DESUSPENSIONS OF $S^1 \wedge (\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\})$

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ABSTRACT. We use the Galois action on $\pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\})$ to show that the homotopy equivalence $S^1 \wedge (\mathbb{G}_{m,\mathbb{Q}} \vee \mathbb{G}_{m,\mathbb{Q}}) \cong S^1 \wedge (\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\})$ coming from purity does not desuspend to a map $\mathbb{G}_{m,\mathbb{Q}} \vee \mathbb{G}_{m,\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$.

1. INTRODUCTION

The étale fundamental group of the scheme $\mathbb{P}^1 - \{0, 1, \infty\}$ contains interesting arithmetic [Del89] [Iha86]. By viewing schemes as objects in the $\mathbb{A}^1$-homotopy category of Morel-Voevodsky [MV99], we may form the simplicial suspension $\Sigma X = S^1 \wedge X$ of a pointed scheme X, and the wedge product of two pointed schemes. After one simplicial suspension, $\mathbb{P}^1 - \{0, 1, \infty\}$ and the wedge $\mathbb{G}_m \vee \mathbb{G}_m$ of two copies of $\mathbb{G}_m$ become canonically $\mathbb{A}^1$-equivalent by the purity theorem [MV99, Theorem 2.23], and this is given in Proposition 3.1. This paper uses calculations of Anderson, Coleman, Ihara and collaborators [And89] [Col89] [Iha91] [IKY87] on the étale fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ to show that this $\mathbb{A}^1$-equivalence does not desuspend, in the sense that there does not exist a map $\mathbb{G}_{m,\mathbb{Q}} \vee \mathbb{G}_{m,\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ whose suspension is equivalent to the map coming from purity. This can be summarized by the statement that the Galois action on the étale fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ is an obstruction to desuspension.

There are topological obstructions to desuspension coming from James-Hopf maps, and they generalize to the setting of $\mathbb{A}^1$-homotopy theory [WW14] [AFWW15] as was known to Morel. They do not a priori have a relationship with the Galois action on $\pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\})$. This paper is motivated by the contrast between the systematic tools from algebraic topology available to obstruct desuspension and the arithmetic of $\pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\})$ which shows that such a desuspension does not exist.

The results of this paper are as follows. Let $\mathbf{Sm}_k$ denote the full subcategory of finite type schemes over a characteristic 0 field k whose objects are smooth, separated schemes. Let $\mathbf{sPre}(\mathbf{Sm}_k)$ denote presheaves of simplicial sets on $\mathbf{Sm}_k$. $\mathbf{sPre}(\mathbf{Sm}_k)$ has the structure of a simplicial model category in several ways. Let $\text{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$ denote the homotopy category of the $\mathbb{A}^1$-local, projective étale (respectively Nisnevich) model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$. See [Isa04, §2]. The results of this paper hold with either the Nisnevich or étale Grothendieck topology, and we will use the same notation for either. The category

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$\mathrm{ho}_{\mathbb{A}^1}\,\mathbf{sPre}(\mathbf{Sm}_k)$ is formed by formally inverting $\mathbb{A}^1$-weak equivalences and local étale (respectively Nisnevich) equivalences. In particular, in the Nisnevich case, $\mathrm{ho}_{\mathbb{A}^1}\,\mathbf{sPre}(\mathbf{Sm}_k)$ is the $\mathbb{A}^1$-homotopy category of [MV99].

In Section 3, we give the canonical isomorphism in $\mathrm{ho}_{\mathbb{A}^1}\,\mathbf{sPre}(\mathbf{Sm}_k)$ discussed above between the unreduced simplicial suspension of $\mathbb{P}^1_k - \{0,1,\infty\}$ and the simplicial suspension $\Sigma(\mathbb{G}_m \vee \mathbb{G}_m) = S^1 \wedge (\mathbb{G}_m \vee \mathbb{G}_m)$. The reduced and unreduced simplicial suspensions are canonically isomorphic in $\mathrm{ho}_{\mathbb{A}^1}\,\mathbf{sPre}(\mathbf{Sm}_k)$, although to form the reduced simplicial suspension, a base point is required. Let $\wp : \Sigma(\mathbb{G}_m \vee G_m) \to \Sigma(\mathbb{P}^1_k - \{0,1,\infty\})$ denote any of the maps in $\mathrm{ho}_{\mathbb{A}^1}\,\mathbf{sPre}(\mathbf{Sm}_k)$ resulting from choosing a base point in $\mathbb{P}^1_k - \{0,1,\infty\}$.

Theorem 1. *Let k be a finite extension of $\mathbb{Q}$ not containing a square root of 2. There is no morphism $\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k} \to \mathbb{P}^1_k - \{0,1,\infty\}$ in $\mathrm{ho}_{\mathbb{A}^1}\,\mathbf{sPre}(\mathbf{Sm}_k)$ whose simplicial suspension is $\wp$.*

Theorem 1 is proved as Theorem 4.2. The proof uses an étale realization, following ideas of Artin-Mazur, Friedlander, Isaksen, Quick, and Schmidt. The needed results are collected in Section 2.

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2. Equivariant Étale homotopy type

Let $\overline{k}$ be an algebraic closure of k, and $G = \mathrm{Gal}(\overline{k}/k)$ denote the absolute Galois group of k. Let $\mathbb{Z}^\wedge$ denote the profinite completion of $\mathbb{Z}$. Let $\chi : G \to (\mathbb{Z}^\wedge)^*$ denote the cyclotomic character. For an integer n, let $\mathbb{Z}^\wedge(n)$ denote $\mathbb{Z}^\wedge$ with the continuous G-action where g in G acts by multiplication by $\chi^n(g)$. For a pro-group $J = \{J_\alpha\}_{\alpha \in A}$, let BP denote the pro-simplicial set $\{BJ_\alpha\}_{\alpha\in A}$ given as the inverse system of the classifying spaces BJ_α of the groups J_α. For a group J, let $\mathrm{pro}-J^\wedge$ denote the pro-group given as the inverse system of the finite quotients of J.

Let $\mathrm{Et} : \mathbf{sPre}(\mathbf{Sm}_k) \to \mathbf{pro}-\mathbf{sSet}$ denote the étale homotopy type of [Isa04, Definition 1]. This étale homotopy type is built using the étale homotopy and topological types of [AM86] and [Fri82]. An alternate extension of the étale homotopy type to the $\mathbb{A}^1$-homotopy category of schemes was constructed independently by Alexander Schmidt [Sch03] [Sch12]. By [Isa04, Theorem 2], Et is a left Quillen functor with respect to the étale or Nisnevich local projective model structure on $\mathbf{sPre}(\mathbf{Sm}_k)$ and the model structure on $\mathbf{pro}-\mathbf{sSet}$ given in [Isa01]. This model structure on $\mathbf{pro}-\mathbf{sSet}$ is such that weak equivalences are the maps $f : X \to Y$ inducing an isomorphism of pro-sets $\pi_0(X) \to \pi_0(Y)$ and inducing an isomorphism from the local systems associated to $\pi_n(X)$ to the pull-back of the local system on Y coming from $\pi_n(Y)$. The cofibrations are maps isomorphic to levelwise cofibrations. See [Isa04, Definition 6.1, 6.2]. The proof that Et is a left Quillen functor uses [Dug01, Prop 2.3]. Let $\mathrm{LEt} : \mathrm{ho}\,\mathbf{sPre}(\mathbf{Sm}_k) \to \mathrm{ho}\,\mathbf{pro}-\mathbf{sSet}$ denote the corresponding homotopy invariant derived functor. By [Isa04, Corollary 4], the functor LEt agrees with the étale homotopy type for a scheme in $\mathbf{Sm}_k$. The standard calculation of the étale homotopy type of $\operatorname{Spec} k$ then gives $\mathrm{LEt}(\operatorname{Spec} k) \cong B(\mathrm{pro}-G^{\wedge})$.

2.1. **Fundamental groupoids.** Let X be a scheme over k. Recall that a geometric point $\overline{x} : \operatorname{Spec} \Omega \to X$ is a map where Ω is a separably closed field. Since we are assuming that k is characteristic 0, this is equivalent to the assumption that Ω be algebraically closed. Let $\Pi_1^{\text{ét}} X$ denote the étale fundamental groupoid of X whose objects are geometric points and morphisms are étale paths, i.e., natural transformations between the associated fiber functors [SGAI, V 7]. The morphisms are topologized with the natural profinite topology.

For a simplicial set X, let Π_1 denote the fundamental groupoid. Since homotopic maps of simplicial sets induce naturally isomorphic functors of fundamental groupoids, Π_1 factors through the homotopy category. For a pro-simplicial set $X = \{X_\alpha\}_{\alpha \in A}$, let $\Pi_1 X$ denote the pro-groupoid $\{\Pi_1 X_\alpha\}_{\alpha \in A}$. For X in $\mathbf{sPre}(\mathbf{Sm}_k)$, use the notation $\Pi_1(X)$ for $\Pi_1 \mathrm{LEt}\, X$.

2.2. **Base points and fundamental groups.** Given a map $* \to X = \{X_\alpha\}_{\alpha \in A}$ in $\mathbf{pro} - \mathbf{sSet}$, each simplicial set X_α has a base-point coming from the definition of morphisms

$$\operatorname{Hom}(*, X) = \varprojlim_{\alpha \in A} \varinjlim \operatorname{Hom}(*, X_\alpha) \cong \varprojlim_{\alpha \in A} \operatorname{Hom}(*, X_\alpha)$$

in the pro-category. Thus there is a distinguished object in each $\Pi_1 X_\alpha$. The endomorphisms of this object fit into a pro-group, defined to be the fundamental group $\pi_1(X)$ of $* \to X$.

The étale homotopy type Et takes a scheme X equipped with a geometric point $\operatorname{Spec} \Omega \to X$ to a pointed pro-simplicial set because $\mathrm{Et}(\operatorname{Spec} \Omega) \cong *$ for Ω an algebraically closed field. The resulting π_1 is independent of the choice of isomorphism $\mathrm{Et}(\operatorname{Spec} \Omega) \cong *$.

Let $\overline{x} : \operatorname{Spec} \Omega \to X$ be a geometric point of a scheme X. Replace Ω by the subfield of Ω given by the algebraic closure of the residue field of the point of X in the image of $\overline{x}$. This replacement has finite transcendence degree, and therefore $\operatorname{Spec} \Omega$ is an essentially smooth k-scheme in the sense of [Mor12, vi], i.e. a Noetherian scheme which is an inverse limit of a left filtering system $\{\Omega_\alpha\}_{\alpha \in A}$ of smooth k-schemes with étale affine transition morphisms. Since X is assumed to be finite type over k, the map $\overline{x}$ is determined by the images of finitely many functions on an open subset of X, and thus determines an element of $\varinjlim X(\Omega_\alpha)$. As in [Mor12], given $X \in \mathbf{sPre}(\mathbf{Sm}_k)$ and an essentially smooth k-scheme $\{Y_\alpha\}_{\alpha \in A}$, define $X(Y) = \varinjlim X(Y_\alpha)$, and call $X(Y)$ the set of Y points of X. For X in $\mathbf{sPre}(\mathbf{Sm}_k)$, a *geometric point* of X indicates an element of $X(\operatorname{Spec} \Omega)$, where Ω is an algebraically closed field of finite transcendence degree over k. Note that for $X \in \mathbf{sPre}(\mathbf{Sm}_k)$, a geometric point $\overline{x} \in X(\operatorname{Spec} \Omega)$ induces a map $\mathrm{Et}(\operatorname{Spec} \Omega) \to \mathrm{LEt}\, X$.

Let $k-\mathbf{Sm}^+$ denote the following category. The objects of $k-\mathbf{Sm}^+$ are pairs $(X, \overline{x})$, where X is a smooth k-scheme and $\overline{x}$ is a geometric point of X equipped with a path between its image under $X \to \operatorname{Spec} k$ and the geometric point $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$ of k. The morphisms $(X, \overline{x}) \to (Y, \overline{y})$ of $k - \mathbf{Sm}^+$ are the morphisms $X \to Y$ in $k - \mathbf{Sm}$. There is no requirement that $\overline{x}$ is taken to $\overline{y}$. Let $k - \mathbf{Sm}^+_c$ denote the full subcategory of $k - \mathbf{Sm}^+$ of objects such that X is connected. Let $\mathbf{Grp}^{\text{out}}_G$ denote the category of topological groups over G and outer homomorphisms, i.e., the objects of $\mathbf{Grp}^{\text{out}}_G$ are morphisms $\pi \to G$ and the morphisms from

$\pi \to G$ to $\pi' \to G$ is the set of equivalence classes of morphisms $\pi \to \pi'$ such that the two morphisms $\pi \to G$ coming from the diagram

$$\begin{array}{ccc} \pi & \longrightarrow & \pi' \\ & \searrow \quad \swarrow & \\ & G & \end{array}$$

differ by an inner automorphism of G and where two such morphims $f, f' : \pi \to \pi'$ are considered equivalent if there exists $\gamma \in \pi'$ such that $f'(x) = \gamma f(x)\gamma^{-1}$ for all x in π.

Given a morphism $\Pi \to \Pi'$ of pro-groupoids and a commutative diagram

$$\begin{array}{ccc} \Pi & \longrightarrow & \Pi' \\ \uparrow & & \uparrow \\ * & \longrightarrow & * \end{array},$$

there is an associated morphism of fundamental pro-groups $\pi \to \pi'$, where π is the endomorphims in Π of the distinguished object and similarly for π'. Given two maps $x_1, x_2 : * \to \Pi'$ and a choice of morphism from x_1 to x_2 we obtain an isomorphism between the fundamental group based at x_1 and the fundamental group based at x_2. A different choice of path changes the isomorphism by an inner isomorphism. To a map between objects of $k - \mathbf{Sm}_c^+$, we may therefore associate an outer homomorphism. We claim that this defines a functor $\pi_1^{\text{ét}} : k - \mathbf{Sm}_c^+ \to \mathbf{Grp}_G^{\text{out}}$. To see this, note that for the object $(X, \overline{x})$ of $k - \mathbf{Sm}_c^+$, the path between the image of $\overline{x}$ under $X \to \operatorname{Spec} k$ and the geometric point $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$ produces a morphism $\pi_1^{\text{ét}}(X, \overline{x}) \to G$. Given $(X, \overline{x}) \to (Y, \overline{y})$, the induced outer homomorphism $\pi_1^{\text{ét}}(X, \overline{x}) \to \pi_1^{\text{ét}}(Y, \overline{y})$ respects the maps to G up to inner automorphism because $X \to Y$ respects the maps to $\operatorname{Spec} k$. This shows that $\pi_1^{\text{ét}}$ determines the claimed functor.

We need a fundamental group on pointed objects of $\mathbf{sPre}(\mathbf{Sm}_k)$ with similar functoriality properties, so we introduce notation in this context analogous to the above. Let $\mathbf{sPre}(\mathbf{Sm}_k)^+$ denote the category whose objects are pairs $(X, \overline{x})$, where X is in $\mathbf{sPre}(\mathbf{Sm}_k)$ and $\overline{x}$ is a geometric point of X whose image in the set of geometric points of $\operatorname{Spec} k$ has a chosen path to $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$, and whose morphisms $(X, \overline{x}) \to (Y, \overline{y})$ are the morphisms $X \to Y$ in $\mathbf{sPre}(\mathbf{Sm}_k)$. There is again no requirement that $\overline{x}$ is taken to $\overline{y}$. Define $\operatorname{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)^+$ similarly, i.e., the morphisms $(X, \overline{x}) \to (Y, \overline{y})$ are the morphisms $X \to Y$ in $\operatorname{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$. Let $\mathbf{sPre}(\mathbf{Sm}_k)_c^+$ denote the full-subcategory of $\mathbf{sPre}(\mathbf{Sm}_k)^+$ on objects such that $\operatorname{LEt} X$ is connected. Similarly define $\operatorname{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)_c^+$ to be the full-subcategory of $\operatorname{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)^+$ on objects such that $\operatorname{LEt} X$ is connected. Let $\text{pro} - \mathbf{Grp}^{\text{out}}_{\text{pro} - G^\wedge}$ denote the category of pro-groups over $\text{pro} - G^\wedge$ and outer homomorphisms.

For $(X, \overline{x})$ in $\mathbf{sPre}(\mathbf{Sm}_k)^+$, define $\pi_1(X, \overline{x} : \operatorname{Spec} \Omega \to X)$ to be π_1 of the pointed prosimplicial set $* \cong \operatorname{Et}(\operatorname{Spec} \Omega) \to \operatorname{LEt} X$. By the same argument as above, π_1 defines a functor $\pi_1 : \mathbf{sPre}(\mathbf{Sm}_k)_c^+ \to \text{pro} - \mathbf{Grp}^{\text{out}}_{\text{pro} - G^\wedge}$.

2.3. **Homotopy invariant functors.**

Proposition 2.1. *The functor* $\pi_1 : \mathbf{sPre}(\mathbf{Sm}_k)^+_c \to \text{pro}-\mathbf{Grp}^{\text{out}}_{\text{pro}-G^\wedge}$ *factors through* $\text{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)^+_c$*. Furthermore, the diagram*

$$\begin{array}{ccc} k-\mathbf{Sm}^+_c & \xrightarrow{\pi_1^{ét}} & \mathbf{Grp}^{\text{out}}_G \\ \downarrow & & \uparrow \underleftarrow{\lim} \\ \text{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)^+_c & \xrightarrow{\pi_1} & \text{pro}-\mathbf{Grp}^{\text{out}}_{\text{pro}-G^\wedge} \end{array}$$

commutes up to isomorphism.

Proof. For a scheme X, let $\mathcal{X}$ in $\mathbf{sPre}(\mathbf{Sm}_k)$ denote the corresponding sheaf. There is a natural isomorphism $\pi_1^{ét}(X) \cong \underleftarrow{\lim}\, \pi_1\, \text{LEt}\, \mathcal{X}$ for every smooth scheme X over k equipped with a geometric point because both sides classify finite étale covers of X. For the left hand side, this is immediate. For the right hand side, this follows from [Fri82, Prop 5.6] and [AM86, 11.1].

To show that π_1 factors through $\text{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)^+_c$, it suffices to show that the functor Π_1 from spaces to pro-groupoids factors through $\text{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)$. Since $\Pi_1 = \Pi_1\, \text{LEt}$, we know that Π_1 factors through the homotopy category of the étale (respectively Nisnevich) local projective model structure. To show the factorization through $\text{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)$, it is thus sufficient to show that $X \times \mathbb{A}^1 \to X$ is sent to an isomorphism for all schemes X. This follows from the analogous claim on étale fundamental groups, which is true in characteristic 0. (One can see that $\pi_1^{ét}(X \times \mathbb{A}^1) \to \pi_1^{ét} X$ is an isomorphism in characteristic 0 by combining [SGAI, IX Théorème 6.1] with the analogous result over $\overline{k}$. Over $\overline{k}$, the map is an isomorphism by invariance of $\pi_1^{ét}$ under algebraically closed extensions of fields [SGAI, XIII Proposition 4.6] and comparison with the topological fundamental group [SGAI, XII Corollaire 5.2].) □

Example 2.2. *We compute* $\pi_1(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}, *) \to G$*. The map* $* \to \mathbb{G}_{m,k}$ *corresponding to the point* 1 *is a flasque cofibration because it is the push-out product of itself and* $\partial\Delta^0 \to \Delta^0$*. See* [Isa05, Definition 3.2]*. Since representable presheaves are projective cofibrant,* $*$ *and* $\mathbb{G}_{m,k}$ *are projective cofibrant, whence also flasque cofibrant. It follows that* $\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}$ *is a homotopy colimit in the flasque model structure. Since* Et *is a left Quillen functor on the Nisnevich (or étale) local flasque model structure* [Qui08, Theorem 3.4] *and since there is a weak equivalence between* Et *derived with respect to the local flasque model structure, and* LEt*, which denotes* Et *derived with respect to the projective local model structure, it follows that*

$$\begin{array}{ccc} \text{LEt}(\text{Spec}\, k) & \longrightarrow & \text{LEt}(\mathbb{G}_{m,k}) \\ \downarrow & & \downarrow \\ \text{LEt}(\mathbb{G}_{m,k}) & \longrightarrow & \text{LEt}(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}) \end{array}$$

is a homotopy push-out square.

Let $\{\langle x, y | x^n = 1 = y^n \rangle\}_n$ *denote the pro-group given as the inverse system over* n *of the free product of* $\mathbb{Z}/n$ *with* $\mathbb{Z}/n$ *and transition maps induced by quotient maps* $\mathbb{Z}/(nm) \to \mathbb{Z}/n$*. Let* G *act on* $\langle x, y | x^n = 1 = y^n \rangle$ *by*

$$gx = x^{\chi(g)} \qquad gy = y^{\chi(g)}.$$

Let I *denote the directed set consisting of pairs* (n, H) *with* n *a positive integer and* H *a finite quotient of* G *which acts on* $k(\mu_n)$*, i.e.,* H *is such that the fixed field of* $\mathrm{Ker}(G \to H)$ *contains the* n*th roots of unity in* k*.* I *is defined so that there is a map* $(n, H) \to (n', H')$ *exactly when* H' *is a quotient of* H *and* n' *is a quotient of* n*.*

We claim that $\mathrm{LEt}(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}) \to \mathrm{LEt}\,\mathrm{Spec}\,k$ *can be identified with the map*

$$\{B(\langle x, y | x^n = 1 = y^n\rangle \rtimes H)\}_{(n,H)\in I} \to B\,\mathrm{pro}\,G^\wedge,$$

induced by the map of pro-groups sending $(\langle x, y|x^n = 1 = y^n\rangle \rtimes H$ *to* H *by the projection.* $\mathrm{LEt}\,\mathbb{G}_m$ *is the étale topological type of* $\mathbb{G}_m$*, and the map* $\mathrm{LEt}\,\mathbb{G}_m \to \mathrm{LEt}\,\mathrm{Spec}\,k$ *can be identified with* $B\,\mathrm{pro}(\mathbb{Z}^\wedge(1) \rtimes G)^\wedge \to B\,\mathrm{pro}\,G^\wedge$*; see, for example,* [Qui11, Prop. 3.2]*, noting that the schemes involved have profinite étale topological types, and thus the referenced result, which is in the context of simplicial profinite sets, implies the given identification of* $\mathrm{LEt}\,\mathbb{G}_m$*. It thus suffices to show that*

(1)
$$\begin{array}{ccc} B\,\mathrm{pro}\,G^\wedge & \longrightarrow & B\,\mathrm{pro}(\mathbb{Z}^\wedge(1) \rtimes G)^\wedge \\ \downarrow & & \downarrow \\ B\,\mathrm{pro}(\mathbb{Z}^\wedge(1) \rtimes G)^\wedge & \longrightarrow & \{B\langle x, y|x^n = 1 = y^n\rangle \rtimes H\}_{(n,H)\in I} \end{array}$$

is a homotopy push-out square. Since $B\,\mathrm{pro}\,G^\wedge \to B\,\mathrm{pro}(\mathbb{Z}^\wedge(1) \rtimes G)^\wedge$ *is a level-wise section of a level-wise fibration of simplicial sets, it is isomorphic to a level-wise monomorphism and is therefore a cofibration. Thus is suffices to show that* (1) *is a push-out.*

To see this, let D *and* F *be finite groups, with actions of a finite group* C*. By Van-Kampen's theorem,*

(2)
$$\begin{array}{ccc} * & \longrightarrow & BF \\ \downarrow & & \downarrow \\ BD & \longrightarrow & B(D * F) \end{array}$$

is a push-out and a homotopy push-out, where $D * F$ *denotes the free product of* D *and* F*. Let* EC *denote a universal cover of* BC*. Applying* $(-) \times_G EC$ *to* (2) *produces another push-out and homotopy push-out. It follows that* (1) *is a push-out, as claimed.*

Thus $\pi_1(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}, *) \to G$ *can be identified with the map*

$$\{\langle x, y|x^n = 1 = y^n\rangle \rtimes H\}_{(n,H)\in I} \to \mathrm{pro}-G^\wedge$$

under the commutative diagram (1)*.*

Define π' *to be the free profinite group on two generators* $\pi' = \langle x, y\rangle^\wedge$*, and let* G *act on* π' *by*

(3)
$$gx = x^{\chi(g)} \qquad gy = y^{\chi(g)}.$$

Lemma 2.3. *Any morphism in* $\mathrm{pro}-\mathbf{Grp}^{\mathrm{out}}_{\mathrm{pro}-G^\wedge}$ *from* $\{\langle x, y|x^n = 1 = y^n\rangle \rtimes H\}_{(n,H)\in I}$ *to an inverse system of finite groups factors through* $\mathrm{pro}-(\pi' \rtimes G)^\wedge$*.*

Proof. Let $\{J_\alpha\}_{\alpha\in A}$ be a pro-group with each J_α finite, and suppose $\{J_\alpha\}_{\alpha\in A}$ is equipped with a map $\{J_\alpha\}_{\alpha\in A} \to \mathrm{pro}-G^\wedge$. Any morphism in $\mathrm{pro}-\mathbf{Grp}^{\mathrm{out}}_{\mathrm{pro}-G^\wedge}$ from $\{\langle x, y|x^n = 1 = y^n\rangle \rtimes$

$H\}_{(n,H)\in I}$ to $\{J_\alpha\}_{\alpha\in A}$ is represented by a morphism of pro-groups

$$\{\langle x, y | x^n = 1 = y^n\rangle \rtimes H\}_{(n,H)\in I} \to \{J_\alpha\}_{\alpha\in A}.$$

Such a morphism is an element of

$$\varprojlim_{\alpha} \varinjlim_{(n,H)} \mathrm{Hom}(\langle x, y | x^n = 1 = y^n\rangle \rtimes H, J_\alpha). \tag{4}$$

Since J_α is finite, this set is in natural bijection with

$$\varprojlim_{\alpha} \varinjlim_{(n,H)} \mathrm{Hom}((\langle x, y | x^n = 1 = y^n\rangle \rtimes H)^\wedge, J_\alpha).$$

Since there is a map $\langle x, y\rangle \to \langle x, y | x^n = 1 = y^n\rangle \rtimes H$ sending x to $x \rtimes 1$ and y to $y \rtimes 1$, there is an induced map $\pi' \to (\langle x, y | x^n = 1 = y^n\rangle \rtimes H)^\wedge$. Since this map is equivariant with respect to the quotient $G \to H$, there is an induced map $\pi' \rtimes G \to (\langle x, y | x^n = 1 = y^n\rangle \rtimes H)^\wedge$. By checking compatibility with the transition maps, it follows that the set of morphisms (4) is in natural bijection with

$$\varprojlim_{\alpha} \mathrm{Hom}(\pi' \rtimes G, J_\alpha).$$

□

Let $H^i(-, \mathbb{Z}/n) : \mathbf{pro-sSet} \to \mathbf{Ab}$ denote the functor which takes a pro-simplicial set $\{X_\alpha\}_{\alpha\in I}$ to the abelian group $\mathrm{colim}_{\alpha\in I} H^i(X_\alpha, \mathbb{Z}/n)$, cf. [Fri82, §5]. By [Isa01, Proposition 18.4], $H^i(-\mathbb{Z}/n)$ passes to the homotopy category and determines a functor

$$H^i(-, \mathbb{Z}/n) : \mathrm{ho\,pro}-\mathbf{sSet} \to \mathbf{Ab}.$$

Let $H^i_{\acute{e}t}(-, \mathbb{Z}/n)$ denote the usual étale cohomology groups of a scheme with coefficients in $\mathbb{Z}/n$.

Proposition 2.4. $H^i_{\acute{e}t}(-, \mathbb{Z}/n) : \mathbf{Sm}_k \to \mathbf{Ab}$ *factors through* $\mathrm{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)$.

Proof. By [Isa04, Corollary 4], $\mathrm{LEt}\, X$ is the étale topological type of [Fri82]. Thus by [Fri82, Proposition 5.9], $H^i(\mathrm{LEt}\, X, \mathbb{Z}/n)$ is naturally isomorphic to the étale cohomology $H^i_{\acute{e}t}(X, \mathbb{Z}/n)$. Thus is suffices to show that

$$H^i(\mathrm{LEt}(-), \mathbb{Z}/n) : \mathrm{ho}\, \mathbf{sPre}(\mathbf{Sm}_k) \to \mathbf{Ab}$$

factors through $\mathrm{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)$. Since the $\mathbb{A}^1$-model structure is obtained by left Bousfield localization at the maps $X \times \mathbb{A}^1 \to X$ for every scheme X, it suffices to show that LEt takes $X \times \mathbb{A}^1 \to X$ to an isomorphism of abelian groups. This is true by [Mil80, VI Corollary 4.20]. □

Let $H^i(-, \mathbb{Z}/n)$ also denote the functor $H^i(-, \mathbb{Z}/n) : \mathbf{sPre}(\mathbf{Sm}_k) \to \mathbf{Ab}$ given by $H^i(\mathrm{LEt}(-), \mathbb{Z}/n)$. As in Proposition 2.4, $H^i(-, \mathbb{Z}/n)$ factors through $\mathrm{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)$.

Proposition 2.5. *There is a natural isomorphism of functors* $H^i(-, \mathbb{Z}/n) \cong H^{i+1}(\Sigma(-), \mathbb{Z}/n)$.

Proof. Let X be an object of $\mathbf{sPre}(\mathbf{Sm}_k)_*$. Since left derived functors commute with homotopy colimits,

$$\begin{array}{ccc} \mathrm{LEt}\,X & \longrightarrow & \mathrm{LEt}\,* \\ \downarrow & & \downarrow \\ \mathrm{LEt}\,* & \longrightarrow & \mathrm{LEt}\,\Sigma X \end{array}$$

is a push-out square in the model structure of [Isa01].

In the model structure of [Isa01], the cofibrations are isomorphic to levelwise cofibrations of systems of simplicial sets of the same shape. Also, levelwise homotopy equivalences are weak equivalences. It follows that

$$\begin{array}{ccc} \{X_\alpha\}_{\alpha\in A} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \{\Sigma X_\alpha\}_{\alpha\in A} \end{array}$$

is a homotopy push-out. In particular, letting $\{X_\alpha\}_{\alpha\in A} = \mathrm{LEt}\,X$, we have that $\mathrm{LEt}\,\Sigma X \cong \{\Sigma X_\alpha\}_{\alpha\in A}$.

The proposition then follows from the fact that in ordinary cohomology of simplicial sets, we have $H^{i+1}(\Sigma A_\alpha, \mathbb{Z}/n) \cong H^i(A_\alpha, \mathbb{Z}/n)$. ☐

Proposition 2.6. *There is a natural isomorphism of functors*

$$H^1(-, \mathbb{Z}/n) \cong \mathrm{Hom}(\pi_1(-), \mathbb{Z}/n) : \mathbf{sPre}(\mathbf{Sm}_k)_c^+ \to \mathbf{Ab}.$$

Proof. The claim is equivalent to exhibiting a natural isomorphism

$$H^1(\mathrm{LEt}(-), \mathbb{Z}/n) \cong \mathrm{Hom}(\pi_1\,\mathrm{LEt}(-), \mathbb{Z}/n).$$

There is a natural isomorphism

$$H^1(-, \mathbb{Z}/n) \cong \mathrm{Hom}(\pi_1(-), \mathbb{Z}/n) : \mathrm{ho}\,\mathbf{sSet} \to \mathbf{Ab}.$$

This induces a natural isomorphism

$$H^1(-, \mathbb{Z}/n) \cong \mathrm{Hom}(\pi_1(-), \mathbb{Z}/n) : \mathrm{ho}\,\mathbf{pro-sSet} \to \mathbf{Ab},$$

where Hom is the homomorphisms in the category of pro-groups. The desired natural isomorphism is obtained by pulling back by LEt. ☐

3. Stable isomorphism $\mathbb{P}^1_k - \{0, 1, \infty\} \cong \mathbb{G}_m \vee \mathbb{G}_m$

Recall that the smash product $X \wedge Y$ of two pointed spaces X and Y is $X \wedge Y = X \times Y/(* \times Y \cup X \times *)$, and that the wedge product $X \vee Y$ is the disjoint union with the two base points identified. These formulas hold sectionwise for simplicial presheaves, e.g. $(X \vee Y)(U) = X(U) \vee Y(U)$. The simplicial suspension ΣX of X in $\mathbf{sPre}(\mathbf{Sm}_k)$ is $\Sigma X = S^1 \wedge X$. Let S denote the unreduced simplicial suspension, $SX = \Delta^1 \times X/\sim$, where Δ^1 denotes the standard 1-simplex, and $\sim$ denotes the equivalence relation defined $0 \times X \sim *_0$ and $1 \times X \sim *_1$, where $*_0$ and $*_1$ are two copies of the terminal object. There is a natural transformation $q : S \to \Sigma$ which for all X in $\mathbf{sPre}(\mathbf{Sm}_k)$ induces a weak equivalence $SX \to \Sigma X$ because it is a sectionwise weak equivalence.

Proposition 3.1. *There is a canonical isomorphism* $\Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \to S(\mathbb{P}^1_k - \{0, 1, \infty\})$ *in* $\mathrm{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)$ *which sends* $*_0$ *to the base point.*

Proof. Let $i : Z \to \mathbb{A}^1_k$ be the reduced closed subscheme corresponding to the closed set $\{0, 1\}$. Note that $\mathbb{A}^1_k - i(Z) \cong \mathbb{P}^1_k - \{0, 1, \infty\}$ is an isomorphism of schemes. Let $\mathcal{N}(i) \to Z$ denote the normal bundle to i, and let $\mathrm{Th}(\mathcal{N}(i))$ denote the Thom space of $\mathcal{N}(i)$, as in [MV99, Definition 2.16]. By [MV99, Theorem 2.23], there is a canonical isomorphism

$$\mathrm{Th}(\mathcal{N}(i)) \cong \mathbb{A}^1_k/(\mathbb{A}^1_k - i(Z)) \tag{5}$$

in $\mathrm{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)$. Since $\mathbb{A}^1_k - i(Z) \to \mathbb{A}^1$ is an open immersion, it is a monomorphism and therefore a cofibration. It follows that $\mathbb{A}^1_k/(\mathbb{A}^1_k - i(Z))$ is equivalent to the homotopy cofiber of $\mathbb{A}^1_k - i(Z) \to \mathbb{A}^1_k$. Since $\mathbb{A}^1_k \to *$ is a weak equivalence, this homotopy cofiber is equivalent to the homotopy cofiber of $\mathbb{A}^1_k - i(Z) \to *$. This later homotopy cofiber is equivalent to the unreduced suspension $S(\mathbb{A}^1_k - i(Z))$.

Let $\mathcal{O}$ denote the structure sheaf of Z, and let $\mathbb{P}\mathcal{N}(i) \to \mathbb{P}(\mathcal{N}(i) \oplus \mathcal{O})$ denote the closed embedding at infinity. The vector bundle $\mathcal{N}(i)$ is trivial of rank 1 over Z. Fix a coordinate z with $\mathbb{A}^1 = \mathrm{Spec}\, k[z]$. For any $p \in k$, the map $k[z]/\langle z - p\rangle \to \langle z - p\rangle/\langle z - p\rangle^2$ sending $f(z)$ to $f(p)(z - p)$ gives a canonical trivialization of the normal bundle of the closed immersion $\mathrm{Spec}\, k[z]/\langle z - p\rangle \to \mathbb{A}^1$. This gives a trivialization of $\mathcal{N}(i)$. We obtain a canonical isomorphism $\mathbb{P}(\mathcal{N}(i) \oplus \mathcal{O})/\mathbb{P}\mathcal{N}(i) \cong \mathbb{P}^1 \vee \mathbb{P}^1$. Use the coordinate z on $\mathbb{A}^1$ and $\mathbb{G}_m = \mathrm{Spec}\, k[z, \frac{1}{z}]$. The reasoning above gives an equivalence $S\mathbb{G}_{m,k} \cong \mathbb{A}^1/\mathbb{G}_{m,k} \to \mathbb{P}^1_k$. Because $q : S\mathbb{G}_{m,k} \to \Sigma\mathbb{G}_{m,k}$ is an isomorphism in the homotopy category, this yields a canonical isomorphism $\Sigma(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}) \cong \mathbb{P}(\mathcal{N}(i) \oplus \mathcal{O})/\mathbb{P}\mathcal{N}(i)$ in $\mathrm{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)$. By [MV99, Proposition 2.17. 3.], there is a canonical equivalence $\mathbb{P}(\mathcal{N}(i) \oplus \mathcal{O})/\mathbb{P}\mathcal{N}(i) \to \mathrm{Th}(\mathcal{N}(i))$. Combining with (5) produces the desired canonical isomorphism. □

Corollary 3.2. *For any choice of base point of* $\mathbb{P}^1_k - \{0, 1, \infty\}$*, there is a canonical isomorphism* $\Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \to \Sigma(\mathbb{P}^1_k - \{0, 1, \infty\})$ *in* $\mathrm{ho}_{\mathbb{A}^1}\, \mathbf{sPre}(\mathbf{Sm}_k)$.

Proof. This corollary follows from Proposition 3.1, and the canonical weak equivalence $q : S(\mathbb{P}^1_k - \{0, 1, \infty\}) \to \Sigma(\mathbb{P}^1_k - \{0, 1, \infty\})$.

□

Let $\mu : \Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \to \Sigma(\mathbb{P}^1_k - \{0, 1, \infty\})$ denote the canonical isomorphism of Corollary 3.2 with some choice of base point. Let $c_i : \mathbb{G}_m \vee \mathbb{G}_m \to \mathbb{G}_m$ for $i = 1$ (respectively $i = 2$) be the map which crushes the first (respectively second) summand of $\mathbb{G}_m$. Let $a_1 : \mathbb{P}^1_k - \{0, 1, \infty\} \to \mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$ denote the open immersion. Let $a_2 : \mathbb{P}^1_k - \{0, 1, \infty\} \cong \mathrm{Spec}\, k[z, \frac{1}{z}, \frac{1}{z-1}] \to \mathbb{G}_m \cong \mathrm{Spec}\, k[z, \frac{1}{z}]$ be given by $a_2^*(z) = z - 1$. Consider these maps as unpointed.

Lemma 3.3. *Let* $i = 1$ *or* 2. *The following is a commutative diagram in the unpointed homotopy category* $\mathrm{ho}_{\mathbb{A}^1}\,\mathbf{sPre}(\mathbf{Sm}_k)$

$$\begin{array}{ccccc} \Sigma(\mathbb{G}_m \vee \mathbb{G}_m) & \xrightarrow{\mu} & \Sigma(\mathbb{P}^1_k - \{0,1,\infty\}) & \xrightarrow[\cong]{q^{-1}} & S(\mathbb{P}^1_k - \{0,1,\infty\}) \\ & \searrow^{\Sigma c_i} & & & \downarrow{\scriptstyle S a_i} \\ & & \Sigma\mathbb{G}_m & \xrightarrow[\cong]{q^{-1}} & S\mathbb{G}_m. \end{array}$$

Proof. We keep the notation of the proof of Proposition 3.1. Let $i_0 : \{0\} \to Z$ and $i_1 : \{1\} \to Z$ be the closed (and open) immersions, and let $j_k = i \circ i_\ell$ for $\ell = 0, 1$. Let $\mathcal{N}(j_\ell)$ denote the normal bundle to j_ℓ. The decomposition of Z as the disjoint union $Z = \{0\} \coprod \{1\}$ gives a decomposition $\mathcal{N}(i) = \mathcal{N}(j_0) \coprod \mathcal{N}(j_1)$. The maps of pairs $(\mathcal{N}(j_\ell), \mathcal{N}(j_\ell) - 0) \to (\mathcal{N}, \mathcal{N} - 0)$ for $\ell = 0, 1$ determine maps $\mathrm{Th}(\mathcal{N}(j_\ell)) \to \mathrm{Th}(\mathcal{N}(i))$ which combine to give an isomorphism

$$\mathrm{Th}(\mathcal{N}(j_0)) \vee \mathrm{Th}(\mathcal{N}(j_1)) \to \mathrm{Th}(\mathcal{N}(i)).$$

Mapping $\mathrm{Th}(\mathcal{N}(j_0))$ to the basepoint thus determines a map $\mathrm{Th}(\mathcal{N}(i)) \to \mathrm{Th}(\mathcal{N}(j_1))$. And we have the analogous map $\mathrm{Th}(\mathcal{N}(i)) \to \mathrm{Th}(\mathcal{N}(j_0))$.

The diagram

$$\begin{array}{ccc} \mathbb{A}^1_k/(\mathbb{A}^1_k - i(Z)) & \longleftarrow & \mathrm{Th}(\mathcal{N}(i)) \\ \downarrow & & \downarrow \\ \mathbb{A}^1_k/(\mathbb{A}^1_k - j_0(\{0\}) & \longleftarrow & \mathrm{Th}(\mathcal{N}(j_\ell)) \end{array}$$

in $\mathrm{ho}_{\mathbb{A}^1}\,\mathbf{sPre}(\mathbf{Sm}_k)$ is commutative by the functoriality of blow-ups and the construction of the canonical isomorphism of [MV99, Theorem 2.23].

Use the trivialization of $\mathcal{N}(j_\ell)$ from the proof of Proposition 3.1. We obtain an isomorphism $\mathrm{Th}(\mathcal{N}(j_\ell)) \to \mathbb{P}^1$. This isomorphism fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{Th}(\mathcal{N}(i)) & \longleftarrow & \mathbb{P}^1 \vee \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \mathrm{Th}(\mathcal{N}(j_\ell)) & \longleftarrow & \mathbb{P}^1 \end{array}$$

where the top horizontal map is as in the proof of Proposition 3.1, and the right vertical morphism crushes the factor not corresponding to ℓ.

Place the two previous commutative diagrams side by side and use the isomorphism

$$\Sigma\mathbb{G}_m \xleftarrow{q} S\mathbb{G}_{m,k} \cong \mathbb{A}^1/\mathbb{G}_{m,k} \to \mathbb{P}^1_k,$$

as in the proof of Proposition 3.1, to replace the $\mathbb{P}^1$'s with $\Sigma\mathbb{G}_m$'s. Then note that the composition

$$\mathbb{A}^1_k/(\mathbb{A}^1_k - i(Z)) \to \mathbb{A}^1_k/(\mathbb{A}^1_k - j_\ell(\{\ell\})) \to \mathbb{P}^1 \to \Sigma\mathbb{G}_m$$

is the composition of $S a_\ell$ with $S\mathbb{G}_m \to \Sigma\mathbb{G}_m$ after identifying $\mathbb{A}^1_k/(\mathbb{A}^1_k - i(Z)) \cong \Sigma(\mathbb{P}^1_k - \{0,1,\infty\})$. This proves the proposition. □

Let $\overline{01}$ denote the tangential base point of $\mathbb{P}^1_k - \{0, 1, \infty\}$ at 0 pointing in the direction of 1, as in [Del89, §15] [Nak99], so $\overline{01}$ determines the fiber functor associated to the geometric point

$$\mathbb{P}^1_k - \{0,1,\infty\} = \operatorname{Spec} k[z, \frac{1}{z}, \frac{1}{1-z}] \leftarrow \operatorname{Spec} \cup_{n \in \mathbb{Z}_{>0}} \overline{k}((z^{1/n}))$$

$$k[z, \frac{1}{z}, \frac{1}{1-z}] \to k(z) \to \cup_{n \in \mathbb{Z}_{>0}} \overline{k}((z^{1/n})).$$

Let $\pi = \pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{k}} - \{0,1,\infty\}, \overline{01})$. Since the étale fundamental group is invariant under algebraically closed base change in characteristic 0, we have a canonical isomorphism $\pi \cong \pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}, \overline{01})$. There is a canonical isomorphism between $\pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}, \overline{01})$ and the profinite completion of the topological fundamental group. Let x be the element of the topological fundamental group represented by a small counter-clockwise loop around 0 based at $\overline{01}$, and let y be the path formed by traveling along $[0,1]$, then traveling along the image of x under $z \mapsto 1-z$, and then traveling back from 1 to 0 along $[0,1]$. Putting this together, we have fixed an isomorphism

$$\pi \cong \langle x, y \rangle^{\wedge}$$

between π and the profinite completion of the free group on two generators x and y. Recall that in Example 2.2, we have defined $\pi' = \langle x, y\rangle^{\wedge}$ and maps out of $\pi_1(\mathbb{G}_{m,\overline{k}} \vee \mathbb{G}_{m,\overline{k}}, *)$ to inverse systems of finite groups factor through π' by Lemma 2.3.

Let $x_n^*, y_n^* \in \operatorname{Hom}(\pi, \mathbb{Z}/n)$ be defined by $x_n^*(x) = 1$, $x_n^*(y) = 0$, $y_n^*(x) = 0$, and $y_n^*(y) = 1$. By Proposition 2.6, $\mathrm{H}^1(\mathbb{P}^1_{\overline{k}} - \{0,1,\infty\}, \mathbb{Z}/n)$ is a free $\mathbb{Z}/n$-module with basis $\{x_n^*, y_n^*\}$. Making the analogous definitions of x_n^* and y_n^* with π' replacing π, Proposition 2.6 and Lemma 2.3 show that $\mathrm{H}^1(\mathbb{G}_{m,\overline{k}} \vee \mathbb{G}_{m,\overline{k}}, \mathbb{Z}/n)$ is a free $\mathbb{Z}/n$-module with basis $\{x_n^*, y_n^*\}$. By Proposition 2.5, we obtain isomorphisms $\mathrm{H}^2(\Sigma X_{\overline{k}}, \mathbb{Z}/n) \cong \mathbb{Z}/n x_n^* \oplus \mathbb{Z}/n y_n^*$ for $X = \mathbb{P}^1_k - \{0,1,\infty\}$, and $\mathbb{G}_m \vee \mathbb{G}_m$.

Let $\wp : \Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \to \Sigma(\mathbb{P}^1_k - \{0,1,\infty\})$ be any map determining the canonical isomorphism of Corollary 3.2.

Proposition 3.4. $\mathrm{H}^2(\wp_{\overline{k}}, \mathbb{Z}/n)$ *is computed by* $\mathrm{H}^2(\wp_{\overline{k}}, \mathbb{Z}/n)(x_n^*) = x_n^*$ *and* $\mathrm{H}^2(\wp_{\overline{k}}, \mathbb{Z}/n)(y_n^*) = y_n^*$.

Proof. By an abuse of notation, let $\wp$ also denote the composite morphism $\Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \to \Sigma(\mathbb{P}^1_k - \{0,1,\infty\}) \to S(\mathbb{P}^1_k - \{0,1,\infty\})$ in $\mathrm{ho}_{\mathbb{A}^1} \mathbf{sPre}(\mathbf{Sm}_k)$, and identify $\mathrm{H}^2(S(\mathbb{P}^1_{\overline{k}} - \{0,1,\infty\}), \mathbb{Z}/n)$ with $\mathrm{H}^2(\Sigma(\mathbb{P}^1_{\overline{k}} - \{0,1,\infty\}), \mathbb{Z}/n)$ and $\mathrm{H}^2(S\mathbb{G}_m, \mathbb{Z}/n)$ with $\mathrm{H}^2(\Sigma\mathbb{G}_m, \mathbb{Z}/n)$ by q, as in Lemma 3.3. Let Σa_i denote the composition of $S a_i$ with the canonical map $S\mathbb{G}_m \to \Sigma\mathbb{G}_m$.

Then Lemma 3.3 says that $\Sigma a_i \circ \wp = \Sigma c_i$. The dual to the counterclockwise loop based at 1 in $\mathbb{G}_m(\mathbb{C})$ determines a canonical element z_n^* of $\mathrm{H}^2(\Sigma\mathbb{G}_{m,\overline{k}}, \mathbb{Z}/n)$ by the comparison between the étale and topological fundamental groups [SGAI, XII Corollaire 5.2], Proposition 2.5, and Proposition 2.6. By the construction of x_n^* and y_n^*, we have that $(\Sigma a_1)^*(z_n^*) = x_n^*$, $(\Sigma a_2)^*(z_n^*) = y_n^*$, $(\Sigma c_1)^*(z_n^*) = x_n^*$ and $(\Sigma c_2)^*(z_n^*) = y_n^*$. This shows the proposition. ☐

4. Desuspending $\Sigma(\mathbb{P}^1_k - \{0, 1, \infty\})$

We use the Galois action on $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{k}} - \{0, 1, \infty\})$ to show that $\mathbb{P}^1_k - \{0, 1, \infty\}$ and $\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}$ are distinct desuspensions of $\Sigma(\mathbb{P}^1_k - \{0, 1, \infty\})$. Recall the definition of π' from Example 2.2. Here are the needed facts about the Galois action on $\pi = \pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{k}} - \{0, 1, \infty\}, \overline{01})$.

An element $g \in G$ acts on π by

$$g(x) = x^{\chi(g)} \qquad g(y) = \mathfrak{f}(g)^{-1} y^{\chi(g)} \mathfrak{f}(g) \tag{6}$$

where $\mathfrak{f} : G \to [\pi]_2$ is a cocycle with values in the commutator subgroup $[\pi]_2$ of π. See [Iha94, Proposition 1.6]. Since $\overline{01}$ is a rational tangential base-point, $\overline{01}$ splits the homomorphism $\pi_1^{\text{ét}}(\mathbb{P}^1_k - \{0, 1, \infty\}, \overline{01}) \to \pi_1^{\text{ét}} \operatorname{Spec} k \cong G$, giving an isomorphism $\pi_1^{\text{ét}}(\mathbb{P}^1_k - \{0, 1, \infty\}, \overline{01}) \cong \pi \rtimes G$.

Let $\pi = [\pi]_1 \supseteq [\pi]_2 \supseteq [\pi]_3 \supseteq \ldots$ denote the lower central series of π, so $[\pi]_n$ is the closure of the subgroup generated by commutators of elements of π with elements of $[\pi]_{n-1}$. Use the analogous notation for the lower central series of any profinite group.

Let $\iota : \pi' \to \pi$ be the homomorphism of groups $\iota(x) = x$ and $\iota(y) = y$, and let $\iota^{ab} : (\pi')^{ab} \to \pi^{ab}$ denote the induced map on abelianizations. Note that ι^{ab} is G-equivariant.

Lemma 4.1. *Let k be a number field not containing the square root of 2. Then there is no continuous homomorphism $\pi' \times G \to \pi \rtimes G$ over G inducing $\iota^{ab} \rtimes 1_G$ after abelianization.*

Proof. Suppose to the contrary that θ is such a map. Because the subgroups of the lower central series are characteristic, θ induces a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & [\pi]_n/[\pi]_{n+1} & \to & \pi/[\pi]_{n+1} \rtimes G & \to & \pi/[\pi]_n \rtimes G & \to & 1 \\ & & \uparrow {\scriptstyle \bar{\theta}_n^{n+1}} & & \uparrow {\scriptstyle \bar{\theta}_{n+1}} & & \uparrow {\scriptstyle \bar{\theta}_n} & & \\ 1 & \to & [\pi']_n/[\pi']_{n+1} & \to & \pi'/[\pi']_{n+1} \rtimes G & \to & \pi'/[\pi']_n \rtimes G & \to & 1 \end{array} \tag{7}$$

Thus if $\bar{\theta}_n^{n+1}$ and $\bar{\theta}_n$ are isomorphisms, so is $\bar{\theta}_{n+1}$. Since π' and π are isomorphic to the profinite completion of the free group on two generators, $[\pi']_n/[\pi']_{n+1}$ and $[\pi]_n/[\pi]_{n+1}$ are isomorphic to the degree n graded component of the free Lie algebra on the same generators over $\mathbb{Z}^\wedge$. Since $\bar{\theta}_2 = \iota^{ab} \rtimes 1_G$ is an isomorphism, it follows that $\bar{\theta}_n^{n+1}$ is an isomorphism. By induction, it follows that $\bar{\theta}_n$ is an isomorphism for all n.

The extension

$$1 \to [\pi]_n/[\pi]_{n+1} \to \pi/[\pi]_{n+1} \rtimes G \to \pi/[\pi]_n \rtimes G \to 1$$

is classified by the element of $H^2(\pi/[\pi]_n \rtimes G, [\pi]_n/[\pi]_{n+1})$ represented by the inhomogeneous cocycle φ_n

$$\varphi_n(\gamma \rtimes g, \eta \rtimes h) = s(\gamma) g s(\eta) s(\gamma g \eta)^{-1}$$

where $s : \pi/[\pi]_n \to \pi/[\pi]_{n+1}$ is a continuous set-theoretic section of the quotient map $\pi/[\pi]_{n+1} \to \pi/[\pi]_n$. See for example [Bro94, IV 3]. Let φ_n' denote the analogous inhomogeneous cocycle obtained by replacing π with π'.

This association of a class in $H^2(\pi/[\pi]_n \rtimes G, [\pi]_n/[\pi]_{n+1})$ to an extension of $\pi/[\pi]_n \times G$ by $[\pi]_n/[\pi]_{n+1}$ induces a bijection between $H^2(\pi/[\pi]_n \rtimes G, [\pi]_n/[\pi]_{n+1})$ and isomorphism classes of extensions [Bro94, IV Theorem 3.12]. Since $\overline{\theta}_n$ is an isomorphism, it follows that φ'_n and $(\overline{\theta}_n)^*\varphi_n$ represent the same class in $H^2(\pi/[\pi]_n \rtimes G, [\pi]_n/[\pi]_{n+1})$.

By (6) and (3),

$$\pi/[\pi]_2 \cong \mathbb{Z}^\wedge(1)x \oplus \mathbb{Z}^\wedge(1)y$$

$$[\pi]_2/[\pi]_3 \cong \mathbb{Z}^\wedge(2)[x,y]$$

$$[\pi]_3/[\pi]_4 \cong \mathbb{Z}^\wedge(3)[[x,y],x] \oplus \mathbb{Z}^\wedge(3)[[x,y],y], \tag{8}$$

and the same isomorphisms hold with π' replacing π.

We claim that $\overline{\theta}_3$ is given by

$$\overline{\theta}_3(x \rtimes 1) = x \rtimes 1 \qquad \overline{\theta}_3(y \rtimes 1) = y \rtimes 1 \qquad \overline{\theta}_3(1 \rtimes g) = [x,y]^{c(g)} \rtimes g \tag{9}$$

for all $g \in G$, where

$$c : G \to \mathbb{Z}^\wedge(2)$$

is a coycle. To see this, note that the hypothesis on $\overline{\theta}_2$ implies that $\overline{\theta}_3(x \rtimes 1) = x[x,y]^{c_1(g)} \rtimes 1$ with $c_1(g)$ in $\mathbb{Z}^\wedge$. Similarly, $\overline{\theta}_3(1 \rtimes g) = [x,y]^{c(g)} \rtimes g$ with $c(g)$ in $\mathbb{Z}^\wedge$. Since θ is a homomorphism, we have $\overline{\theta}_3(gx) = \overline{\theta}_3(g)\overline{\theta}_3(x)$. Since $gx = x^{\chi(g)} \rtimes g$ and $[x,y]$ is in the center, we have

$$\overline{\theta}_3(gx) = \overline{\theta}_3(x)^{\chi(g)}\overline{\theta}_3(g) = x^{\chi(g)}[x,y]^{c_1(g)\chi(g)+c(g)} \rtimes g.$$

On the other hand,

$$\overline{\theta}_3(g)\overline{\theta}_3(x) = ([x,y]^{c(g)} \rtimes g)(x[x,y]^{c_1(g)} \rtimes 1) = x^{\chi(g)}[x,y]^{c_1(g)\chi(g)^2+c(g)} \rtimes g.$$

Thus $c_1(g)\chi(g)^2 + c(g) = c_1(g)\chi(g) + c(g)$ for all g in G. It follows that $c_1(g) = 0$. Since $\mathfrak{f}(g)$ is in $[\pi]_2$ and elements of $[\pi]_2$ are all in the center of $\pi/[\pi]_3$, switching x and y induces an isomorphism on $\pi/[\pi]_3$. The same argument therefore implies that $\overline{\theta}_3(y) = y$. Since $\overline{\theta}_3$ is a homomorphism when restricted to $1 \rtimes G$, it follows that c is a cocycle, showing (9).

By (8), we have a direct sum decomposition

$$H^2(\pi/[\pi]_3 \rtimes G, [\pi]_3/[\pi]_4) \cong H^2(\pi/[\pi]_3 \rtimes G, \mathbb{Z}^\wedge(3))[[x,y],x] \oplus H^2(\pi/[\pi]_3 \rtimes G, \mathbb{Z}^\wedge(3))[[x,y],y].$$

Define $\varphi_{3,[[x,y],x]}$ and $\varphi_{3,[[x,y],y]}$ so that under this isomorphism φ_3 decomposes as $\varphi_3 = \varphi_{3,[[x,y],x]} \oplus \varphi_{3,[[x,y],y]}$. It was calculated in [Wic12] that $\varphi_{3,[[x,y],x]}$ is represented by the cocycle maping $(y^{a_1}x^{b_1}[x,y]^{c_1} \rtimes g_1, y^{a_2}x^{b_2}[x,y]^{c_2} \rtimes g_2)$ to

$$c_1\chi(g_1)b_2 + \binom{b_1+1}{2}\chi(g_1)a_2 + b_1\chi(g_1)^2a_2b_2 - \frac{\chi(g_1)-1}{2}\chi(g_1)^2c_2$$

and that $\varphi_{3,[[x,y],y]}$ is represented by the cocycle maping $(y^{a_1}x^{b_1}[x,y]^{c_1} \rtimes g_1, y^{a_2}x^{b_2}[x,y]^{c_2} \rtimes g_2)$ to

$$c_1\chi(g_1)a_2 + b_1\binom{\chi(g_1)a_2+1}{2} - \chi(g_1)\binom{\chi(g_1)}{2}c_2 - f(g_1)\chi(g_1)a_2$$

where $f : G \to \mathbb{Z}^\wedge(2)$ is such that $\mathfrak{f}(g) = [x,y]^{f(g)}$ in $\pi/[\pi]_3$.

We may similarly decompose φ_3' as $\varphi_3' = \varphi'_{3,[[x,y],x]} \oplus \varphi'_{3,[[x,y],y]}$. By the above calculation of $\overline{\theta}_3$, and the expressions (6) and (3) for the G-action on π and π', we have that $\varphi'_{3,[[x,y],x]}$ and $\varphi'_{3,[[x,y],y]}$ are obtained from the expressions for $\varphi_{3,[[x,y],x]}$ and $\varphi_{3,[[x,y],y]}$ by setting $f = 0$.

It follows that $\varphi_3' - (\overline{\theta}_3)^*\varphi_3$ is represented by the direct sum of two cocycles, given by sending $(y^{a_1}x^{b_1}[x,y]^{c_1} \rtimes g_1, y^{a_2}x^{b_2}[x,y]^{c_2} \rtimes g_2)$ to

$$(-c(g_1)\chi(g_1)b_2 + \frac{\chi(g_1)-1}{2}\chi(g_1)^2 c(g_2))[[x,y],x]$$

and

$$(-c(g_1)\chi(g_1)a_2 + \chi(g_1)\binom{\chi(g_1)}{2}c(g_2) + f(g_1)\chi(g_1)a_2)[[x,y],y]$$

respectively.

Using the above direct sum decomposition of $H^2(\pi/[\pi]_3 \rtimes G, [\pi]_3/[\pi]_4)$, this implies that

$$\varphi'_{3,[[x,y],x]} - (\overline{\theta}_3)^*\varphi_{3,[[x,y],x]} = -c \cup b + \frac{\chi(g)-1}{2} \cup c$$

and

$$\varphi'_{3,[[x,y],y]} - (\overline{\theta}_3)^*\varphi_{3,[[x,y],y]} = -c \cup a + \frac{\chi(g)-1}{2} \cup c + f \cup a,$$

where these equalities are in $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3))$, and where $f : G \to \mathbb{Z}^\wedge(2)$ is considered via pullback as an element of $H^1(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(2))$, $a : \pi'/[\pi']_3 \rtimes G \to \mathbb{Z}^\wedge(1)$ is the cocyle $y^a x^b [x,y]^c \rtimes g \mapsto a$, b is defined similarly, and $\frac{\chi(g)-1}{2}$ is the cocycle $g \mapsto \frac{\chi(g)-1}{2}$ taking values in $\mathbb{Z}^\wedge(1)$ pulled back to $\pi'/[\pi']_3 \rtimes G$.

As shown above, the existence of θ therefore implies that $-c \cup b + \frac{\chi(g)-1}{2} \cup c = 0$ and $-c \cup a + \frac{\chi(g)-1}{2} \cup c + f \cup a = 0$ in $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3))$.

Consider first the equality $-c \cup b + \frac{\chi(g)-1}{2} \cup c = 0$. Since the cup product is graded-commutative, we may rewrite this equality as $(b + \frac{\chi(g)-1}{2}) \cup c = 0$. The quotient map $\mathbb{Z}^\wedge(3) \to \mathbb{Z}/2$ determines map $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3)) \to H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}/2)$. Passing to the image under this map, we have an equality $\overline{(b + \frac{\chi(g)-1}{2})} \cup \overline{c} = 0$, where $\overline{c}$ denotes the image of c and $\overline{(b + \frac{\chi(g)-1}{2})}$ denotes the image $b + \frac{\chi(g)-1}{2}$. Recall that for any $\beta \in k^*$ with a chosen compatible system of nth roots of β, there is a Kummer cocycle $\kappa(\beta) : G \to \hat{\mathbb{Z}}(1) \cong \varprojlim_n \mu_n(\overline{k})$ defined by $g\sqrt[n]{\beta} = \kappa(b)(g)_n \sqrt[n]{\beta}$ where $\kappa(b)(g)_n$ is the element of $\mu_n(\overline{k})$ determined by $\kappa(b)(g)$. We may define a homomorphism $G \to \pi'/[\pi']_3 \rtimes G$ by $g \mapsto y^{\kappa(\beta)} \rtimes g$. Pulling back the equality $\overline{(b + \frac{\chi(g)-1}{2})} \cup \overline{c} = 0$ by this homomorphism gives the equality $(\kappa(b) + \overline{\frac{\chi(g)-1}{2}}) \cup \overline{c} = 0$ in $H^2(G, \mathbb{Z}/2)$ because c and $\frac{\chi(g)-1}{2}$ are pulled back from G. Note that any element of $H^1(G, \mathbb{Z}/2)$ is of the form $(\kappa(b) + \frac{\chi(g)-1}{2})$ for an appropriate choice of β. By the non-degeneracy of the cup product $H^1(G, \mathbb{Z}/2) \otimes H^1(G, \mathbb{Z}/2) \to H^2(G, \mathbb{Z}/2)$, it follows that $\overline{c} \in H^2(G, \mathbb{Z}/2)$ is zero.

Consider now the second equality $-c \cup a + \frac{\chi(g)-1}{2} \cup c + f \cup a = 0$ in $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3))$, and again pass to the image under $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3)) \to H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}/2)$. Since $\overline{c} = 0$ in $H^2(G, \mathbb{Z}/2)$, we have $\overline{f} \cup \overline{a} = 0$ in $H^2(\pi'/[\pi']_3 \rtimes G, \mathbb{Z}^\wedge(3))$. On the other hand, $f : G \to \mathbb{Z}^\wedge(2)$ is known by work of Ihara [Iha91, 6.3 Thm p.115] [IKY87], Anderson [And89], and Coleman [Col89], and we can show that this is inconsistent with $\overline{f} \cup \overline{a} = 0$ in the following manner. Namely, $f(g) = \frac{1}{24}(\chi(g)^2 - 1)$. See [Wic12, 12.5.2]. By [Wic12, Lemma 31], the image of f under the map $H^1(G, \mathbb{Z}^\wedge(2)) \to H^1(G, \mathbb{Z}/2) \cong k^*/(k^*)^2$ is represented by $2 \in k^*$. For any $\alpha \in k^*$ we may choose a compatible system of nth roots of α and define a homomorphism $G \to \pi'/[\pi']_3 \rtimes G$ by $g \mapsto y^{\kappa(\alpha)} \rtimes g$. Pulling back $\overline{f} \cup \overline{a}$ by this homomorphism gives $\overline{f} \cup \kappa(\alpha) \in H^2(G, \mathbb{Z}/2)$. Thus $\kappa(2) \cup \kappa(\alpha) = 0 \in H^2(G, \mathbb{Z}/2)$ for all $\alpha \in k^*$. Since k does not contain the square root of 2, this contradicts the nondegeneracy of the the cup product, giving the desired contradiction. □

Theorem 4.2. *Let k be a finite extension of $\mathbb{Q}$ not containing the square root of 2. There is no morphism $\rho : \mathbb{G}_{m,k} \vee \mathbb{G}_{m,k} \to \mathbb{P}^1_k - \{0, 1, \infty\}$ in $\mathrm{ho}_{\mathbb{A}^1}$ **sPre(Sm$_k$)** such that $\Sigma\rho = \wp$ in $\mathrm{ho}_{\mathbb{A}^1}$ **sPre(Sm$_k$)**.*

Proof. Suppose to the contrary that we have such a morphism ρ. The geometric point $\overline{01}$ of $\mathbb{P}^1_k - \{0, 1, \infty\}$ and the extension of the k-basepoint of $\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}$ to a geometric point allow us to consider $(\mathbb{P}^1_k - \{0, 1, \infty\}, \overline{01})$ and $(\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}, *)$ as objects of $\mathbf{sPre}(\mathbf{Sm}_k)^+$. Since $\mathbb{P}^1_k - \{0, 1, \infty\}$ and $\mathbb{G}_{m,k} \vee \mathbb{G}_{m,k}$ have connected étale homotopy type, ρ is a morphism in $\mathbf{sPre}(\mathbf{Sm}_k)^+_c$. Thus ρ induces an outer continuous homomorphism $\rho_* : \pi' \rtimes G \to \pi \rtimes G$ by Proposition 2.1, Lemma 2.3, and taking the inverse limit. We may choose a continuous homomorphism over G representing ρ_*. By a slight abuse of notation, we call this representative ρ_* as well.

Let $(\rho_{\overline{k}})_*$ denote the induced map $\pi' \to \pi$. It follows from Proposition 2.6 that the induced map $\rho^* : H^1(\mathbb{P}^1_{\overline{k}} - \{0, 1, \infty\}, \mathbb{Z}/n) \to H^1(\mathbb{G}_{m,\overline{k}} \vee \mathbb{G}_{m,\overline{k}}, \mathbb{Z}/n)$ is computed $\rho^* = \mathrm{Hom}((\rho_{\overline{k}})_*, \mathbb{Z}/n)$. By Proposition 2.5, $H^2(\wp_{\overline{k}}) = H^1(\rho_{\overline{k}})$. Combining the two previous, we have $H^2(\wp_{\overline{k}}) = \mathrm{Hom}((\rho_{\overline{k}})_*, \mathbb{Z}/n)$. By Proposition 3.4, it follows that $\mathrm{Hom}(\rho_*, \mathbb{Z}/n)(x_n^*) = x_n^*$ and $\mathrm{Hom}(\rho_*, \mathbb{Z}/n)(y_n^*) = y_n^*$. Since n is arbitrary, it follows that $(\rho_{\overline{k}})_*^{ab} = \iota^{ab}$.

We claim that after modifying ρ_* by an inner automorphism, the map $\rho_*^{ab} : \pi'/[\pi']_2 \rtimes G \to \pi/[\pi]_2 \rtimes G$ induced by ρ_* is $\iota^{ab} \rtimes 1_G$. Note the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi/[\pi]_2 & \longrightarrow & \pi/[\pi]_2 \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\
 & & \uparrow {\scriptstyle (\rho_{\overline{k}})_*^{ab}} & & \uparrow {\scriptstyle \rho_*^{ab}} & & \uparrow {\scriptstyle 1_G} & & \\
1 & \longrightarrow & \pi'/[\pi']_2 & \longrightarrow & \pi'/[\pi']_2 \rtimes G & \longrightarrow & G & \longrightarrow & 1.
\end{array}$$

Since 1_G and $(\rho_{\overline{k}})_*^{ab}$ are isomorphisms, so is ρ_*^{ab}. It follows by induction that $\overline{(\rho_*)}_n : \pi'/[\pi']_n \rtimes G \to \pi/[\pi]_n \rtimes G$ is an isomorphism, cf. (7).

Let $\varphi_2 \in H^2(\pi/[\pi]_2 \rtimes G, [\pi]_2/[\pi]_3)$ be the element classifying

$$1 \to [\pi]_2/[\pi]_3 \to \pi/[\pi]_3 \rtimes G \to \pi/[\pi]_2 \rtimes G \to 1,$$

and define φ_2' by replacing π with π' in the definition of φ_2. Since $\overline{(\rho_*)}_3$ is an isomorphism, $\overline{(\rho_*)}_2^* \varphi_2 = \varphi_2'$. By [Wic12, Proposition 7], $\varphi_2 = b \cup a$, where $b : \pi/[\pi]_2 \rtimes G \to \mathbb{Z}^\wedge(1)$ is the cocycle $y^a x^b \rtimes g \mapsto b$ and $a : \pi/[\pi]_2 \rtimes G \to \mathbb{Z}^\wedge(1)$ is the cocycle $y^a x^b \rtimes g \mapsto a$. Since conjugation by $f(g)$ is trivial in $\pi/[\pi]_3$, it follows that $\varphi_2' = b \cup a$, where a and b are defined by replacing π' with π in the previous. Because $(\rho_{\bar{k}})_*^{ab} = \iota^{ab}$, we have $\overline{(\rho_*)}_2(y^a x^b \rtimes g) = y^{a+\alpha(g)} x^{b+\beta(g)} \rtimes g$ where $\alpha, \beta : G \to \mathbb{Z}^\wedge(1)$ are cocycles. Thus $\overline{(\rho_*)}_2^* \varphi_2 = (b+\beta) \cup (a+\alpha)$. Thus $(b+\beta) \cup (a+\alpha) = b \cup a$. Since the cup product is non-degenerate, it follows that β and α are trivial in cohomology. Thus after modifying ρ by an inner automorphism, we may assume $\rho_*^{ab} = \iota^{ab} \rtimes 1_G$. This contradicts Lemma 4.1. □

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