tag{3}$$

where π denotes the topological fundamental group of Y, based at some point not included in the notation, and $\pi_1(Y)$ denotes the orbifold fundamental group, which can be identified with the topological fundamental group of $EG \times_G Y$ or with the group of automorphisms of the universal cover of Y lying over an automorphism of Y induced by an element of G. There is a natural map $\pi_0(Y^{hG}) \to \mathcal{S}_{\pi_1(Y/G)}$, which is a bijection if Y is a $K(\pi, 1)$.

For X a geometrically connected, finite type scheme over $\mathbb{R}$, the map κ for the étale fundamental group is the composition

$$\pi_0(X(\mathbb{C})^{G_{\mathbb{R}}}) \to \pi_0(X(\mathbb{C})^{hG_{\mathbb{R}}}) \to \mathcal{S}_{\pi_1(X(\mathbb{C})/G_{\mathbb{R}})} \to \mathcal{S}_{\pi_1(X/\mathbb{R})} \tag{4}$$

where the last map is induced by the canonical isomorphism from the profinite completion of (3) to (1) [SGAI, XII Cor 5.2], and the map κ for the topological fundamental group is the composition of the first two maps of (4). For $X(\mathbb{C})$ a $K(\pi, 1)$, as in the section conjecture, the second map is a bijection.

The Sullivan conjecture [Sul05], proven by Miller [Mil84], Dwyer-Miller-Neisendorfer [DMN89], Carlsson [Car91], and Lannes [Lan92], shows that the first map of (4) is a bijection. Precisely, it says that the natural map from the p-completion of the fixed points to the homotopy fixed points of the p-completion is a weak equivalence for a finite p-group G acting on a finite G-CW complex, but proven at the same time is the fact that applying π_0 to $Y^G \to Y^{hG}$, as in the first map, is a bijection [Car91, Theorem B (a)]. So if one overlooks the map $\mathcal{S}_{\pi_1(X(\mathbb{C})/G_{\mathbb{R}})} \to \mathcal{S}_{\pi_1(X/\mathbb{R})}$ comparing the topological to the étale fundamental group, i.e. if one uses κ for the topological fundamental group, the real section conjecture is π_0 of Sullivan's conjecture applied to a $K(\pi, 1)$. Also see [Pál11] for a nice proof of the real section conjecture in the étale and topological case which does not appeal to Sullivan's conjecture.

The proof of Theorem 1.1 can be summarized as follows. Let $\pi_0(X(\mathbb{R}))^-$ denote $\pi_0(X(\mathbb{R}))$ with the component containing the base point removed. Let $\pi_0((-)^{G_{\mathbb{R}}})$ denote the functor taking $G_{\mathbb{R}}$-fixed points and then applying π_0. Applying $\pi_0((-)^{G_{\mathbb{R}}})$ to the map $X(\mathbb{C}) \to \mathrm{Sym}^\infty X(\mathbb{C})$ from $X(\mathbb{C})$ to its infinite symmetric product expresses $\pi_0(X(\mathbb{R}))^-$ as a basis for the $\mathbb{Z}/2$-vector space $\pi_0((\mathrm{Sym}^\infty X(\mathbb{C}))^{G_{\mathbb{R}}})$ (Proposition 2.4). For a smooth curve X with generalzed Jacobian $\mathrm{Pic}^0 X^+$, there is a natural map $\mathrm{Sym}^\infty X(\mathbb{C}) \to \mathrm{Pic}^0 X^+(\mathbb{C})$ which is an affine or projectivized vector bundle, whence a bijection after applying $\pi_0((-)^{G_{\mathbb{R}}})$ (Proposition 2.7). The conjugacy classes of sections of (2) for $n = 2$ are canonically identified with the connected components of the homotopy fixed points $(\mathrm{Pic}^0 X^+(\mathbb{C}))^{hG_{\mathbb{R}}}$ as above, since $\mathrm{Pic}^0 X^+_{\mathbb{C}}$ is a $K(\pi/[\pi]_2, 1)$, but κ also identifies them with the connected components of the fixed points of $\mathrm{Pic}^0 X^+(\mathbb{C})$, as is shown without appealing to Sullivan's conjecture (Proposition 2.2). In total, it follows that $\pi_0(X(\mathbb{R}))^-$ is a $\mathbb{Z}/2$-basis for the conjugacy classes of sections of (2) for $n = 2$, and in fact this holds without the assumption that X is smooth (Proposition 2.1).

By a standard interpretation of group cohomology, the conjugacy classes of sections of (2) for $n = 2$ are identified with $H^1(G_{\mathbb{R}}, \pi/[\pi]_2)$. The obstruction to lifting to a section of (2) for $n = 3$ is a map

$$\delta_2 : H^1(G_{\mathbb{R}}, \pi/[\pi]_2) \to H^2(G_{\mathbb{R}}, [\pi]_2/[\pi]_3),$$

which is quadratic with associated bilinear form

$$H^1(G_{\mathbb{R}}, \pi/[\pi]_2) \wedge H^1(G_{\mathbb{R}}, \pi/[\pi]_2) \to H^2(G_{\mathbb{R}}, [\pi]_2/[\pi]_3)$$

induced by the cup-product and the commutator pairing

$$\pi/[\pi]_2 \wedge \pi/[\pi]_2 \to [\pi]_2/[\pi]_3,$$

as follows from a result of Zarkhin [Zar74, p 242]. We show the associated bilinear form is injective (Lemmas 3.3 and 3.4), which implies Theorem 1.1 (Theorem 3.5).

It is not true in general that Sullivan's conjecture holds for the infinite symmetric product of a finite G-CW complex, nor that

$$\pi_0((\mathrm{Sym}^\infty Y)^G) \to \pi_0((\mathrm{Sym}^\infty Y)^{hG}) \quad (5)$$

is a bijection. An interesting consequence of the proof of the 2-nilpotent real section is that for $Y = X(\mathbb{C})$ with X a real based algebraic curve such that each irreducible component of its normalization has $\mathbb{R}$-points, the map (5) is a bijection. See Remark 2.9.

δ_2 was studied by Jordan Ellenberg as an obstruction to rational points of a curve's Jacobian lying in the image of an Abel-Jacobi map [Ell00], and also studied by Zarhin [Zar74]. Theorem 1.1 was guessed by Jordan Ellenberg in the proper, smooth case, as he told me after I had observed it held in several examples, and it can naturally be expressed in terms of his ideas in [Ell00]: a smooth based curve embeds into its generalized Jacobian by its Abel-Jacobi map. Those rational points y of the Jacobian which are the image of a point of the curve satisfy the condition that $\kappa(y)$ lifts to a section of (1) where X denotes the curve. Filtering $\pi_1(X_{\mathbb{C}})$ by its lower central series provides a series of obstructions, the first of which is the quadratic form, here denoted δ_2 and discussed in §3.

We give a topological interpretation of Ellenberg's point of view in Section 5, constructing a diagram of finite $G_{\mathbb{R}}$-CW complexes

(6)
$$\begin{array}{ccc} & & \mathrm{Alb}_2 \\ & \nearrow & \downarrow q \\ X(\mathbb{C}) & \longrightarrow & \mathrm{Alb}_1 \end{array}$$

for an arbitrary geometrically connected curve X over $\mathbb{R}$ with a chosen real base point, such that Alb_2 is a $K(\pi_1(X(\mathbb{C}))/[\pi_1(X(\mathbb{C}))]_3, 1)$, Alb_1 is a $K(\pi_1(X(\mathbb{C}))/[\pi_1(X(\mathbb{C}))]_2, 1)$, q is a fiber bundle, and all maps induce the obvious quotient maps on topological fundamental groups. Sullivan's conjecture gives an equivalence between Theorem 1.1 and the statement that the connected components of the real points of the curve are those of the abelian approximation Alb_1 which can be lifted to the 2-nilpotent approximation Alb_2. See Theorem 5.5.

Relation to other work: Grothendieck's section conjecture is part of his anabelian conjectures predicting that certain schemes are determined by their étale fundamental groups. Birational variants of the anabelian conjectures replace π_1^{et} by the absolute Galois group of the function field. There has been considerable work describing varieties using small quotients of their fundamental groups or the Galois groups of their function fields. Pop has shown a meta-abelian birational section conjecture over p-adic fields [Pop10a]. Bogomolov and Tschinkel developed an approach to recognize the function field of certain varieties using the 2-nilpotent quotient of the absolute Galois group. Work of Bogomolov, Pop, and Tschinkel shows that it is possible to recover the function field of certain varieties of dimension ≥ 2 over algebraically closed fields from the 2-nilpotent quotient of the absolute Galois group [Bog91b] [Bog91a] [BT02] [BT08] [Pop10b] [Pop12] – see [BT12] and [Pop11] for more discussion. There is also interesting work limiting when such minimalistic anabelian results can hold. Yuichiro Hoshi has found examples where any section of a pro-p homotopy exact sequence of the Jacobian lifts to a section of a pro-p homotopy exact sequence of the curve [Hos10].

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between this problem and Sullivan's conjecture. It is a pleasure to express my deepest gratitude and admiration to both Carlsson and Ellenberg. I thank an anonymous reviewer of a previous version of this paper for Lemma 3.3 and other helpful suggestions. I would also like to thank Florian Pop for suggesting that Theorem 1.1 could be used to show a birational result, as well as for many helpful comments.

2. ABELIAN APPROXIMATION

We compute κ^{ab}, which is considered as an abelian approximation to $\pi_0(X(\mathbb{R}))$.

Let X be a geometrically connected scheme with geometric point b and étale fundamental group π. The base point b determines a natural bijection $\mathcal{S}_{\pi_1(X/k)} = H^1(G_k, \pi)$. Under this identification, $\kappa(x)$ is represented by the cocycle

$$G_k \ni g \mapsto \gamma^{-1}(g\gamma) \in \pi,$$

where γ is a path from the base point to $x \in X(k)$, and composition in the fundamental group is written so that $\gamma^{-1}(g\gamma)$ is the loop starting at the base point obtained by first traversing $g\gamma$ and then traversing γ^{-1}. The analogous statements of course hold for the topological space $X(\mathbb{C})$.

Let $\mathcal{I}$ denote the forgetful functor from vector spaces over $\mathbb{Z}/2$ to pointed sets, sending a vector space to its underlying set, pointed by the identity. Let $\mathcal{V}$ denote its left adjoint, called the *free vector space* on the pointed set. The *unit* is the canonical map of pointed sets $(S, s_0) \to \mathcal{I}\mathcal{V}(S, s_0)$, so $\mathcal{V}(S, s_0)$ has basis $S - \{s_0\}$, and the unit map sends $S - \{s_0\}$ to this basis and s_0 to 0.

Note that for a based curve X over $\mathbb{R}$, the set $\pi_0(X(\mathbb{R}))$ is naturally pointed, as the base point's image either lies in a particular connected component or the associated tangent vector points towards one.

2.1. *Proposition. — Let X be a geometrically connected, based curve over $\mathbb{R}$, such that each irreducible component of its normalization has $\mathbb{R}$-points. Then*

$$\kappa^{ab} : \pi_0(X(\mathbb{R})) \to H^1(G_{\mathbb{R}}, \pi^{ab})$$

is canonically isomorphic to the unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on the pointed set $\pi_0(X(\mathbb{R}))$.

In Proposition 2.1, π can denote either the étale or topological fundamental group of $X_{\mathbb{C}}$ or $X(\mathbb{C})$, respectively.

The numerical version of Proposition 2.1 for X smooth and proper saying that

$$2^{|\pi_0(X(\mathbb{R}))|-1} = |H^1(G_{\mathbb{R}}, \pi^{ab})|$$

follows from combining [GH81, Prop 1.3, Prop 3.2] with the Kummer exact sequence of $\mathrm{Pic}^0 X$. There is a similar numerical computation of $|\pi_0(X(\mathbb{R}))|$ in terms of $\mathbb{Z}/2$-homology of X in [GH81, Prop 4.4]. The definition of κ^{ab} allows for the natural version above.

Proof. We first reduce to the case where X is smooth. Let $f : \tilde{X} \to X$ be the normalization of X. Note that $\tilde{X}$ is a disjoint union of smooth curves satisfying the hypotheses of the proposition. There is a finite $G_{\mathbb{R}}$-equivariant set $D \subset X(\mathbb{C})$ such that $X(\mathbb{C})$ is homeomorphic to the push-out $\tilde{X}(\mathbb{C}) \coprod_{\tilde{D}} D$ where $\tilde{D} = f^{-1}(D)$ and the push-out is taken with respect to $f : \tilde{D} \to D$ and the inclusion $\tilde{D} \subset \tilde{X}(\mathbb{C})$. It follows that

$$\pi_0 X(\mathbb{R}) \cong \pi_0 \tilde{X}(\mathbb{R}) \coprod_{\tilde{D}^{G_{\mathbb{R}}}} D^{G_{\mathbb{R}}}. \tag{7}$$

By Mayer-Vietoris, there is an exact sequence

$$0 \to H_1(\tilde{X}(\mathbb{C})) \to H_1(X(\mathbb{C})) \to \oplus_{\tilde{D}} \mathbb{Z} \to \oplus_{D \coprod \pi_0(\tilde{X}(\mathbb{C}))} \mathbb{Z} \to \mathbb{Z} \to 0 \tag{8}$$

of $\mathbb{Z}[G_{\mathbb{R}}]$-modules, where $H_1(-)$ denotes singular homology with $\mathbb{Z}$-coefficients. For the étale case, substitute $\mathbb{Z}^{\wedge}$ for $\mathbb{Z}$ and the abelianization of π_1 for $H_1(-)$. The sequence (8) remains exact and the following argument is valid with these substitutions.

Let τ denote complex conjugation. Consider the double complex

$$(E_{ij}, d_{ij} : E_{ij} \to E_{i+1,j}, \delta_{ij} : E_{ij} \to E_{i,j+1})$$

with identical rows (E_{*j}, d_{*j}) equal to (8) and differentials $\delta_{i,2j} = 1 - \tau$ and $\delta_{i,2j} = 1 + \tau$. This double complex gives a spectral sequence $(E^r_{ij}, D^r_{ij} : E^r_{ij} \to E^r_{i+r,j-r+1})$, $r = 0, 1, \ldots$ and $d^0_{ij} = \delta_{ij}$. This spectral sequence converges with $E^5_{ij} = E^\infty_{ij} = 0$, because (8) is exact.

Since $\oplus_{\tilde{D}} \mathbb{Z}$ and $\oplus_{D \coprod \pi_0(\tilde{X}(\mathbb{C}))} \mathbb{Z}$ are of the form $\mathbb{Z}[G_{\mathbb{R}}]^a \oplus \mathbb{Z}^b$, it follows that (E^1_{*j}, d^1_{*j}) for $j = 2n, 2n+1$ is

$$H^1(G_{\mathbb{R}}, H_1(\tilde{X}(\mathbb{C}))) \longrightarrow H^1(G_{\mathbb{R}}, H_1(X(\mathbb{C}))) \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

$$\hat{H}^0(G_{\mathbb{R}}, H_1(\tilde{X}(\mathbb{C}))) \longrightarrow \hat{H}^0(G_{\mathbb{R}}, H_1(X(\mathbb{C}))) \longrightarrow \oplus_{\tilde{D}^{G_{\mathbb{R}}}} \mathbb{Z}/2 \longrightarrow \oplus_{D^{G_{\mathbb{R}}} \coprod \pi_0(\tilde{X}(\mathbb{C}))} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$$

where $\hat{H}^0 = H^2$ is Tate cohomology. The map κ^{ab} applied to $D^{G_{\mathbb{R}}}$ and the images of chosen base points for each component of $\tilde{X}(\mathbb{C})$ gives a splitting of

$$d^2_{11} : H^1(G_{\mathbb{R}}, H_1(X(\mathbb{C}))) \to E^2_{03},$$

inducing an exact sequence

$$0 \to \oplus_{\tilde{D}^{G_{\mathbb{R}}}} \mathbb{Z}/2 \to H^1(G_{\mathbb{R}}, H_1(\tilde{X}(\mathbb{C}))) \oplus (\oplus_{D^{G_{\mathbb{R}}} \coprod \pi_0(\tilde{X}(\mathbb{C}))} \mathbb{Z}/2) \to H^1(G_{\mathbb{R}}, H_1(X(\mathbb{C}))) \oplus \mathbb{Z}/2 \to 0$$

which is compatible with κ^{ab} applied to the coproduct decomposition (7) of $\pi_0(X(\mathbb{R}))$ and the resulting short exact sequence

$$0 \to \oplus_{\tilde{D}^{G_{\mathbb{R}}}} \mathbb{Z}/2 \to \oplus_{\pi_0(\tilde{X}(\mathbb{R})) \coprod D^{G_{\mathbb{R}}}} \mathbb{Z}/2 \to \oplus_{\pi_0(X(\mathbb{R}))} \mathbb{Z}/2 \to 0.$$

It follows that it is sufficient to prove the claim for each connected component of $\tilde{X}$, i.e. we may assume that X is smooth.

We may also assume that the base point b of X is a geometric point whose image is an $\mathbb{R}$-point, i.e b is not tangential: for b and b' different choices of base point,

$$\kappa^{ab}_b(x) = \kappa^{ab}_{b'}(x) + \kappa^{ab}_b(b')$$

for all x in $\pi_0(X(\mathbb{R}))$. In particular, κ_b^{ab} is canonically isomorphic to $\kappa_{b'}^{ab}$ for any b' which determines the same path component of $X(\mathbb{R})$.

Since X is smooth, X embeds into its generalized Jacobian: for X smooth, non-proper, let X^+ denote the coproduct in schemes $X^c \coprod_{X^c - X} \operatorname{Spec} \mathbb{R}$ where X^c denotes the smooth compactification of X, and let $X^+ = X$ for X smooth, proper. In other words, X^+ is the one-point compactification formed by crushing $X^c - X$ to a point. Sending a point x of X to the invertible sheaf of rational functions on X^+ with at worst a simple pole at x determines a map $X \to \operatorname{Pic}^1 X^+$ from X to the degree 1 Picard scheme of X^+. Translation by the $\mathbb{R}$-point of $\operatorname{Pic}^1 X^+$ equal to the image of b gives an isomorphism $\operatorname{Pic}^n X^+ \cong \operatorname{Pic}^0 X^+$ for all n. The resulting map

$$\alpha : X \to \operatorname{Pic}^0 X^+$$

is the Abel-Jacobi embedding of X into its generalized Jacobian.

The Abel-Jacobi map induces $\pi_1(\alpha) : \pi \to \pi_1(\operatorname{Pic}^0 X^+_{\mathbb{C}})$ which is the abelianization in either the étale or topological case [Moc10, Prop A.8 (iii)]. By functoriality of κ, the diagram

$$\begin{array}{ccc} \pi_0(X(\mathbb{R})) & \xrightarrow{\pi_0(\alpha(\mathbb{R}))} & \pi_0(\operatorname{Pic}^0 X^+(\mathbb{R})) \\ \downarrow \kappa & & \downarrow \kappa_J \\ H^1(G_{\mathbb{R}}, \pi) & \xrightarrow{\pi_1(\alpha)_*} & H^1(G_{\mathbb{R}}, \pi_1(\operatorname{Pic}^0 X^+_{\mathbb{C}})) \end{array}$$

commutes, where κ_J is based at the identity of $\operatorname{Pic}^0 X^+$, giving an isomorphism between κ^{ab} and the composition

$$\pi_0(X(\mathbb{R})) \xrightarrow{\pi_0(\alpha(\mathbb{R}))} \pi_0(\operatorname{Pic}^0 X^+(\mathbb{R})) \xrightarrow{\kappa_J} H^1(G_{\mathbb{R}}, \pi_1(\operatorname{Pic}^0 X^+_{\mathbb{C}})) .$$

Proposition 2.1 follows from the fact that κ_J is an isomorphism (Proposition 2.2) and $\pi_0(\alpha(\mathbb{R}))$ is canonically isomorphic to the unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on the pointed set $\pi_0(X(\mathbb{R}))$, as follows from Proposition 2.4 and 2.7.

□

Let X be a smooth geometrically connected, based curve over $\mathbb{R}$, and choose a geometric point over the identity of $\operatorname{Pic}^0 X^+_{\mathbb{C}}$, giving $\kappa_J : \pi_0(\operatorname{Pic}^0 X^+(\mathbb{R})) \to H^1(G_{\mathbb{R}}, \pi_J)$, where π_J denotes either the étale or topological fundamental group of $\operatorname{Pic}^0 X^+_{\mathbb{C}}$.

2.2. *Proposition. — The map κ_J is an isomorphism of $\mathbb{Z}/2$-vector spaces.*

The content of the proof of Proposition 2.2 is identical in the étale and topological setting using [VW11, §3.3, 4.1]. We give the étale proof.

Proof. Let $\tilde{J}$ denote the universal cover of $J = \operatorname{Pic}^0 X^+$, which is automatically an abelian group. The canonical exact sequence

$$0 \to \pi_J \to \tilde{J} \to J_{\mathbb{C}} \to 0$$

gives a short exact sequence of abelian groups with $G_{\mathbb{R}}$-action

$$0 \to \pi_J \to \tilde{J}(\mathbb{C}) \to J(\mathbb{C}) \to 0.$$

Applying Tate cohomology gives the exact sequence

$$\hat{H}^0(G_{\mathbb{R}}, \tilde{J}(\mathbb{C})) \to \hat{H}^0(G_{\mathbb{R}}, J(\mathbb{C})) \to \hat{H}^1(G_{\mathbb{R}}, \pi_J) \to \hat{H}^1(G_{\mathbb{R}}, \tilde{J}(\mathbb{C})). \tag{9}$$

Since $\tilde{J} \to J \xrightarrow{[2]} J$ and $\tilde{J} \to J$ are simply connected covering spaces of J, where $[2] : J \to J$ denotes multiplication by 2, they are isomorphic [VW11, Prop. 3.1, Thm 3.1], whence multiplication by 2 is an isomorphism $[2] : \tilde{J} \to \tilde{J}$ and $\hat{H}^i(G_{\mathbb{R}}, \tilde{J}) = 0$ for all i.

It is straight-forward to verify that the diagram

$$\begin{array}{ccc} \hat{H}^0(G_{\mathbb{R}}, J(\mathbb{C})) & \xrightarrow{\cong} & \hat{H}^1(G_{\mathbb{R}}, \pi_J) \\ \uparrow & & \uparrow \kappa_J \\ J(\mathbb{R}) & \longrightarrow & \pi_0(J(\mathbb{R})) \end{array}$$

is commutative. Since $J(\mathbb{R}) \to \hat{H}^0(G_{\mathbb{R}}, J(\mathbb{C}))$ is surjective, κ_J is surjective. By [GH81, Prop 1.3], $\hat{H}^0(G_{\mathbb{R}}, J(\mathbb{C}))$ and $\pi_0(J(\mathbb{R}))$ have the same cardinality, so κ_J is bijective. □

2.3. *Remark.* Proposition 2.2 also follows from the real section conjecture.

The unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on $\pi_0(X(\mathbb{R}))$ is computed by the following proposition, whose proof is is omitted here, but is essentially contained in the proof of Proposition 3.2 in [GH81], which Gross and Harris credit to Shimura. For a topological space $\mathcal{X}$ with a base point, addition of the base point defines a map $\operatorname{Sym}^n \mathcal{X} \to \operatorname{Sym}^{n+1} \mathcal{X}$, and the infinite symmetric product $\operatorname{Sym}^\infty \mathcal{X}$ is defined to be the direct limit $\operatorname{Sym}^\infty \mathcal{X} = \varinjlim_n \operatorname{Sym}^n \mathcal{X}$. Taking the union of two finite sets of points of $\mathcal{X}$ determines an addition on $\operatorname{Sym}^\infty \mathcal{X}$ whose identity is the base point.

2.4. *Proposition. — Let $\mathcal{X}$ be a path connected, Hausdorff, topological space with an action of $G = \mathbb{Z}/2$ equipped with a G-fixed base point. Assume that the path components of $\mathcal{X}^G$ are closed in $\mathcal{X}^G$. Then the monoid structure on $\pi_0((\operatorname{Sym}^\infty \mathcal{X})^G)$ determined by the monoid structure on $\operatorname{Sym}^\infty \mathcal{X}$ is a $\mathbb{Z}/2$ vector space structure, and $\pi_0((-)^G)$ applied to $\mathcal{X} \to \operatorname{Sym}^\infty \mathcal{X}$ is canonically isomorphic to the unit of the adjunction $(\mathcal{V}, \mathcal{I})$ on the pointed set $\pi_0(\mathcal{X}^G)$.*

Since $\operatorname{Pic}^0 X^+$ is an abelian group scheme, α determines a map $\operatorname{Sym}^n X \to \operatorname{Pic}^0 X^+$ from the nth symmetric product of X to its generalized Jacobian. The following Proposition is well-known, but a proof is provided in the appendix for completeness.

2.5. *Proposition. — For n sufficiently large, $\operatorname{Sym}^n X \to \operatorname{Pic}^0 X^+$ is an affine bundle (projectivized vector bundle) for X non-proper (respectively proper).*

2.6. *Remark.* It follows that when the real points of $\mathrm{Sym}^n X$ and $\mathrm{Pic}^0 X^+$ are given the analytic topology, $\mathrm{Sym}^n X(\mathbb{R}) \to \mathrm{Pic}^0 X^+(\mathbb{R})$ is an affine bundle or projectivized vector bundle for n sufficiently large, giving an alternate way to see [GH81, Prop 3.2 (2) $n(W^d) = n(S^d X)$ for large d] and [Bro96, 2.7.4 $\mathcal{O}(D') \simeq \mathcal{O}(D)$].

The natural map from the nth symmetric power of $X(\mathbb{C})$ to $\mathrm{Sym}^n X(\mathbb{C})$ is a homeomorphism, so both may be denoted $\mathrm{Sym}^n X(\mathbb{C})$. Let $\mathrm{Sym}^\infty X(\mathbb{C}) = \varinjlim_n \mathrm{Sym}^n X(\mathbb{C})$ be the infinite symmetric product of the topological space $X(\mathbb{C})$. The maps $\mathrm{Sym}^n X \to \mathrm{Pic}^0 X^+$ are compatible with the map $\mathrm{Sym}^n X \to \mathrm{Sym}^{n+1} X$ given by addition of the base point, defining a $G_\mathbb{R}$-equivariant map $\mathrm{Sym}^\infty X(\mathbb{C}) \to \mathrm{Pic}^0 X^+(\mathbb{C})$.

2.7. *Proposition.* — $\pi_0((\mathrm{Sym}^\infty X(\mathbb{C}))^{G_\mathbb{R}}) \to \pi_0(\mathrm{Pic}^0 X^+(\mathbb{R}))$ *is a bijection.*

Proof. The natural map from $\varinjlim_n \mathrm{Sym}^n X(\mathbb{R})$ to the fixed points of $\mathrm{Sym}^\infty X(\mathbb{C})$ is a homeomorphism. By Proposition 2.5, $\pi_0(\mathrm{Sym}^n X(\mathbb{R})) \to \pi_0(\mathrm{Pic}^0 X^+(\mathbb{R}))$ is a bijection for sufficiently large n, whence $\pi_0(\varinjlim_n \mathrm{Sym}^n X(\mathbb{R})) = \varinjlim_n \pi_0(\mathrm{Sym}^n X(\mathbb{R})) \to \pi_0(\mathrm{Pic}^0 X^+(\mathbb{R}))$ is a bijection. □

2.8. *Remark.* The hypothesis of Proposition 2.1 and the resulting hypothesis of Theorem 1.1 is different from that of the real section conjecture, which is that $X(\mathbb{C})$ be a $K(\pi, 1)$. The normalization $\tilde{X} \to X$ induces a continuous map $\pi : \tilde{X}(\mathbb{C}) \to X(\mathbb{C})$, which factors through the quotient $\tilde{X}(\mathbb{C}) \to Y$ where Y is obtained by identifying points of $\tilde{X}(\mathbb{C})$ with equal images under π. The homeomorphism $Y \to X(\mathbb{C})$ shows that $X(\mathbb{C})$ is homotopy equivalent to the wedge of the connected components of $\tilde{X}(\mathbb{C})$ and a certain number of circles S^1. Since a wedge of $K(\pi, 1)$'s is a $K(\pi, 1)$, one sees that $X(\mathbb{C})$ is a $K(\pi, 1)$ if and only if none of the connected components of $\tilde{X}(\mathbb{C})$ are $\mathbb{P}^1_\mathbb{C}$.

2.9. *Remark.* Proposition 2.1 and Proposition 2.4 show that

$$\pi_0((\mathrm{Sym}^\infty X(\mathbb{C}))^{G_\mathbb{R}}) \to \pi_0((\mathrm{Sym}^\infty X(\mathbb{C}))^{hG_\mathbb{R}}) \to \mathcal{S}_{\pi_1((\mathrm{Sym}^\infty X(\mathbb{C}))/G_\mathbb{R})}$$

is a bijection. The second map is injective by the Dold-Thom theorem and the spectral sequence $H^i(G_\mathbb{R}, \pi_j(\mathrm{Sym}^\infty X(\mathbb{C}))) \Rightarrow \pi_{j-i}(\mathrm{Sym}^\infty X(\mathbb{C}))^{hG}$ [BK72, IX §4]. Thus the map from the fixed points to the homotopy fixed points of $\mathrm{Sym}^\infty X(\mathbb{C})$ is a bijection on π_0. Note that $\mathrm{Sym}^\infty X(\mathbb{C})$ is not a finite complex, so Sullivan's conjecture does not apply.

3. 2-NILPOTENT OBSTRUCTION

Recall that for an extension of profinite groups

$$1 \to \pi \to \tilde{\pi} \to G \to 1, \tag{10}$$

the conjugacy class of a section $s : G \to \tilde{\pi}$ refers to the set of sections of the form

$$g \mapsto \gamma s(g)\gamma^{-1}$$

where γ is in π. Pushing out (10) by $\pi \to \pi/[\pi]_n$ gives the extension

$$1 \to \pi/[\pi]_n \to \tilde{\pi}/[\pi]_n \to G \to 1. \tag{11}$$

When (10) is equipped with a splitting, the extensions (11) inherit splittings, which induce bijections between $H^1(G, \pi/[\pi]_n)$ and the conjugacy classes of sections of (11). Given a section $s : G \to \tilde{\pi}/[\pi]_2$ of (11) for $n = 2$, there exists a section $\tilde{s} : G \to \tilde{\pi}/[\pi]_3$ of (11) for $n = 3$ such that the composition $G \xrightarrow{\tilde{s}} \tilde{\pi}/[\pi]_3 \to \tilde{\pi}/[\pi]_2$ is in the conjugacy class of s if and only if the class of s vanishes under

$$\delta_2 : H^1(G, \pi/[\pi]_2) \to H^2(G, [\pi]_2/[\pi]_3)$$

where δ_2 is the boundary map in continuous group cohomology from the extension

$$1 \to [\pi]_2/[\pi]_3 \to \pi/[\pi]_3 \to \pi/[\pi]_2 \to 1.$$

We show that $\operatorname{Ker} \delta_2 = \operatorname{Image} \kappa^{ab}$ (Theorem 3.5) for $G = G_{\mathbb{R}}$ and $\tilde{\pi}$ equal to the étale or orbifold fundamental group of X or $X(\mathbb{C})/G_{\mathbb{R}}$, respectively, with X a curve over $\mathbb{R}$ satisfying the hypotheses of Theorem 1.1. This proves Theorem 1.1, since κ^{ab} is injective by Proposition 2.1.

δ_2 is quadratic with associated bilinear form given by the following cup product [Zar74, p 242]: let

$$[-,-] : \pi/[\pi]_2 \otimes \pi/[\pi]_2 \to [\pi]_2/[\pi]_3$$

be defined $[\gamma, \delta] = \tilde{\gamma}\tilde{\delta}\tilde{\gamma}^{-1}\tilde{\delta}^{-1}$ where $\tilde{\gamma}$, $\tilde{\delta}$ in $\pi/[\pi]_3$ map to γ, δ in $\pi/[\pi]_2$, respectively. (Since the choices of $\tilde{\gamma}$ differ by an element of the center, this map is well-defined on $\pi/[\pi]_2 \times \pi/[\pi]_2$. Bilinearity follows from [MKS04, Thm 5.1 p. 290].) The cup product

$$H^1(G, \pi/[\pi]_2) \otimes H^1(G, \pi/[\pi]_2) \to H^2(G, \pi/[\pi]_2 \otimes \pi/[\pi]_2)$$

can be pushed forward by $[-,-]$ to give a map

$$H^1(G, \pi/[\pi]_2) \otimes H^1(G, \pi/[\pi]_2) \to H^2(G, [\pi]_2/[\pi]_3). \tag{12}$$

3.1. *Proposition* (Zarkhin). — *For all* x, y *in* $H^1(G, \pi/[\pi]_2)$,

$$\delta_2(x + y) = \delta_2(x) + \delta_2(y) + [-,-]_* x \cup y.$$

3.2. *Remark.* For $\tilde{\pi}$ the étale fundamental group of a smooth, based, algebraic curve over a field and G the absolute Galois group, Jordan Ellenberg introduced and studied δ_2 as an obstruction to rational points of the Jacobian lying on the curve [Ell00].

We will show injectivity of the bilinear form $(x, y) \mapsto [-,-]_* x \cup y$, for which we need the following notation and lemmas.

For an abelian group $\mathcal{L}$, let $\mathcal{L} \wedge \mathcal{L}$ denote the quotient of $\mathcal{L} \otimes \mathcal{L}$ by the relation $\ell \otimes \ell = 0$ for all ℓ in $\mathcal{L}$. When $G = \mathbb{Z}/2$, the pairing

$$\cup : H^1(G, \mathcal{L}) \otimes H^1(G, \mathcal{L}) \to H^2(G, \mathcal{L} \wedge \mathcal{L})$$

induced by the cup product satisfies

$$\ell \cup \ell = 0$$

by a straight forward computation, giving a natural map

$$H^1(G, \mathcal{L}) \wedge H^1(G, \mathcal{L}) \to H^2(G, \mathcal{L} \wedge \mathcal{L}).$$

3.3. *Lemma. — Let $\mathcal{L}$ be an abelian group or profinite abelian group with no 2-torsion and an action of $G = \mathbb{Z}/2$. Then the natural map*

$$H^1(G, \mathcal{L}) \wedge H^1(G, \mathcal{L}) \to H^2(G, \mathcal{L} \wedge \mathcal{L})$$

induced by the cup product is injective.

Proof. Let τ denote the generator of G. Since G has order 2, $H^1(G, \mathcal{L})$ is a $\mathbb{Z}/2$-vector space. The short exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{2} \mathcal{L} \longrightarrow \mathcal{L}/2\mathcal{L} \longrightarrow 0$$

shows that

$$H^1(G, \mathcal{L}) \hookrightarrow H^1(G, \mathcal{L}/2\mathcal{L})$$

is injective. Similarly,

$$H^2(G, \mathcal{L} \wedge \mathcal{L}) \hookrightarrow H^2(G, \mathcal{L}/2 \wedge \mathcal{L}/2)$$

is injective. Thus, it suffices to show that

$$H^1(G, W) \wedge H^1(G, W) \to H^2(G, W \wedge W)$$

is injective for any $\mathbb{Z}/2$-vector space W with a G action. This map is described by

$$[w_1] \wedge [w_2] \mapsto [w_1 \wedge \tau w_2]$$

where w_i is in the kernel of $\tau + 1 : W \to W$, and $[w_i]$ denotes the corresponding cohomology class via the cyclic G resolution of $\mathbb{Z}$ [Bro94, V §1 pg. 108].

Note that $(1+\tau)(w_1 \wedge w_2) = w_1 \wedge w_2 + \tau w_1 \wedge \tau w_2 = (w_1 + \tau w_1) \wedge w_2 + \tau w_1 \wedge (w_2 + \tau w_2)$. Therefore, we have a map $H^2(G, W \wedge W) \to (W/I) \wedge (W/I)$ where I denotes the image of $\tau + 1 : W \to W$.

Let K denote the kernel of $\tau + 1 : W \to W$. Since the automorphisms $\tau + 1$ and $\tau - 1$ of W are equal, $H^1(G, W) \cong K/I$. The inclusion $K \hookrightarrow W$ induces an injection $K/I \wedge K/I \hookrightarrow W/I \wedge W/I$ because W is a vector space.

The the desired injectivity follows from the commutative diagram

$$\begin{array}{ccc} H^1(G, W) \wedge H^1(G, W) & \xrightarrow{\cup} & H^2(G, W \wedge W) \\ \downarrow \cong & & \downarrow \\ K/I \wedge K/I & \longrightarrow & (W/I) \wedge (W/I) \end{array}$$

∎

Note that $[-,-]$ factors through $\pi/[\pi]_2 \wedge \pi/[\pi]_2$ giving a map, also denoted $[-,-]$,

$$[-,-] : \pi/[\pi]_2 \wedge \pi/[\pi]_2 \to [\pi]_2/[\pi]_3.$$

3.4. *Lemma.* — *Let* X *be a geometrically connected, based curve over* $\mathbb{R}$ *and let* π *denote the étale or topological fundamental group of* $X_{\mathbb{C}}$, $X(\mathbb{C})$ *respectively. Then* $[-,-]_* : H^2(G_{\mathbb{R}}, \pi/[\pi]_2 \wedge \pi/[\pi]_2) \to H^2(G_{\mathbb{R}}, [\pi]_2/[\pi]_3)$ *is injective.*

Proof. By [Dwy75, Lemma 1.3], for any group π, there is a right exact sequence

$$H_2(\pi) \to H_2(\pi/[\pi]_2) \to [\pi]_2/[\pi]_3 \to 1, \tag{13}$$

where H_2 denotes group homology with $\mathbb{Z}$-coefficients. When $\pi/[\pi]_2$ is a free $\mathbb{Z}$-module, $H_2(\pi/[\pi]_2)$ is canonically identified with $\pi/[\pi]_2 \wedge \pi/[\pi]_2$ and the surjection of (13) is $[-,-]$.

Let $\pi = \pi_1^{\mathrm{top}} X(\mathbb{C})$, so $\pi/[\pi]_2$ is a free $\mathbb{Z}$-module. Since complex conjugation is orientation reversing, the canonical surjection $H_2(X(\mathbb{C})) \to H_2(\pi)$ [Bro94, Thm 5.2 p. 41] shows that $G_{\mathbb{R}}$ acts on $H_2(\pi)$ by multiplication by -1. Thus $G_{\mathbb{R}}$ acts on the kernel K of $[-,-] : \pi/[\pi]_2 \wedge \pi/[\pi]_2 \to [\pi]_2/[\pi]_3$ by multiplication by -1. Since $\pi/[\pi]_2 \wedge \pi/[\pi]_2$ is free, so is K. Thus $H_2(G_{\mathbb{R}}, K) = 0$, giving the desired injection in this case.

For $\mathcal{L}$ a free $\mathbb{Z}$-module with $\mathbb{Z}/2$-action, $H^i(\mathbb{Z}/2, \mathcal{L}) \to H^i(\mathbb{Z}/2, \mathcal{L}^\wedge)$ induced by profinite completion $\mathcal{L} \to \mathcal{L}^\wedge \cong \mathcal{L} \otimes_Z \mathbb{Z}^\wedge$ is an isomorphism because $\mathbb{Z}^\wedge$ is torsion-free whence flat over $\mathbb{Z}$. Since $\pi/[\pi]_2$ and $[\pi]_2/[\pi]_3$ are free $\mathbb{Z}$-modules, the étale case follows. □

Let X be as in Theorem 1.1 and define δ_2 with the extension (1) or its topological analogue.

3.5. *Theorem.* — $\operatorname{Ker} \delta_2 = \operatorname{Image} \kappa^{\mathrm{ab}}$.

Proof. For p in $X(\mathbb{R})$, $\kappa(p)$ determines a section of (11) for $n = 3$, so $\delta_2(\kappa^{\mathrm{ab}}(p)) = 0$.

By Proposition 2.1, an arbitrary element of $H^1(G_{\mathbb{R}}, \pi^{\mathrm{ab}})$ is of the form $x_1 + x_2 + \ldots + x_n$ where the x_i are in the image of κ^{ab} and $\{x_1, x_2, \ldots, x_n\}$ is linearly independent. By Proposition 3.1,

$$\delta_2(x_1 + x_2 + \ldots + x_n) = \sum_{i=1}^{n} \delta_2(x_i) + \sum_{1 \le i < j \le n} [-,-]_* x_i \cup x_j.$$

Furthermore, $\delta_2(x_i) = 0$ since $x_i \in \operatorname{Image} \kappa^{\mathrm{ab}}$. If $n > 1$, then $\sum_{1 \le i < j \le n} x_i \wedge x_j$ is a non-zero element of $H^1(G_{\mathbb{R}}, \pi^{\mathrm{ab}}) \wedge H^1(G_{\mathbb{R}}, \pi^{\mathrm{ab}})$, whence $\sum_{1 \le i < j \le n} [-,-]_* x_i \cup x_j$ is a non-zero element of $H^2(G_{\mathbb{R}}, [\pi]_2/[\pi]_3)$ by Lemmas 3.3 and 3.4. Thus $n = 1$ for any element of $\operatorname{Ker} \delta_2$, showing $\operatorname{Ker} \delta_2 \subset \operatorname{Image} \kappa^{\mathrm{ab}}$. □

4. Birational 2-nilpotent real section conjecture

Let X be a smooth, proper, geometrically connected curve over $\mathbb{R}$ such that $X(\mathbb{R}) \neq \emptyset$. Let $X(\mathbb{R})^{\pm}$ denote the set of real points of X equipped with a real tangent direction.

Choose a local parameter $z \in \mathbb{R}(X)$ at a point of $X(\mathbb{R})$, and note that z embeds the function field $\mathbb{R}(X)$ into the field of Puiseux series

$$\mathbb{C}((z^{\mathbb{Q}})) := \cup_{n \in \mathbb{Z}_{>0}} \mathbb{C}((z^{1/n})),$$

which is algebraically closed. Taking the algebraic closure Ω_z of $\mathbb{R}(X)$ in $\mathbb{C}((z^{\mathbb{Q}}))$ and considering the intermediate extension $\mathbb{R}(X) \subset \mathbb{C}(X)$ gives

$$1 \to \mathrm{Gal}(\Omega_z/\mathbb{C}(X)) \to \mathrm{Gal}(\Omega_z/\mathbb{R}(X)) \to G_{\mathbb{R}} \to 1 \tag{14}$$

which should be viewed as an analogue of the homotopy exact sequence (1). The coefficientwise action of $G_{\mathbb{R}}$ on $\mathbb{C}((z^{\mathbb{Q}}))$ defines a section of (14).

Given a second local parameter w and the associated embeddings

$$\mathbb{R}(X) \subset \overline{\mathbb{R}(X)} =: \Omega_w \subset \mathbb{C}((w^{\mathbb{Q}}))$$

choose an isomorphism $\Omega_z \cong \Omega_w$ which is the identity on the inclusion of $\mathbb{C}(X)$ in both fields, determining an isomorphism $\mathrm{Gal}(\Omega_z/\mathbb{R}(X)) \cong \mathrm{Gal}(\Omega_w/\mathbb{R}(X))$. The map $G_{\mathbb{R}} \to \mathrm{Gal}(\Omega_w/\mathbb{R}(X))$ defined by the coefficientwise action of $G_{\mathbb{R}}$ on $\mathbb{C}((w^{\mathbb{Q}}))$ then gives a second section of (14). It is not difficult to check that the conjugacy class of this section depends only on the element of $X(\mathbb{R})^{\pm}$ determined by the tangent vector

$$\mathrm{Spec}\,\mathbb{R}[w]/\langle w^2 \rangle \to X$$

associated to w.

It follows that $b \in X(\mathbb{R})^{\pm}$ determines a map $\kappa : X(\mathbb{R})^{\pm} \to \mathcal{S}_{\mathrm{Gal}(X/\mathbb{R})}$, where $\mathcal{S}_{\mathrm{Gal}(X/\mathbb{R})}$ denotes the set of conjugacy classes of sections of

$$1 \to G_{\mathbb{C}(X)} \to G_{\mathbb{R}(X)} \to G_{\mathbb{R}} \to 1.$$

We introduce some notation. Let $\mathcal{S}^n_{\mathrm{Gal}(X/\mathbb{R})}$ denote the set of conjugacy classes of sections of the push-out sequence

$$1 \to G_{\mathbb{C}(X)}/[G_{\mathbb{C}(X)}]_n \to G_{\mathbb{R}(X)}/[G_{\mathbb{C}(X)}]_n \to G_{\mathbb{R}} \to 1, \tag{15}$$

and for $m > k$, let $\mathcal{S}^{k \leftarrow m}_{\mathrm{Gal}(X/\mathbb{R})}$ denote the set of conjugacy classes of sections of (15) for $n = k$ which have a lift to a section of (15) for $n = m$. For $V \subseteq X$ Zariski open, let $\mathcal{S}^n_{\pi_1(V/\mathbb{R})}$ denote the set of conjugacy classes of sections of

$$1 \to \pi_1(V_{\mathbb{C}})/[\pi_1(V_{\mathbb{C}})]_n \to \pi_1(V)/[\pi_1(V_{\mathbb{C}})]_n \to G_{\mathbb{R}} \to 1, \tag{16}$$

where π_1 denotes the étale fundamental group, and let $\mathcal{S}^{k \leftarrow m}_{\pi_1(V/\mathbb{R})}$ denote the conjugacy classes of sections of (16) for $n = k$ which have a lift to a section of (16) for $n = m$.

4.1. *Corollary. — κ gives a natural bijection from $X(\mathbb{R})^{\pm}$ to $\mathcal{S}^{2 \leftarrow 3}_{\mathrm{Gal}(X/\mathbb{R})}$.*

Proof. For U a Zariski open of X, applying π_1 to the inclusion of the generic point gives a map $G_{\mathbb{R}(X)} \to \pi_1(U, b)$ [SGAI, V Proposition 8.1]. By functoriality of π_1, for any $V \subset U \subset X$, we have compatible maps

$$\mathcal{S}^n_{\mathrm{Gal}(X/\mathbb{R})} \to \mathcal{S}^n_{\pi_1(V/\mathbb{R})} \to \mathcal{S}^n_{\pi_1(U/\mathbb{R})}$$
$$\mathcal{S}^{n\leftarrow m}_{\mathrm{Gal}(X/\mathbb{R})} \to \mathcal{S}^{n\leftarrow m}_{\pi_1(V/\mathbb{R})} \to \mathcal{S}^{n\leftarrow m}_{\pi_1(U/\mathbb{R})}$$

for any n and $m > n$. These maps are compatible with κ in that the diagram

$$\begin{array}{ccc} X(\mathbb{R})^{\pm} & \longrightarrow & \mathcal{S}^{2\leftarrow 3}_{\mathrm{Gal}(X/\mathbb{R})} \\ \downarrow & & \downarrow \\ \varprojlim_U \pi_0(U(\mathbb{R})) & \longrightarrow & \varprojlim_U \mathcal{S}^{2\leftarrow 3}_{\pi_1(U/\mathbb{R})} \end{array}$$

commutes, where $X(\mathbb{R})^{\pm} \to \pi_0(U(\mathbb{R}))$ sends an element of $X(\mathbb{R})^{\pm}$ to the connected component the tangent direction points to, and where U runs over the Zariski opens of X.

By Theorem 1.1, $\kappa : \pi_0(U(\mathbb{R})) \to \mathcal{S}^{2\leftarrow 3}_{\pi_1(U/\mathbb{R})}$ is a bijection, showing that the bottom horizontal arrow is a bijection. The left vertical arrow is a bijection by inspection. It thus suffices to show that $\mathcal{S}^2_{\mathrm{Gal}(X/\mathbb{R})} \to \varprojlim_U \mathcal{S}^2_{\pi_1(U/\mathbb{R})}$ is injective, which is equivalent to showing that $\mathrm{H}^1(G_{\mathbb{R}}, G^{\mathrm{ab}}_{\mathbb{C}(X)}) \to \varprojlim_U \mathrm{H}^1(G_{\mathbb{R}}, \pi_1(U_{\mathbb{C}})^{\mathrm{ab}})$ is injective [Bro94, IV 2.3].

By [SGAI, V Proposition 8.2] the natural map $G_{\mathbb{C}(X)} \to \varprojlim_U \pi_1(U_{\mathbb{C}}, b)$ is an isomorphism. It follows that $G^{\mathrm{ab}}_{\mathbb{C}(X)} \to \varprojlim_U \pi_1(U_{\mathbb{C}}, b)^{\mathrm{ab}}$ is an isomorphism. Since $\varprojlim$ is exact as a functor on inverse systems of compact abelian groups, the natural map

$$\mathrm{H}^1(G_{\mathbb{R}}, \varprojlim_U \pi_1(U_{\mathbb{C}})^{\mathrm{ab}}) \to \varprojlim_U \mathrm{H}^1(G_{\mathbb{R}}, \pi_1(U_{\mathbb{C}})^{\mathrm{ab}})$$

is an isomorphism, showing the corollary. □

5. 2-NILPOTENT TOPOLOGICAL APPROXIMATION

Let X be a geometrically connected curve over $\mathbb{R}$ equipped with a base point in $X(\mathbb{R})$, and let π denote the topological fundamental group of $X(\mathbb{C})$.

5.1. *Abelian approximation.* The action of $\pi/[\pi]_2$ on itself by left translation gives an injective homomorphism

$$\pi/[\pi]_2 \to \mathrm{Affine}(\pi/[\pi]_2),$$

where $\mathrm{Affine}(\pi/[\pi]_2)$ denotes the group of invertible affine transformations of the free abelian group $\pi/[\pi]_2$. (The image of $\pi/[\pi]_2$ is contained in the subgroup of translations, but the notation is set up for the larger group.) Tensoring with $\mathbb{R}$ gives an action of $\pi/[\pi]_2$ on $\pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}$, and taking the quotient gives a model for $\mathrm{K}(\pi/[\pi]_2, 1)$ denoted

$$\mathrm{Alb}_1 = (\pi/[\pi]_2)\backslash(\pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}).$$

The Galois group $G_{\mathbb{R}}$ acts on $\pi/[\pi]_2$, giving a linear action on $\pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}$. For $g \in G_{\mathbb{R}}$ and $\gamma \in \pi/[\pi]_2$, we have an element $g\gamma$ of $\pi/[\pi]_2$, and the equality $g(\gamma v) = (g\gamma)(gv)$ for all $v \in \pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}$. Thus Alb_1 inherits a $G_{\mathbb{R}}$-action.

5.2. *2-Nilpotent approximation.* Choose a set of generators $x_1, \ldots, x_n$ for π which are a basis for $\pi/[\pi]_2$. Let s be the set-theoretic section of the quotient map $q : \pi/[\pi]_3 \to \pi/[\pi]_2$ given by taking $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ in $\pi/[\pi]_2$ to $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ in $\pi/[\pi]_3$. The section s determines a bijection between $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$ and $\pi/[\pi]_3$ by sending $v \oplus z \in \pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$ to $s(v)z$. Via this bijection, the group law on $\pi/[\pi]_3$ gives a composition law on $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$, denoted $\circ$. Let $+$ denote addition on the free abelian group $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$. There is a bilinear pairing

$$\langle -, - \rangle : \pi/[\pi]_2 \oplus \pi/[\pi]_2 \to [\pi]_2/[\pi]_3$$

such that

$$v \circ w = v + w + \langle qv, qw \rangle,$$

for v and w in $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$. Here and later, we slightly abuse the notation q by letting q also denote the projection $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3 \to \pi/[\pi]_2$. Thus the action of $\pi/[\pi]_3$ on itself by left translation gives an injective homomorphism

$$\pi/[\pi]_3 \to \mathrm{Affine}(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \subset \mathrm{Affine}((\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}).$$

For example, x_2 is sent to the affine transformation

$$(a_1, a_2, \ldots, a_n) \times z \mapsto (a_1, a_2 + 1, \ldots, a_n) \times (z - a_1[x_1, x_2]).$$

Taking the quotient of $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$ by $\pi/[\pi]_3$ gives a model for $K(\pi/[\pi]_3, 1)$, denoted

$$\mathrm{Alb}_2 = (\pi/[\pi]_3)\backslash((\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}).$$

$G_{\mathbb{R}}$ acts on $\pi/[\pi]_3$. Give $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$ the $G_{\mathbb{R}}$-action inherited from its bijection with $\pi/[\pi]_3$. Since for all $z \in 0 \oplus [\pi]_2/[\pi]_3$ we have $z \circ w = w \circ z = w + z$, it follows that

$$g(v \circ w) = g((v + w) \circ \langle qv, qw \rangle) = g(v + w) \circ g\langle qv, qw \rangle = g(v + w) + g\langle qv, qw \rangle,$$

$$(gv) \circ (gw) = (gv) + (gw) + \langle qgv, qgw \rangle.$$

Since $g(v \circ w) = (gv) \circ (gw)$ for $g \in G_{\mathbb{R}}$, we have that

$$g(w + v) - g(w) - g(v) = \langle gqv, gqw \rangle - g\langle qv, qw \rangle$$

is bilinear. Let $B_g(v, w) = \langle gqv, gqw \rangle - g\langle qv, qw \rangle$ denote this bilinear form. It follows that

$$g(v) = \frac{1}{2}B_g(v, v) + L_g(v) \tag{17}$$

where L_g is a linear endomorphism of $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$. In particular, the $G_{\mathbb{R}}$ action on $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$ extends to $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$. As above, the equality $g(\gamma v) = (g\gamma)(gv)$ for all $v \in (\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$, $g \in G_{\mathbb{R}}$ and $\gamma \in \pi/[\pi]_3$, implies that $G_{\mathbb{R}}$ acts on Alb_2.

The map $\mathrm{Alb}_2 \to \mathrm{Alb}_1$ induced by the projection q is a $G_{\mathbb{R}}$-equivariant fiber bundle with fiber $K([\pi]_2/[\pi]_3, 1)$.

5.3. *Mapping $X(\mathbb{C})$ to its approximations.* Equip Alb_2 with the base point induced by the origin of $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$, and say that a map between spaces with base points is pointed if the image of the base point of the domain is the base point of the codomain. Choose a pointed map $f : X(\mathbb{C}) \to \mathrm{Alb}_2$ such that the induced map f_* on fundamental groups is the quotient. Let $\widetilde{X(\mathbb{C})} \to X(\mathbb{C})$ and $\widetilde{\mathrm{Alb}_2} \to \mathrm{Alb}_2$ denote the universal pointed covering maps of $X(\mathbb{C})$ and Alb_2 respectively. Note that $\widetilde{\mathrm{Alb}_2} \to \mathrm{Alb}_2$ is $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{Alb}_2$, up to unique pointed isomorphism, where the origin is the base point of $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$. There is a unique lift of f to a pointed map $\tilde{f} : \widetilde{X(\mathbb{C})} \to \widetilde{\mathrm{Alb}_2}$. For all $\gamma \in \pi$, we can view γ as an automorphism of $\widetilde{X(\mathbb{C})}$ or $\widetilde{\mathrm{Alb}_2}$, and the choice of f_* implies that $\gamma\tilde{f} = \tilde{f}\gamma$. Give $\widetilde{X(\mathbb{C})}$ and $\widetilde{\mathrm{Alb}_2}$ the $G_{\mathbb{R}}$-actions lifting those of $X(\mathbb{C})$ and Alb_2 and such that the base points are fixed under $G_{\mathbb{R}}$. This $G_{\mathbb{R}}$-action on $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$ is consistent with the one above. Let τ denote complex conjugation and note that $\tau f \tau^{-1} : X(\mathbb{C}) \to \mathrm{Alb}_2$ is a pointed map such that the induced map on π_1 is the quotient. $\tau\tilde{f}\tau^{-1}$ is the unique lift of $\tau f \tau^{-1}$ to a pointed map between the pointed universal covering spaces, and $\gamma\tau\tilde{f}\tau^{-1} = \tau\tilde{f}\tau^{-1}\gamma$.

Define $\tilde{g} : \widetilde{X(\mathbb{C})} \to (\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R} = \widetilde{\mathrm{Alb}_2}$ by

$$\tilde{g} = \frac{1}{2}\tilde{f} + \frac{1}{2}\tau\tilde{f}\tau^{-1}.$$

By the above, γ acts by affine transformations on $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$, which implies that $\gamma\tilde{g} = \frac{1}{2}\gamma\tilde{f} + \frac{1}{2}\gamma\tau\tilde{f}\tau^{-1} = \frac{1}{2}\tilde{f}\gamma + \frac{1}{2}\tau\tilde{f}\tau^{-1}\gamma = \tilde{g}\gamma$. Thus, $\tilde{g}$ induces a pointed map $g : X(\mathbb{C}) \to \mathrm{Alb}_2$ such that g_* the quotient map on fundamental groups. Since τ acts linearly on the universal cover $\pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}$ of Alb_1, we have that $q\tilde{g} : \widetilde{X(\mathbb{C})} \to \pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R}$ is G_{R}-equivariant. Thus for any $\tilde{x}$ in $\widetilde{X(\mathbb{C})}$, the points $\tilde{g}$ and $\tau\tilde{g}\tau^{-1}$ are contained in the same fiber of q. The calculation (17) implies that τ determines an affine transformation between the fibers of $q : \widetilde{\mathrm{Alb}_2} \to \widetilde{\mathrm{Alb}_1}$ over $q\tilde{g}(\tilde{x})$ and $\tau q\tilde{g}(\tilde{x}) = q\tilde{g}(\tau\tilde{x})$.

Define $\tilde{h} : \widetilde{X(\mathbb{C})} \to \widetilde{\mathrm{Alb}_2}$ by $\tilde{h} = \frac{1}{2}\tilde{g} + \frac{1}{2}\tau\tilde{g}\tau^{-1}$. As above, $\tilde{h}$ induces a pointed map $h : X(\mathbb{C}) \to \mathrm{Alb}_2$ such that h_* the quotient map on fundamental groups. Furthermore,

$$\tau\tilde{h}\tau^{-1} = \tau(\frac{1}{2}\tilde{g}\tau^{-1} + \frac{1}{2}\tau\tilde{g}) = \frac{1}{2}\tau\tilde{g}\tau^{-1} + \frac{1}{2}\tilde{g} = \tilde{h}$$

where the second to last equality follows because τ is affine on the fiber over $q\tilde{g}(\tau^{-1}\tilde{x})$ for all $\tilde{x}$ in $\widetilde{X(\mathbb{C})}$. Thus $\tilde{h}$ and h and $G_{\mathbb{R}}$-equivariant. Use the notation α_2 for h, so $\alpha_2 = h$. Let $\alpha : X(\mathbb{C}) \to \mathrm{Alb}_1$ denote the composition of α_2 with the projection. The notation is chosen to recall that for the Abel-Jacobi map.

5.4. *Remark.* $X(\mathbb{C})$, Alb_1, and Alb_2 can be given the structure of finite $G_{\mathbb{R}}$-CW complexes in the sense of [Bre67]; by the main theorem of [Ill78], a smooth compact manifold with a $\mathbb{Z}/2$ action can be given such a structure, showing the existence of finite $G_{\mathbb{R}}$-CW complex structures for Alb_1, Alb_2, and the complex points of the smooth compactification of the normalization of X. Removing and or identifying finitely many points of a finite $G_{\mathbb{R}}$-CW complex yields a finite $G_{\mathbb{R}}$-CW complex, so $X(\mathbb{C})$ also has the structure of a finite $G_{\mathbb{R}}$-CW complex.

We obtain the commutative diagram of $G_{\mathbb{R}}$-equivariant maps between finite $G_{\mathbb{R}}$-CW complexes

$$\begin{array}{ccc} & & \mathrm{Alb}_2 \\ & \overset{\alpha_2}{\nearrow} & \downarrow \\ X(\mathbb{C}) & \xrightarrow{\alpha} & \mathrm{Alb}_1 \end{array} \tag{18}$$

and view Alb_1 as an abelian approximation to X and Alb_2 as a 2-nilpotent approximation to X. For X smooth and proper, integration gives a natural map from $\pi/[\pi]_2$ to the $\mathbb{C}$-linear dual of the global holomorphic one-forms on X, denoted $\mathrm{H}^0(X, \Omega)^*$. This map extends to a $G_{\mathbb{R}}$-equivariant $\mathbb{R}$-linear isomorphism

$$\widetilde{\mathrm{Alb}_1} = \pi/[\pi]_2 \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{H}^0(X, \Omega)^*$$

which identifies Alb_1 with the $G_{\mathbb{R}}$-topological space underlying the complex points of the Albanese variety of X. Thus Alb_1 in (18) can be replaced with $\mathrm{Pic}^0 X(\mathbb{C})$. The notation Alb comes from the view point that a 2-nilpotent approximation to X is analogous to a higher Albanese variety. See [Hai87], [HZ87].

Theorem 1.1 can be rephrased as the statement that the connected components of real points of the curve are those of the Albanese which can be lifted to the 2-nilpotent approximation:

5.5. *Theorem. — Let* X *be a geometrically connected, based curve over* $\mathbb{R}$*, such that each irreducible component of its normalization has* $\mathbb{R}$*-points. Let* Alb_1*,* Alb_2*, and* α *be as constructed above to obtain* (18)*. Then* α *induces a bijection from* $\pi_0(X(\mathbb{R}))$ *to the image of* $\pi_0(\mathrm{Alb}_2^{G_{\mathbb{R}}}) \to \pi_0(\mathrm{Alb}_1^{G_{\mathbb{R}}})$.

Proof. By Remark 5.4, Alb_1 is a finite $G_{\mathbb{R}}$-CW complexes. By [Car91, Thm B(a)], the natural map $\pi_0(\mathrm{Alb}_1^{G_{\mathbb{R}}}) \to \pi_0(\mathrm{Alb}_1^{hG_{\mathbb{R}}})$ is a bijection. Since Alb_1 is a $\mathrm{K}(\pi/[\pi]_2, 1)$, the natural map $\pi_0(\mathrm{Alb}_1^{hG_{\mathbb{R}}}) \to \mathcal{S}_{\pi_1(\mathrm{Alb}_1/G_{\mathbb{R}})}$ is a bijection. Under these bijections, α is identified with κ^{ab}. The same reasoning applied to Alb_2 identifies the image of $\pi_0(\mathrm{Alb}_2^{G_{\mathbb{R}}}) \to \pi_0(\mathrm{Alb}_1^{G_{\mathbb{R}}})$ with the image of $\mathcal{S}_{\pi_1(\mathrm{Alb}_2/G_{\mathbb{R}})} \to \mathcal{S}_{\pi_1(\mathrm{Alb}_1/G_{\mathbb{R}})}$. This shows that Theorem 5.5 and Theorem 1.1 are equivalent. ☐

5.6. *Example.* Let $X = \mathbb{P}^1_{\mathbb{R}} - \{0, 1, \infty\}$ equipped with a real base point b in $(0, 1)$, say $b = \frac{1}{2}$. The fundamental group π is freely generated by x_1 and x_2, where x_1 is represented by the loop $t \mapsto e^{2\pi i t}/2$ for $t \in [0, 1]$ and x_2 is the image of x_1 under the automorphism of X given by $z \mapsto 1 - z$.

The set $\{x_1, x_2, [x_1, x_2]\}$ is a basis for $\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3$, so points of $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$ can be labeled $(a_1, a_2, a_{12}) \in \mathbb{R}^3$ as an abbreviation for $a_1 x_1 + a_2 x_2 + a_{12}[x_1, x_2]$. The action of $\pi/[\pi]_3$ on $(\pi/[\pi]_2 \oplus [\pi]_2/[\pi]_3) \otimes_{\mathbb{Z}} \mathbb{R}$ is given by $x_1(a_1, a_2, a_{12}) = (a_1 + 1, a_2, a_{12})$ and $x_2(a_1, a_2, a_{12}) = (a_1, a_2 + 1, a_{12} - a_1)$. Note that $[x_1, x_2](a_1, a_2, a_{12}) = (a_1, a_2, a_{12} + 1)$ as well. It follows that Alb_3 is the quotient of the unit cube in $\mathbb{R}^3$ given by identifying the $a_1 = 0$ face with the $a_1 = 1$ face via the translation x_1, identifying the $a_{12} = 0$ face with the $a_{12} = 1$

face via the translation $[x_1, x_2]$, and identifying the $a_2 = 0$ face with the $a_2 = 1$ face via the translation-shear x_2. The $G_{\mathbb{R}}$-action on Alb_3 is given by $(a_1, a_2, a_{12}) \mapsto (-a_1, -a_2, a_{12})$.

Alb_2 is the torus given as the quotient of the unit square in $(\pi/[\pi]_2) \otimes_{\mathbb{Z}} \mathbb{R}$ with respect to the basis $\{x_1, x_2\}$ by the translations $(a_1, a_2) \mapsto (a_1 + 1, a_2)$ and $(a_1, a_2) \mapsto (a_1, a_2 + 1)$, and $G_{\mathbb{R}}$ acts by multiplication by -1.

X deformation retracts $G_{\mathbb{R}}$-equivariantly onto the union of the two circles which are the images of the representative loops for x_1 and x_2 described above. α_2 is the map taking the point $e^{2\pi i t}/2$ in the image of x_1 to tx_1 in Alb_3 and similarly for the points in the image of x_2.

Note that there are four $G_{\mathbb{R}}$-fixed points of Alb_2 given by the 2-torsion points

$$\{(0,0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$$

of Alb_2, and that the first three lift to fixed points of Alb_3 as they constitute the image of the fixed points of X in Alb_2. The fourth point $(\frac{1}{2}, \frac{1}{2})$ does not lift to a fixed point of Alb_3 as $G_{\mathbb{R}}$ acts on the fiber above $(\frac{1}{2}, \frac{1}{2})$ by translation by $(0, 0, \frac{1}{2})$. The fact that $(\frac{1}{2}, \frac{1}{2})$ does not lift can also be seen by applying Theorem 5.5. Example 5.6 is illustrated with figure 1. The fixed points are shown in red.

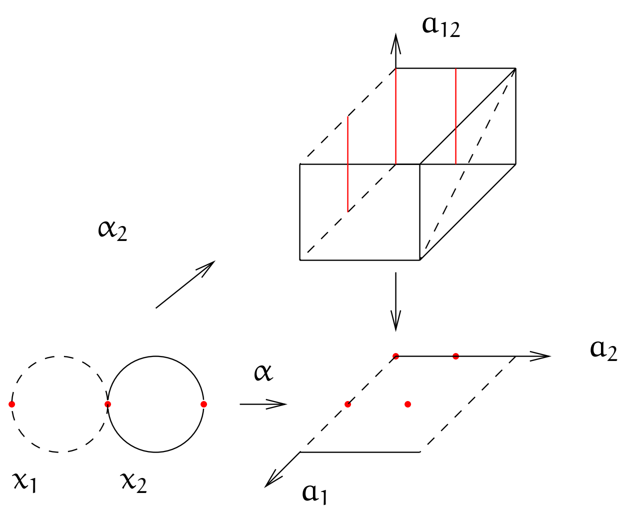

FIGURE 1. Approximations of $\mathbb{P}^1_{\mathbb{R}} - \{0, 1, \infty\}$

APPENDIX A. SYMMETRIC POWERS OF SMOOTH CURVES OVER THEIR GENERALIZED JACOBIANS

We provide a proof of Proposition 2.5 in the case where X is non-proper. See [ACGH85, VII §2 Prop 2.1] for the case where X is proper. This proposition and the following proof are both well-known, but we were unable to locate a reference in the literature.

For clarity, consider the more general situation in which Y is a geometrically integral, proper curve over k equipped with a k-point y_0. Let Div^n_Y denote the functor taking a locally Noetherian scheme T over k to the closed subschemes D of $Y \times T$, flat over T of degree n, and with invertible ideal sheaf $\mathcal{I}$ (see [BLR90, 8.2 p 212]). The association $D \mapsto \mathcal{I}^{-1}$ for $D \in \mathrm{Div}_Y(T)$ determines a map

$$\mathrm{Div}_Y^n \to \mathrm{Pic}_Y^n. \tag{19}$$

Let Div_{Y,y_0}^n be the functor taking T to the subset of $\mathrm{Div}_Y^n(T)$ consisting of those closed subschemes containing $y_0 \times T$. The map $\mathrm{Div}_{Y,y_0}^n \to \mathrm{Div}_Y^n$ is represented as follows.

By [BLR90, 8.1 Prop. 4], there is a unique universal invertible sheaf $\mathcal{P}$ over $Y \times \mathrm{Pic}_Y$ whose restriction to $y_0 \times \mathrm{Pic}_Y$ is trivial. Let $p : Y \times \mathrm{Pic}_Y \to \mathrm{Pic}_Y$ and $p' : Y \times \mathrm{Pic}_Y \to Y$ denote the projections. The closed immersion $y_0 : \mathrm{Spec}\, k \to Y$ corresponds to a short exact sequence of sheaves on Y

$$0 \to \mathcal{I}_{y_0} \to \mathcal{O}_Y \to \mathcal{O}_{y_0} \to 0.$$

Applying $(p')^*$ and tensoring with $\mathcal{P}$ yields the short exact sequence

$$0 \to \mathcal{P} \otimes (p')^*\mathcal{I}_{y_0} \to \mathcal{P} \to \mathcal{P} \otimes (p')^*\mathcal{O}_{y_0} \to 0. \tag{20}$$

By [BLR90, 8.1 Thm 7], since $\mathcal{P}$ is flat over Pic_Y, there is a coherent sheaf $\mathcal{F}$ on Pic_Y with functorial isomorphisms

$$p_*(\mathcal{P} \otimes p^*\mathcal{M}) \cong \mathcal{H}om(\mathcal{F}, \mathcal{M})$$

for all quasi-coherent sheaves $\mathcal{M}$ on Pic_Y. The morphism $\mathcal{P} \to \mathcal{P} \otimes (p')^*\mathcal{O}_{y_0}$ defines a natural map

$$p_*(\mathcal{P} \otimes p^*\mathcal{M}) \to p_*(\mathcal{P} \otimes (p')^*\mathcal{O}_{y_0} \otimes p^*\mathcal{M}) \cong \mathcal{M}, \tag{21}$$

where the isomorphism comes from the trivialization of $\mathcal{P}$ restricted to $y_0 \times \mathrm{Pic}_Y$. The map (21) defines a map $\mathcal{O}_{\mathrm{Pic}_Y} \to \mathcal{F}$. Let $\mathcal{E}$ denote the cokernel.

For a coherent sheaf $\mathcal{M}$, let $\mathbb{P}(\mathcal{M}) = \underline{\mathrm{Proj}}\,\mathrm{Sym}\,\mathcal{M}$ as in [EGAII, 3.1.3]. When $\mathcal{M}$ is locally free, $\mathbb{P}(\mathcal{M})$ is called a projectivized vector bundle.

The proof of [BLR90, 8.2 Prop 7] can be modified to show:

A.1. *Proposition.* — $\mathrm{Div}_{Y,y_0} \hookrightarrow \mathrm{Div}_Y$ *is represented by the closed immersion* $\mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{F}$.

Proof. It is sufficient to show the claim after restricting Div_{Y,y_0} and Div_Y to functors on schemes $T \to \mathrm{Pic}_Y$. We will let p, p' also denote their pullbacks to $Y \times T$, $\mathcal{P}_T$ denote the pullback of $\mathcal{P}$ to $Y \times T$, and $\mathcal{F}_T$, and $\mathcal{E}_T$ denote the pullbacks to T of $\mathcal{F}$, and $\mathcal{E}$ respectively. For a point $t : \mathrm{Spec}\, k(t) \to T$, let $Y_{k(t)} = (Y \times T) \times_T \mathrm{Spec}\, k(t)$. For an invertible sheaf $\mathcal{L}$, let $\mathcal{L}^{-1}$ be the dual invertible sheaf $\mathcal{L}^{-1} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$.

A section $s : \mathcal{O}_{Y \times T} \to \mathcal{P}_T$ induces a map

$$s^{-1} : \mathcal{P}_T^{-1} \to \mathcal{O}_{Y \times T}^{-1} = \mathcal{O}_{Y \times T}$$

which is an injection such that the corresponding closed subscheme D is flat over T if and only if the restriction of s^{-1} to $Y_{k(t)}$ is injective for all t by [EGAIV$_3$, Prop. 11.3.7]. Since $Y_{k(t)}$ is reduced and irreducible, the restriction of s^{-1} to $Y_{k(t)}$ is injective if and only if it is non-zero. By the definition of $\mathcal{F}$, the set of such s is in natural bijection with morphisms $\mathcal{F}_T \to \mathcal{O}_T$ which are non-zero on all stalks. By Nakayama's lemma, a morphism $\mathcal{F}_T \to \mathcal{O}_T$ is non-zero on all stalks if and only if it is surjective.

D contains $y_0 \times T$ if and only if the image of s^{-1} is contained in the ideal sheaf $(p')^*\mathcal{I}_{y_0}$ of $y_0 \times T$. This occurs if and only if the image of s is contained in $\mathcal{P}_T \otimes (p')^*\mathcal{I}_{y_0}$, which by (20) occurs if and only if the composition of s with

$$\mathcal{P}_T \to \mathcal{P} \otimes (p')^*\mathcal{O}_{y_0}$$

is 0. Thus, D contains $y_0 \times T$ if and only if the morphism $\mathcal{F}_T \to \mathcal{O}_T$ corresponding to s is pulled back from a morphism $\mathcal{E}_T \to \mathcal{O}_T$.

It follows that the association $s \mapsto D$ induces a bijection between the set of elements D of Div_{Y,y_0} equipped with an isomorphism $\mathcal{I} \to \mathcal{P}_T^{-1}$, where $\mathcal{I}$ denotes the ideal sheaf of D, and surjections $\mathcal{E}_T \to \mathcal{O}_T$. Let $\mathrm{Hom}(\mathcal{E}_T, \mathcal{O}_T)^{\mathrm{surj}} \subset \mathrm{Hom}(\mathcal{E}_T, \mathcal{O}_T)$ denote the subsheaf of surjections. As the automorphisms of $\mathcal{P}_T^{-1}$ are canonically isomorphic to the global sections $\Gamma(Y \times T, \mathcal{O}_{Y\times T}^*) = \Gamma(T, \mathcal{O}_T^*)$, the association $s \mapsto D$ induces a bijection between the set of elements D of Div_{Y,y_0} such that $\mathcal{I} \cong \mathcal{P}_T^{-1}$ and $\Gamma(T, \mathrm{Hom}(\mathcal{E}_T, \mathcal{O}_T)^{\mathrm{surj}})/\Gamma(T, \mathcal{O}^*)$.

Since two invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ on $Y \times T$ induce the same map $T \to \mathrm{Pic}_Y$ if and only if $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ is pulled back from T [BLR90, 8.1 Prop 4], it follows that

$$\Gamma(T, \mathrm{Hom}(\mathcal{E}_T, \mathcal{O}_T)^{\mathrm{surj}}/\mathcal{O}_T^*) = (\mathrm{Div}_Y \times_{\mathrm{Pic}_Y} T)(T).$$

Note that $\Gamma(T, \mathrm{Hom}(\mathcal{E}_T, \mathcal{O}_T)^{\mathrm{surj}}/\mathcal{O}_T^*)$ is in natural bijection with the set of all equivalence classes of pairs of an invertible sheaf $\mathcal{L}$ on T and a surjection $\varphi : \mathcal{E}_T \to \mathcal{L}$ where $(\mathcal{L}, \varphi)$ and $(\mathcal{L}', \varphi')$ are equivalent if there is an isomorphism $\theta : \mathcal{L} \to \mathcal{L}'$ such that $\varphi' = \theta\varphi$. By [EGAII, 4.2.3], it follows that the canonical closed immersion $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{F})$ represents $\mathrm{Div}_{Y,y_0} \to \mathrm{Div}_Y$. □

Let f be the global section of $\mathcal{F}$ corresponding to the map $\mathcal{O}_{\mathrm{Pic}_Y} \to \mathcal{F}$. By Proposition A.1,

$$\mathrm{Div}_Y - \mathrm{Div}_{Y,y_0} \cong \underline{\mathrm{Spec}}_{\mathrm{Pic}\,Y} \mathrm{Sym}\,\mathcal{F}[f^{-1}]_0, \tag{22}$$

where $\mathrm{Sym}\,\mathcal{F}[f^{-1}]_0$ denotes the degree 0 elements of the graded algebra given by inverting f in the symmetric algebra of $\mathcal{F}$. We will show that for large enough n, $(\mathrm{Div}_Y^n - \mathrm{Div}_{Y,y_0}^n) \to \mathrm{Pic}^n Y$ is an affine bundle, i.e. $\mathrm{Sym}\,\mathcal{F}[f^{-1}]_0$ is locally isomorphic to a polynomial algebra over $\mathcal{O}_{\mathrm{Pic}_Y^n}$ with affine transition maps between the local isomorphisms. For this, we will need the following lemma.

A.2. *Lemma. — For a coherent sheaf $\mathcal{M}$ on* Y, *let $\mathcal{G}$ be the coherent sheaf* $(p')^*\mathcal{M} \otimes \mathcal{P}$ *on* $Y \times \mathrm{Pic}_Y$. *Then:*

1. *For large n, the restriction of $R^i p_* \mathcal{G}$ to Pic_Y^n is 0 for all $i > 0$, and locally free for $i = 0$.*
2. *For large n, the restriction of $\mathcal{G}$ to $Y \times \mathrm{Pic}_Y^n$ is cohomologically flat over Pic_Y^n in all dimensions.*

Proof. The $i > 0$ case of (1): Since Y is projective, there is a relatively very ample invertible sheaf $\mathcal{L}$ for $p : Y \times \mathrm{Pic}_Y \to \mathrm{Pic}_Y$. Let $\mathcal{L}^{\otimes n}$ denote the n-fold tensor product of $\mathcal{L}$. Let $\mathcal{G}^k$ denote the restriction of $\mathcal{G}$ to $Y \times \mathrm{Pic}_Y^k$. For a fixed k, there is N such that for $n > N$, we have $R^i p_*(\mathcal{G}^k \otimes \mathcal{L}^{\otimes n}) = 0$ for all $i > 0$ by [EGAIII$_1$, Thm 2.2.1]. The invertible sheaf $\mathcal{P} \otimes \mathcal{L}^{\otimes m}$ induces an isomorphism $t : \mathrm{Pic}_Y^k \to \mathrm{Pic}_Y^{k+md}$ such that $t^*\mathcal{G}^{k+md} = \mathcal{G}^k \otimes \mathcal{L}^{\otimes m}$, where d denotes the degree of $\mathcal{L}$. Since $t^*R^ip_*(\mathcal{G}^{k+md}) = R^ip_*(t^*\mathcal{G}^{k+md})$, we have that $R^ip_*\mathcal{G}^{k+md} = 0$ for a fixed k, and all m sufficiently large and $i > 0$. Taking $k = 0, 1, \dots, d-1$ shows that for m sufficiently large we have $R^ip_*\mathcal{G}^m = 0$ for all $i > 0$.

(2): By the $i > 0$ case of (1), the restriction to $Y \times \mathrm{Pic}^n_Y$ of $R^ip_*(\mathcal{G})$ for n sufficiently large is locally free for all $i \geq 1$. By [EGAIII$_2$, Prop 7.8.5] it follows that this restriction of $\mathcal{G}$ is cohomologically flat in dimensions $i \geq 1$. For a point z of Pic_Y, let $\mathcal{G}_z$ denote the pullback of $\mathcal{G}$ by the closed immersion $Y_{k(z)} = Y \times \operatorname{Spec} k(z) \to Y \times \mathrm{Pic}^n_Y$ corresponding to z. By [EGAIII$_2$, Prop 7.8.4], it follows that for a fixed for $i \geq 1$, the function $z \mapsto d_i(z) = \dim_{k(z)} H^i(Y_{k(z)}, \mathcal{G}_z)$ on the points of Pic^n_Y for n sufficiently large is locally constant. Since the Euler characteristic of $\mathcal{G}_z$ is locally constant [EGAIII$_2$, Thm 7.9.4], it follows that d_0 is locally constant when restricted to Pic^n_Y for n sufficiently large. Since Pic^n_Y is reduced, (2) follows from [EGAIII$_2$, Prop 7.8.4].

The $i = 0$ case of (1), follows from (2) and [EGAIII$_2$, 7.8.5]. □

By [BLR90, 8.1 Thm 7], when n is large enough so that $\mathcal{P}$ is cohomologically flat over Pic^n_Y in dimension 0, the dual of $p_*(\mathcal{P})$ is canonically isomorphic to $\mathcal{F}$. Restrict (20) to $Y \times \mathrm{Pic}^n Y$ and apply p_*. By Lemma A.2, we obtain the short exact sequence of locally free sheaves

$$0 \to p_*(\mathcal{P} \otimes (p')^*\mathcal{I}_{y_0}) \to p_*(\mathcal{P}) \to \mathcal{O}_{\mathrm{Pic}^n_Y} \to 0, \tag{23}$$

for sufficiently large n. Dualizing yields the short exact sequence of locally free sheaves

$$0 \to \mathcal{O}_{\mathrm{Pic}^n_Y} \to \mathcal{F} \to \mathcal{E} \to 0.$$

By (22), it follows that for large enough n, $(\mathrm{Div}^n_Y - \mathrm{Div}^n_{Y,y_0}) \to \mathrm{Pic}^n Y$ is an affine bundle.

By [SGA4$_{\mathrm{III}}$, 6.3.9], $\mathrm{Sym}^n X$ represents the functor taking a locally Noetherian scheme T over k to the closed subschemes D of $X \times T$ which are flat and finite over T of degree n with invertible ideal sheaf. Thus $\mathrm{Sym}^n X \cong \mathrm{Div}^n_{X^+} - \mathrm{Div}^n_{X^+,\infty}$, where ∞ is the unique k point in $X^+ - X$. It follows that $\mathrm{Sym}^n X \to \mathrm{Pic}^n X^+$ is an affine bundle, which also shows Proposition 2.5.

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