K-Theory, Land-Tamme, and Levy

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1. K-Theory and Localizing Invariants

Given a stable ∞-category $\mathcal{C}$, one extracts the non-connective and connective K-theory spectra, denoted $\mathbb{K}(\mathcal{C})$ and $K(\mathcal{C})$ respectively. Here, $K_0(\mathcal{C})$ admits a tractable description as

$$K_0(\mathcal{C}) = \Big\{\text{free abelian group on symbols } [X] \text{ for } X \in \mathcal{C}\Big\}/\sim$$

where $[X] = [X'] + [X'']$ if there exists a cofiber sequence $X' \to X \to X''$ in $\mathcal{C}$.

We begin by stating the universal property of K-theory of Blumberg, Gepner, and Tabuada. Recall that an ∞-category $\mathcal{C}$ is *idempotent-complete* if its image under the Yoneda embedding is closed under retracts. Let $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ denote the ∞-category of small stable ∞-categories, and $\mathrm{Cat}_\infty^{\mathrm{perf}}$ the full subcategory of idempotent-complete small stable ∞-categories. The inclusion $\mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Cat}_\infty^{\mathrm{Ex}}$ admits a left adjoint denoted $\mathrm{Idem} : \mathrm{Cat}_\infty^{\mathrm{Ex}} \to \mathrm{Cat}_\infty^{\mathrm{perf}}$. A functor $\mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ is said to be a *Morita equivalence* if $\mathrm{Idem}(\mathcal{C}) \to \mathrm{Idem}(\mathcal{D})$ is an equivalence.

Example 1. Let A be an $\mathbf{E}_1$-ring, and let $\mathrm{perf}(A)$ denote the ∞-category of compact objects of Mod_A. Then $\mathrm{perf}(A) \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, and the *connective K-theory of A* is defined to be

$$K(A) := K(\mathrm{perf}(A))$$

Definition 1.1. A sequence $\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{p} \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is *exact* if $\mathcal{A} \to \mathcal{B}$ is fully faithful, the composite $\mathcal{A} \to \mathcal{C}$ is 0, and $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence. The exact sequence is *split* if both i and p admit left adjoints which compose with i and p to give the respective identities. A sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ is *exact* (*split exact*) if $\mathrm{Idem}(\mathcal{A}) \to \mathrm{Idem}(\mathcal{B}) \to \mathrm{Idem}(\mathcal{C})$ is exact (*split exact*) in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

Definition 1.2. A functor $E : \mathrm{Cat}_\infty^{\mathrm{perf}} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$ is *localizing* if it sends exact sequences to fiber sequences.

functor $E : \mathrm{Cat}_\infty^{\mathrm{Perf}} \to \mathrm{Sp}$ is an *additive invariant* if it sends split exact sequences to fiber sequences.

Example 2. The functors $\mathbb{K}$, THH, and TC are localizing. Every localizing invariant is additive, but the converse is not true. For example, connective K-theory is additive, but not localizing.

Here, we follow Land and Tamme's terminology. Blumberg, Gepner and Tabuada also require that localizing invariants preserve filtered colimits, which would exclude TC.

Theorem 3 (Blumberg-Gepner-Tabuada)**.** [3] *There exist stable presentable ∞-categories $\mathcal{M}_{\mathrm{loc}}$ and $\mathcal{M}_{\mathrm{add}}$, and localizing and additive invariants $\mathcal{U}_{\mathrm{loc}} : \mathrm{Cat}_\infty^{\mathrm{Ex}} \to$*

$\mathcal{M}_{\mathrm{loc}}$ *and* $\mathcal{U}_{\mathrm{add}} : \mathrm{Cat}_\infty^{\mathrm{Ex}} \to \mathcal{M}_{\mathrm{add}}$ *respectively, which are universal in the following sense: given any stable presentable* ∞*-category* $\mathcal{D}$*, post-composition induces equivalences*

$$\mathcal{U}_{\mathrm{loc}}^* : \mathrm{Fun}^L(\mathcal{M}_{\mathrm{loc}}, \mathcal{D}) \to \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Cat}_\infty^{\mathrm{Ex}}, \mathcal{D})$$
$$\mathcal{U}_{\mathrm{add}}^* : \mathrm{Fun}^L(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) \to \mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{Ex}}, \mathcal{D})$$

where $\mathrm{Fun}^L(\mathcal{M}_{\mathrm{loc}}, \mathcal{D})$ *denotes the* ∞*-category of colimit preserving functors, and* $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Cat}_\infty^{\mathrm{Ex}}, \mathcal{D})$ *and* $\mathrm{Fun}_{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{Ex}}, \mathcal{D})$ *denote the* ∞*-categories of localizing and additive invariants, which preserve filtered colimits and invert Motiva equivalences, respectively.*

Theorem 4 (Blumberg-Gepner-Tabuada)**.** [3] *For any* $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$*, there is a natural equivalence of spectra*

$$\mathrm{Map}_{\mathcal{M}_{\mathrm{loc}}}(\mathcal{U}_{\mathrm{loc}}(\mathrm{Sp}^\omega), \mathcal{U}_{\mathrm{loc}}(\mathcal{C})) \simeq \mathbb{K}(\mathcal{C})$$
$$\mathrm{Map}_{\mathcal{M}_{\mathrm{add}}}(\mathcal{U}_{add}(\mathrm{Sp}^\omega), \mathcal{U}_{\mathrm{add}}(\mathcal{C})) \simeq K(\mathcal{C})$$

Moreover, for any additive invariant, which inverts Morita equivalences and preserves filtered colimits, $E : \mathrm{Cat}_\infty^{\mathrm{Ex}} \to \mathrm{Sp}$*, there is a natural equivalence*

$$\mathrm{Map}(K, E) \simeq E(\mathrm{Sp}^\omega)$$

In particular, taking $E = \mathrm{THH}$, we obtain $\pi_0\mathrm{Map}(K, \mathrm{THH}) \simeq \pi_0\mathrm{THH}(\mathrm{Sp}^\omega) \simeq \pi_0(\mathbf{S}) \simeq \mathbf{Z}$. The natural transformation $K \to \mathrm{THH}$ given by the image of 1 refines to the Dennis trace $K \to \mathrm{TC}$.

Theorem 5 (Dundas-Goodwillie-McCarthy)**.** [5] *Let* $B \to A$ *be a morphism of connective* $\mathbf{E}_1$*-ring spectra such that* $\pi_0(B) \to \pi_0(A)$ *is surjective, with kernel a nilpotent ideal. Then the Dennis Trace induces a pullback*

$$\begin{array}{ccc} \mathrm{K}(B) & \longrightarrow & \mathrm{TC}(B) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{K}(A) & \longrightarrow & \mathrm{TC}(A) \end{array}$$

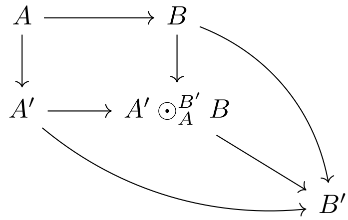
Taking $A = \pi_0 B$, we see that computing the spectrum $K(B)$ can be reduced to the more tractable problems of computing $\mathrm{TC}(A)$, $\mathrm{TC}(B)$, and $K(\pi_0 B)$.

When computing with K-theory, one is naturally led to the question of when a pullback of rings induces a pullback on K-theory spectra. As noted at the beginning of [2], Swann showed that there is no functor K_2 for which Milnor squares (which are pullback squares of rings $A' \times_B' B$ with $B \to B'$ surjective) give rise to the long exact excision sequence. Land and Tamme [2] proved that one can obtain pullback diagrams in K-theory, or more generally any localizing invariant E, by equipping the spectrum $A' \otimes_A B$ with a different ring structure.

Theorem 6 (Land-Tamme)**.** *Any pullback square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

of $\mathbf{E}_1$*-ring spectra refines naturally to a commutative square*



such that any localizing invariant sends the outer square to a pullback. Furthermore, the underlying spectrum of $A' \odot_A^{B'} B$ *is* $A' \otimes_A B$*.*

Definition 1.3. A localizing invariant E is *truncating* if $E(A) \to E(\pi_0 A)$[1] is an equivalence for any connective $\mathbf{E}_1$-algebra A.

Example 7. The localizing invariant $K^{\mathrm{inv}} := \mathrm{fib}(K \to \mathrm{TC})$ is truncating by Dundas-Goodwillie-McCarthy.

2. Topological Cyclic Homology

Ishan Levy extends the Dundas-Goodwillie-McCarthy theorem to the fixed points of connective ring spectra by $\mathbf{Z}$-actions.

The ∞-category of spectra Sp has a t-structure whose n-connective objects can be described as $\mathrm{Sp}_{\geq n} = \{E \in \mathrm{Sp} : \pi_i(E) = 0 \text{ for } i < n\}$. If R is an $\mathbf{E}_1$-ring, then there exists a t-structure on $\mathrm{Mod}(R)$ whose connective and coconnective objects admit the following description: $\mathrm{Mod}(R)_{\geq 0}$ is the stable subcategory of $\mathrm{Mod}(R)$ generated by R under colimits and extensions, and $\mathrm{Mod}(R)_{<0}$ consists of those R-modules whose underlying spectrum is in $\mathrm{Sp}_{<0}$.

Lemma 8. [1, 3.1] *Let* R *be a* (-1)*-connective* $\mathbf{E}_1$*-ring. Let* M *be any* R*-module which is connective as a spectrum. Then*

(1) $M \in \mathit{Mod}(R)_{\geq 0}$.
(2) *For any right* R*-module* N *with* $N \in \mathit{Sp}_{\geq 0}$*, we have* $M \otimes_R N \in \mathit{Sp}_{\geq 0}$.

Proof. (1) The t-structure on $\mathrm{Mod}(R)$ supplies a cofiber sequence $\tau_{\geq 0} M \to M \to \tau_{<0} M$. As $\tau_{\geq 0} M \in \mathrm{Mod}(R)$ is built from R by colimits and extensions, and as R is (-1)-connective, it follows that the underlying spectrum of $\tau_{\geq 0} M$ is (-1)-connective. As M is connective as an underlying spectrum by assumption, it follows that $\tau_{<0} M$ is as well. Since $\tau_{<0} M \in \mathrm{Sp}_{<0}$, it follows that $\tau_{<0} M = 0$, thus $\tau_{\geq 0} M \to M$ is an equivalence; in particular, $M \in \mathrm{Mod}(R)_{\geq 0}$.

(2) By assumption M is generated by R by colimits and extensions, and as $- \otimes_R N$ preserves such constructions, it follows that $M \otimes_R N$ is build out of colimits and extensions by $R \otimes_R N \simeq N$. If $N \in \mathrm{Sp}_{\geq 0}$, it follows that $M \otimes_R N \in \mathrm{Sp}_{\geq 0}$ as well. $\square$

[1] As with K-theory, we denote $E(A) := E(\mathrm{perf}(A))$.

Lemma 9. [1, 3.2] *Let R, S be $\mathbf{E}_1$-rings in $\mathrm{Sp}_{\geq -1}$. Suppose that $f : R \to S$ is an i-connective map of $\mathbf{E}_1$-rings for $i \geq -1$. Let M, N be right and left S-modules respectively, with $M, N \in \mathrm{Mod}(S)_{\geq 0}$. Then $M \otimes_R N \to M \otimes_S N$ is $(i+1)$-connective.*

Proposition 10. (Waldhausen)[1, 3.3] Let $f : R \to S$ be an i-connective map of connective $\mathbf{E}_1$-spectra for $i \geq 1$. Then $\mathrm{fib}(\mathbb{K}(f))$ is $(i+1)$-connective.

Theorem 11. [1, 3.5] *Let*

$$\begin{array}{ccccc} R_0 & \longrightarrow & R_1 & \longleftarrow & R_2 \\ \downarrow & & \downarrow & & \downarrow \\ S_0 & \longrightarrow & S_1 & \longleftarrow & S_2 \end{array}$$

be a map of cospans of connective $\mathbf{E}_1$-rings that is levelwise i-connective for $i \geq 1$. Then for any truncating localizing invariant E, $E(R_0 \times_{R_1} R_2) \to E(S_0 \times_{S_1} S_2)$ is an equivalence, and $\mathrm{TC}(R_0 \times_{R_1} R_2) \to \mathrm{TC}(S_0 \times_{S_1} S_2)$ is i-connective.

Proof. Let $R_3 = R_0 \times_{R_1} R_2$ and $S_3 = S_0 \times_{S_1} S_2$, and let $\mathcal{U}'_{\mathrm{loc}}$ denote the version of the universal localizing invariant of [3] that does not necessarily preserve filtered colimits. Note that R_3 is (-1)-connective. By [2], we have a pullback square

$$\begin{array}{ccc} \mathcal{U}'_{\mathrm{loc}}(R_3) & \longrightarrow & \mathcal{U}'_{\mathrm{loc}}(R_0) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{U}'_{\mathrm{loc}}(R_2) & \longrightarrow & \mathcal{U}'_{\mathrm{loc}}(R_0 \odot_{R_3}^{R_1} R_2) \end{array}$$

where the underlying spectrum of $R_0 \odot_{R_3}^{R_1} R_2$ is equivalent to $R_0 \otimes_{R_3} R_2$. Applying Lemma 3.1, we see that $R_0 \odot_{R_3}^{R_1} R_2$ is connective. By assumption, $\mathrm{fib}(R_j \to S_j)$ is i-connective for $i \geq 1$, hence $\pi_0(\mathrm{fib}(R_j \to S_j)) \simeq 0$, so that $\pi_0(R_j) \to \pi_0(S_j)$ is an equivalence. Therefore $E(\pi_0(R_j)) \to E(\pi_0(S_j))$ is an equivalence, and as E is truncating, we find that $E(R_j) \to E(S_j)$ is an equivalence. Then $R_3 \to S_3$ is $(i-1)$-connective, and the map $R_0 \otimes_{R_3} R_2 \to S_0 \otimes_{R_3} S_2$ is i-connective by Lemma 8. Moreover, by Lemma 8 and Lemma 9, the map $S_0 \otimes_{R_3} S_2 \to S_0 \otimes_{S_3} S_2$ is also i-connective. It follows that the composite $R_0 \otimes_{R_3} R_2 \to S_0 \otimes_{S_3} S_2$ is i-connective. On underlying spectra, this agrees with the map $R_0 \odot_{R_3}^{R_1} R_2 \to S_0 \odot_{S_3}^{S_1} S_2$, which we conclude is also i-connective. Thus $E(R_0 \odot_{R_3}^{R_1} R_2) \to E(S_0 \odot_{S_3}^{S_1} S_2)$ is an equivalence and $\mathrm{TC}(R_0 \odot_{R_3}^{R_1} R_2) \to \mathrm{TC}(S_0 \odot_{S_3}^{S_1} S_2)$ is $(i+1)$-connective by Theorem 5. Finally, by Theorem 6 we deduce that $E(R_3) \to E(S_3)$ is an equivalence, and that $\mathrm{TC}(R_3) \to \mathrm{TC}(S_3)$ is i-connective. □

Remark 12. Giving a ring R a $\mathbf{Z}$-action is the same as giving an automorphism $\phi : R \to R$. Given the latter, $R^{h\mathbf{Z}}$ fits into the pullback square

$$\begin{array}{ccc} R^{h\mathbf{Z}} & \longrightarrow & R \\ \big\downarrow & \lrcorner & \big\downarrow \Delta \\ R & \xrightarrow[(1,\phi)]{} & R \times R \end{array}$$

Applying Theorem 11 to the cospan $R \xrightarrow{\Delta} R \times R \xleftarrow{(1,\phi)} R$ we get the following.

Theorem 13. [1, B] *Let $f : R \to S$ be a map of connective $\mathbf{E}_1$-rings with $\mathbf{Z}$-actions, such that f is 1-connective. Then for any truncating invariant E, we have that $E(R^{h\mathbf{Z}}) \to E(S^{h\mathbf{Z}})$ is an equivalence. Moreover, if f is i-connective, then $\mathrm{TC}(R^{h\mathbf{Z}}) \to \mathrm{TC}(S^{h\mathbf{Z}})$ is also i-connective.*

3. Purity Theorem

We discuss the Purity results of Land, Mathew, Meier, and Tamme [4]. Fix a prime p and $n \geq 1$, and let $K(i)$ denote the i-th Morava K-theory at the prime p. Let V_n be a type n complex; that is, a pointed finite CW-complex with $K(i) \otimes V_n = 0$ for $i < n$, and $K(n) \otimes V_n \neq 0$. A self map $\nu_n : \Sigma^d V_n \to V_n$ is called a *ν_n-map* if $K(i)_*(\nu_n)$ is an equivalence for $i = n$, and nilpotent for $i \neq n$. Let $T(n) := \Sigma^\infty V_n[\nu_n^{-1}]$ be the telescope of ν_n. For a spectrum E, let $L_E : \mathrm{Sp} \to \mathrm{Sp}_E$ denote the Bousfield localization functor, where Sp_E denotes the ∞-category of E-local spectra.

Theorem 14 (Land, Mathew, Meier, Tamme)**.** *Let A be a ring spectrum. For $n \geq 1$, the canonical map $A \to L_{T(n-1)\oplus T(n)} A$ induces an equivalence in $T(n)$-local K-theory.*

One can use such purity results to reprove the following theorem of Mitchell.

Theorem 15 (Mitchell)**.** *Let E be a module over $K(\mathbf{Z})$. Then $K(n)_* E \simeq 0$ for all $n \geq 0$.*

Lemma 16. [4, Lemma 2.3] *Let R be a ring spectrum and $n \geq 1$. Then R is $K(n)$-acyclic if and only if R is $T(n)$-acyclic.*

Corollary 16.1. *For any ordinary ring R, $L_{T(n)} K(R) \simeq 0$ and $L_{T(n)} \mathrm{TC}(R) \simeq 0$.*

Proof. Note that both $K(R)$ and $\mathrm{TC}(R)$ are modules over $K(\mathbf{Z})$. By Theorem 15, $K(n)_* K(R) \simeq 0$ and $K(n)_* \mathrm{TC}(R) \simeq 0$, and by Lemma 16, we find that $T(n)_* K(R) \simeq 0$ and $T(n)_* \mathrm{TC}(R) \simeq 0$. □

Proposition 17. [4, Cor 4.30] Let $n \geq 2$ and let A be a commutative ring spectrum. Then $L_{T(n)} K(A) \to L_{T(n)} \mathrm{TC}(A)$ is an equivalence.

Proof. By the Dundas-Goodwillie-McCarthy theorem, we have a bicartesian square

$$\begin{array}{ccc} K(A) & \longrightarrow & \mathrm{TC}(A) \\ \downarrow & & \downarrow \\ K(\pi_0(A)) & \longrightarrow & \mathrm{TC}(\pi_0(A)) \end{array}$$

As the localization functor $L_{T(n)}$ is a left adjoint, applying it to the diagram above yields a bicartesian square

$$\begin{array}{ccc} L_{T(n)}K(A) & \longrightarrow & L_{T(n)}\mathrm{TC}(A) \\ \downarrow & & \downarrow \\ L_{T(n)}K(\pi_0(A)) & \longrightarrow & L_{T(n)}\mathrm{TC}(\pi_0(A)) \end{array}$$

Finally, by Theorem 15, $L_{T(n)}K(\pi_0(A)) \simeq 0$ and $L_{T(n)}\mathrm{TC}(\pi_0(A)) \simeq 0$, so that $\mathrm{cofib}(L_{T(n)}K(A) \to L_{T(n)}\mathrm{TC}(A)) \simeq 0$, and thus $L_{T(n)}K(A) \to L_{T(n)}\mathrm{TC}(A)$ is an equivalence. □

Proposition 18. Let $n \geq 2$ and let A be a connective $\mathbf{E}_1$-ring spectrum with $\mathbf{Z}$-action. Then $L_{T(n)}K(A^{h\mathbf{Z}}) \to L_{T(n)}\mathrm{TC}(A^{h\mathbf{Z}})$ is an equivalence.

Proof. Let $f : A \to \pi_0(A)$ denote the canonical map; as $\mathrm{fib}(f) \in \mathrm{Sp}_{\geq 1}$, f is 1-connective. Then by Theorem 13, $K^{\mathrm{inv}}(A^{h\mathbf{Z}}) \to K^{\mathrm{inv}}(\pi_0(A)^{h\mathbf{Z}})$ is an equivalence. By definition, $K^{\mathrm{inv}}(A^{h\mathbf{Z}})$ fits in a fiber sequence $K^{\mathrm{inv}}(A^{h\mathbf{Z}}) \to K(A^{h\mathbf{Z}}) \to \mathrm{TC}(A^{h\mathbf{Z}})$; applying $L_{T(n)}$ yields a fiber sequence

$$L_{T(n)}K^{\mathrm{inv}}(A^{h\mathbf{Z}}) \to L_{T(n)}K(A^{h\mathbf{Z}}) \to L_{T(n)}\mathrm{TC}(A^{h\mathbf{Z}})$$

To prove that $L_{T(n)}K(A^{h\mathbf{Z}}) \to L_{T(n)}\mathrm{TC}(A^{h\mathbf{Z}})$ is an equivalence, it suffices to show that $L_{T(n)}K^{\mathrm{inv}}(A^{h\mathbf{Z}}) \simeq 0$, or equivalently $L_{T(n)}K^{\mathrm{inv}}(\pi_0(A)^{h\mathbf{Z}}) \simeq 0$. Observe that $\pi_0(A)^{h\mathbf{Z}}$ is a $\mathbf{Z}$-module, hence both $K(\pi_0(A)^{h\mathbf{Z}})$ and $\mathrm{TC}(\pi_0(A)^{h\mathbf{Z}})$ are $K(\mathbf{Z})$-modules. By Mitchell's theorem, $K(n)_*K(\pi_0(A)^{h\mathbf{Z}})$ and $K(n)_*\mathrm{TC}(\pi_0(A)^{h\mathbf{Z}})$ both vanish, which by Theorem 16 implies $L_{T(n)}K(\pi_0(A)^{h\mathbf{Z}})$ and $L_{T(n)}\mathrm{TC}(\pi_0(A)^{h\mathbf{Z}})$ vanish, thus $L_{T(n)}K^{\mathrm{inv}}(\pi_0(A)^{h\mathbf{Z}}) \simeq 0$. □

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