A CLASSICAL PROOF THAT THE ALGEBRAIC HOMOTOPY CLASS OF A RATIONAL FUNCTION IS THE RESIDUE PAIRING

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ABSTRACT. Cazanave has identified the algebraic homotopy class of a rational function of 1 variable with an explicit nondegenerate symmetric bilinear form. Here we show that Hurwitz's proof of a classical result about real rational functions essentially gives an alternative proof of the stable part of Cazanave's result. We also explain how this result can be interpreted in terms of the residue pairing and that this interpretation relates the result to the signature theorem of Eisenbud, Khimshiashvili, and Levine, showing that Cazanave's result answers a question posed by Eisenbud for polynomial functions in 1 variable. Finally, we announce results answering this question for functions in an arbitrary number of variables.

1. INTRODUCTION

In this paper we show that a classical proof of Hurwitz can be modified to give a new proof of Cazanave's description of the degree of a rational function in $\mathbf{A}^1$-homotopy theory. In [Hur95], Hurwitz computed the topological degree of a real rational function $f/g \in \operatorname{Frac} \mathbf{R}[x]$ where we assume f is monic of degree $\mu > \deg(g)$ and $f(x)$ and $g(x)$ are relatively prime. The real rational function f/g defines a self-map of the real projective line $\mathbf{P}^1_{\mathbf{R}}$, and the space $\mathbf{P}^1_{\mathbf{R}}(\mathbf{R})$ of real points is homeomorphic to the 1-sphere S^1. This self-map has a well-defined topological degree. Hurwitz's theorem [Hur95, Section 3] computes the degree as

Theorem 1 (Hurwitz). *The real topological degree* $\deg^{\mathbf{R}}(f/g)$ *of* $f/g\colon \mathbf{P}^1_{\mathbf{R}}(\mathbf{R}) \to \mathbf{P}^1_{\mathbf{R}}(\mathbf{R})$ *equals the signature of the real symmetric matrix*

$$S(f/g) = \begin{pmatrix} s_1 & s_2 & \cdots & s_{\mu-1} & s_\mu \\ s_2 & s_3 & \cdots & s_\mu & s_{\mu-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_\mu & s_{\mu+1} & \cdots & s_{2\mu-2} & s_{2\mu-1} \end{pmatrix}$$

for $s_1, s_2, \ldots$ *defined by* $g(x)/f(x) = s_1/x + s_2/x^2 + s_3/x^3 + \ldots$.

Cazanave proved an enrichment of Hurwitz's result in $\mathbf{A}^1$-homotopy theory. $\mathbf{A}^1$-homotopy theory is a theory of algebraic topology for algebraic varieties over an arbitrary field that plays a role analogous to classical homotopy as applied to real or complex varieties. One celebrated result in the field is Morel's construction of a degree map. Morel's construction was indirect, but Cazanave proved the $\mathbf{A}^1$-degree has the following explicit description.

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Theorem 2 (Cazanave). *The* $\mathbf{A}^1$*-homotopical degree* $\deg^{\mathbf{A}^1}(f/g)$ *of* $f/g\colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ *equals the class of*

$$\text{Béz}(f/g) := \begin{pmatrix} b_{1,1} & \cdots & b_{1,\mu} \\ \vdots & \ddots & \vdots \\ b_{\mu,1} & \cdots & b_{\mu,\mu} \end{pmatrix}$$

for $b_{i,j}$ *defined by*

$$f(x)g(y) - f(y)g(x) = (x-y)\cdot\left(\sum_{i,j=1}^{\mu} b_{i,j}x^{i-1}y^{j-1}\right). \tag{1}$$

This is not explicitly stated in [Caz08, Caz12], but it follows from [Caz12, Theorem 1.2, Corollary 3.10], as we explain below. Cazanave also proved results stronger than Theorem 2 which we omit from this discussion.

The equality $\deg^{\mathbf{A}^1}(f/g) = \text{Béz}(f/g)$ in Theorem 2 takes place in the Grothendieck–Witt group $\mathrm{GW}(k)$ of nondegenerate symmetric bilinear forms over k. Recall that isomorphism classes of nondegenerate symmetric bilinear forms are a monoid under direct sum, and the Grothendieck–Witt group is defined by formally adding inverses to make the monoid into a group. The group can be described explicitly in terms of generators and relations. If $\langle A\rangle$ denotes the isomorphism class of the 1-dimensional bilinear form $(a,b) \mapsto A \cdot ab$, then $\mathrm{GW}(k)$ has the presentation given by the generators $\langle A\rangle$, $A \neq 0$, and relations $\langle A\rangle = \langle AB^2\rangle$ for $A, B \neq 0$ and $\langle A\rangle + \langle B\rangle = \langle A+B\rangle + \langle AB(A+B)\rangle$ for $A, B, A+B \neq 0$.

When $k = \mathbf{R}$, the only invariants of a nondegenerate symmetric bilinear form are the signature and rank, and they define an injection of $\mathrm{GW}(\mathbf{R})$ in $\mathbf{Z}^{\oplus 2}$. Like the signature, the rank of $S(f/g)$ can be interpreted topologically: it is the topological degree $\deg^{\mathbf{C}}(f/g)$ of the map $f/g\colon \mathbf{P}^1_{\mathbf{R}}(\mathbf{C}) \to \mathbf{P}^1_{\mathbf{R}}(\mathbf{C})$ of the complex projective line. Thus Hurwitz's theorem implies that the pair (real topological degree, complex topological degree) equals the class of $\deg^{\mathbf{A}^1}(f/g) \in \mathrm{GW}(\mathbf{R}) \subset \mathbf{Z}^{\oplus 2}$. Cazanave's theorem can be viewed as extending this result to an arbitrary field.

While Hurwitz's theorem and Cazanave's theorem involves different matrices, the first theorem is a consequence from the second. Indeed, as we will see, the matrices $\text{Béz}(f/g)$ and $S(f/g)$ are conjugate, so they have the same signature. In particular, Cazanave's theorem implies that the signature of $S(f/g)$ equals the signature of $\deg^{\mathbf{A}^1}(f/g)$. The signature of $\deg^{\mathbf{A}^1}(f/g)$ equals the real topological degree $\deg^{\mathbf{R}}(f/g)$ by an application of the real realization functor relating $\mathbf{A}^1$-homotopy theory to classical homotopy theory ([MV99, Section 3.3]). Over a more general field, the $\mathbf{A}^1$-homotopical degree is not determined by classical topological invariants, but it can be understood in analogy with the topological degree. Recall $\deg^{\mathbf{R}}(f/g)$ equals the signed count of the points in the preimage of a regular value with a point r being counted with the sign of the derivative $(f/g)'(r)$. The degree $\deg^{\mathbf{A}^1}(f/g)$ can computed analogously except the signed of $(f/g)'(r)$ is replaced by the class of $(f/g)'(r)$ modulo $k^*)^2$, considered as elements of $\mathrm{GW}(k)$. (To formulate this precisely, one must take care with the definition of regular value and take into account points in the preimage that are defined over a field extension. The exact formula is (3) below).

The equality in Cazanave's theorem is more subtle than the equality in Hurwitz's theorem. Over the real numbers, an equality of elements w_1, w_2 of $\mathrm{GW}(\mathbf{R})$ can be verified by computing ranks and signatures, but in general, equality can be hard to check. Some information can be extracted from easily computed invariants like the rank, signature, and discriminant (which which is the determinant of a Gram matrix considered as an element of $k^*/(k^*)^2$). These invariants determine when two classes are equal over very simple fields like $\mathbf{R}$ and $\mathbf{F}_q$ but not in general. The situation is nicely illustrated by the rational functions

$$f_0/g_0 = (x^3 - x)/(5x^2 + 1) \text{ and } f_1/g_1 = (x^3 - x)/(7x^2 - 8x - 3)$$

over a field k of characteristic $\neq 2, 3$ (so that both rational functions are in lowest terms). Both functions have 0 as a regular value with preimage $\{\pm 1, 0\}$, and by computing derivatives, we get that

$$\deg^{\mathbf{A}^1}(f_0/g_0) = 2 \cdot \langle 1/3 \rangle + \langle -1 \rangle \text{ and}$$
$$\deg^{\mathbf{A}^1}(f_1/g_1) = \langle -2 \rangle + \langle 3 \rangle + \langle 6 \rangle.$$

These two $\mathbf{A}^1$-degrees have rank 3, discriminant -1, and when $k = \mathbf{R}$ signature 1. We conclude that the classes are equal when $k = \mathbf{C}, \mathbf{R}$, or a finite field $\mathbf{F}_q$ because, for these fields, the invariants determine an element $w \in \mathrm{GW}(k)$. The two degrees are not, however, equal in general. For example, when $k = \mathbf{Q}$, the classes can be distinguished by an invariant that is more difficult to define: the p-adic Hasse–Witt invariant for p a prime. The invariant is defined on [Mil73, page 79], and for $p = 3$, it takes value -1 on $\deg^{\mathbf{A}^1}(f_0/g_0)$ and value $+1$ on $\deg^{\mathbf{A}^1}(f_1/g_1)$.

The connection with Grothendieck–Witt group also has interesting consequences for the real topological degree of a rational function. For example, suppose k is a number field. Given a real embedding $\sigma\colon k \to \mathbf{R}$, we can form the topological degree $\deg^{\mathbf{R},\sigma}(f/g)$ of a rational function f/g by considering it as a real function via σ. Labeling the real embeddings $\sigma_1, \ldots, \sigma_r$, we thus obtain a tuple $(\deg^{\mathbf{R},\sigma_1}(f/g), \ldots, \deg^{\mathbf{R},\sigma_r}(f/g))$ of integers. Not every tuple of integers arises in this way because $\deg^{\mathbf{R},\sigma_i}(f/g) = \deg^{\mathbf{R},\sigma_j}(f/g)$ modulo 2 for all i, j (both terms are congruent to the algebraic degree of f). However, every tuple with even components is a degree sequence. Indeed, a given such sequence equals the sequence of signatures of a nondegenerate symmetric bilinear form by [Mil73, (2.10) Example], and every such form is equivalent some $S(f/g)$ (see e.g. Remark 13).

Let us now turn to the proofs of Cazanave and Hurwitz. While Hurwitz's theorem is a consequence of Cazanave's theorem, the proofs of the two theorems are fundamentally different. To contrast them, let us recall Cazanave's proof. Cazanave deduces the result from a computation of a certain monoid of homotopy classes, specifically the monoid $[\mathbf{P}^1_k, \mathbf{P}^1_k]^N$ of naive $\mathbf{A}^1$-homotopy classes. This set is defined by naively imitating the definition of homotopy equivalence in topology by replacing the unit interval $[0, 1]$ with the affine line $\mathbf{A}^1_k$ so that naive $\mathbf{A}^1$-equivalence is the equivalence relation generated by requiring that rational functions f_0/g_0 and f_1/g_1 are equivalent when there exists a rational function $F(x, t)/G(x, t)$ such that $F(x, 0)/G(x, 0) = f_0/g_0$, $F(x, 1)/g(x, 1) = f_1/g_1$, and the resultant $\mathrm{Res}_x(F, G) \in k[t]$ is a unit. (The resultant condition implies that F and G are relatively prime and this property persists after specializing t to any constant; this condition corresponds to F/G defining a morphism $\mathbf{P}^1_k \times_k \mathbf{A}^1_k \to \mathbf{P}^1_k$).

There is a tautological map $[\mathbf{P}^1_k, \mathbf{P}^1_k]^N \to [\mathbf{P}^1_k, \mathbf{P}^1_k]^{\mathbf{A}^1}$ because the inclusion of a point $* \to \mathbf{A}^1_k$ is an equivalence in $\mathbf{A}^1$-homotopy theory. Cazanave constructs a monoid structure on $[\mathbf{P}^1_k, \mathbf{P}^1_k]^N$ that makes this map into a monoid homomorphism that identifies $[\mathbf{P}^1_k, \mathbf{P}^1_k]^{\mathbf{A}^1}$ as the group completion of $[\mathbf{P}^1_k, \mathbf{P}^1_k]^N$. He further shows that $f/g \mapsto \mathrm{Béz}(f/g)$ defines a monoid homomorpism, so by general formalism, there is an induced map $[\mathbf{P}^1_k, \mathbf{P}^1_k]^{\mathbf{A}^1} \to \mathrm{GW}(k)$ that sends the homotopy class of f/g to $\mathrm{Béz}(f/g)$. Both this homomorphism and $\deg^{\mathbf{A}^1}$ send the rational function x/A to $\langle A \rangle$ and the homotopy classes of functions of the form x/A generate $[\mathbf{P}^1_k, \mathbf{P}^1_k]^{\mathbf{A}^1}$, so the two homomorphism must be equal.

By contrast, Hurwitz's proof uses matrix theory as follows. The topological degree of f/g equals the sum, over the real zeros of f/g, of the local topological degrees. In general, the local degree of the germ of a continuous function $h\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ is defined to be the topological degree of

$$f/||f||\colon S^{n-1}_\epsilon \to S^{n-1}_1,$$

where $||f||$ is the absolute value of f, $S^n_\epsilon \subset \mathbf{R}^n$ is a sphere of sufficiently small radius $\epsilon > 0$. When $n = 1$ and f/g is a rational function, the local degree is given explicitly by

$$\deg_r(f/g) = \begin{cases} +1 & \text{if } f/g \text{ is increasing at } r; \\ -1 & \text{if } f/g \text{ is decreasing at } r; \\ 0 & \text{otherwise.} \end{cases}$$

To identify the sum of local degrees with the signature of $S(f/g)$, we pass to the complex numbers. Over the complex numbers, $S(f/g)$ is conjugate to a block diagonal matrix $\mathrm{New}(f/g)$ with blocks indexed by the complex roots of $f(x)$. The proof is completed by computing the signature of $S(f/g)$ in terms of the blocks, computing that a pair of blocks corresponding to a complex conjugate pair of roots contributes 0 to the signature and a block corresponding to a real root contributes the local degree at the root.

We show that the same argument, suitably modified, gives an independent proof that $\deg^{\mathbf{A}^1}(f/g)$ is the class of $S(f/g)$ or equivalently $\mathrm{Béz}(f/g)$. This argument, however, proves a result that is strictly weaker than the result proven by Cazanave. The $\mathbf{A}^1$-degree of f/g determines the *stable* homotopy class of f/g, and Cazanave proves that in fact $\mathrm{Béz}(f/g)$ represents the *unstable* homotopy class of f/g. Recall Morel proved that the unstable homotopy class of f/g is determined by $\deg^{\mathbf{A}^1}(f/g)$ together with a scalar $d(f/g) \in k^*$ representing the class of the discriminant of $\deg^{\mathbf{A}^1}(f/g)$. Here the discriminant of $w \in \mathrm{GW}(k)$ is the determinant of a Gram matrix that represents w. The discriminant is only defined modulo $(k^*)^2$ since conjugating a Gram matrix by a matrix A changes the determinant by $\det(A)^2$. Cazanave proves that $d(f/g)$ equals the determinant of $\mathrm{Béz}(f/g)$. This second result is not established by our modification of Hurwitz's proof, as we explain at the end of Section 2. We also do not study the monoid structure on $[\mathbf{P}^1_k, \mathbf{P}^1_k]^N$. While the monoid structure plays a fundamental role in Cazanave's proof, it does not play a role in Hurwitz's proof.

A secondary goal of this paper is to explain the relation of the work of Hurwitz and Cazanave to the beautiful signature theorem of Eisenbud–Levine and Khimshiashvili. The signature formula identifies the local degree of a real polynomial function (possibly in many variables) as the signature of the residue pairing. Recall the local degree of $f\colon \mathbf{R}^n \to \mathbf{R}^n$ at the origin is defined to be the topological degree of

$$f/||f||\colon S^{n-1}_{\epsilon} \to S^{n-1}_{1},$$

where $||f||$ is the absolute value of f, $S^{n}_{\epsilon} \subset \mathbf{R}^n$ is a sphere of sufficiently small radius $\epsilon > 0$. In Section 3, we explain that the matrices $\mathrm{Béz}(f/g)$ and $S(f/g)$ are Gram matrices for the residue pairing and that the signature formula generalizes Hurwitz's theorem. We announce the following theorem which generalizes the signature theorem and answers a question posed by Eisenbud in [Eis78]:

Theorem 3 (The main result of [KW16])**.** *The local $\mathbf{A}^1$-degree of $f\colon \mathbf{A}^n_k \to \mathbf{A}^n_k$ is the class of the residue pairing $\beta_0(\mathrm{Res}_0)$.*

This is Theorem 17 in Section 3. As we explain in that section, $\beta_0(\mathrm{Res}_0)$ is a bilinear pairing on the localization of the algebra $Q_0(f/g) = k[x, g^{-1}]/(fg^{-1})$ at the ideal (x). The reader familiar with Cazanave's work may recall that he identifies $\mathrm{Béz}(f/g)$ as a Gram matrix of a bilinear pairing under suitable hypotheses, but his bilinear pairing is not $\beta_0(\mathrm{Res}_0)$. We explain the relation between the pairings in the same section.

In this paper we will assume $\mathrm{char}\, k \neq 2$, primarily to simplify exposition. Both Cazanave's theorem and Theorem 3 remain valid in characteristic 2, but this assumption simplifies the proof of Theorem 12. At one step in the proof, we reduce to the case where the preimage $(f/g)^{-1}(0)$ of the origin is étale so that we can apply a formula (Equation (3) below) relating the global degree to the local degree, and the reduction makes use of the assumption $\mathrm{char}\, k \neq 2$ (specifically the consequence that a purely inseparable extension has odd degee).

CONVENTIONS

k is a field of characteristic $\neq 2$.

A rational function is a nonconstant k-morphism $f/g\colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ satisfying $f/g(\infty) = \infty$. We always represent such a morphism by a fraction $f(x)/g(x) \in \mathrm{Frac}\, k[x]$ with f monic, f relatively prime to g, and $\deg(f) > \deg(g)$. Observe that our rational functions are more properly called pointed rational functions, but we omit the term "pointed."

The Gram matrix of a symmetric bilinear form β on a finite dimensional k-vector space V with respect to a basis $e_1, e_2, \dots, e_\mu$ is the matrix $\big(\beta(e_i, e_j)\big)$.

The Grothendieck–Witt group of k will be denoted by $\mathrm{GW}(k)$. It is the group completion of the monoid of isomorphism classes of symmetric nondegenerate bilinear forms $\beta : V \times V \to k$, where V is a finite dimensional vector space over k, under perpendicular direct sum. Tensor product of forms gives $\mathrm{GW}(k)$ a ring structure.

We write $\langle A\rangle \in \mathrm{GW}(k)$ for the class of the symmetric bilinear form with Gram matrix (A). $\mathrm{GW}(k)$ is generated by $\langle A\rangle$ for A in k^* and has an explicit presentation given in [Lam05, II Theorem 4.1, page 39] and [EKM08, Theorem 4.7, page 23].

For a field extension $k \subset L$ of finite rank, we will use the trace map $\mathrm{Tr}_{L/k} : \mathrm{GW}(L) \to \mathrm{GW}(k)$. For $k \subseteq L$ separable, this map takes the bilinear pairing $\beta : V \times V \to L$ to the composition $\mathrm{Tr}_{L/k} \circ \beta$ of β with the field trace.

2. A classical proof

Here we adapt Hurwitz's proof of Theorem 1 to prove Cazanave's result, Theorem 2. The proof is closely modeled on the Kreĭn–Naĭmark's exposition of Hurwitz's result in [KN81], and after giving the proof, we closely compare our argument with the argument in loc. cit. We then explain why Hurwitz's argument does not prove the stronger result that $\mathrm{Béz}(f/g)$ represents the unstable homotopy class of f/g.

We begin by summarizing the basic properties of the $\mathbf{A}^1$-degree that we use. The properties are analogues of familiar properties of the usual topological degree of a real rational function $f/g\colon \mathbf{P}^1_{\mathbf{R}} \to \mathbf{P}^1_{\mathbf{R}}$. The reader unfamiliar with the degree map in $\mathbf{A}^1$-homotopy theory should be able to understand the proof of Theorem 12 using just the properties we summarize below (up to Lemma 4).

The degree map in $\mathbf{A}^1$-homotopy theory takes values in the Grothendieck–Witt $\mathrm{GW}(k)$. This is the group completion of the monoid of nondegenerate quadratic forms. The theory supports both a global degree $\deg^{\mathbf{A}^1}(f/g)$ of a rational function f/g and a local degree $\deg_r^{\mathbf{A}^1}(f/g)$ at a closed point $r \in \mathbf{P}^1_k$. These degrees satisfy the following properties:

- the degree of a composition is the product of the degrees,
- the global degree is expressed as a sum of traces of local degrees by Equation (3),
- the local degree can be computed after extending scalars by a separable extension by Equation (2),
- both the local and global degrees are invariant under naive $\mathbf{A}^1$-homotopies,
- the global degree is normalized so that $\deg^{\mathbf{A}^1}(x/A) = \langle A\rangle$.

The expression computing the local degree $\deg_r(f/g)$ after extending scalars is the following. The closed point $r \in \mathbf{P}^1_k$ has a distinguished lift to a closed point of $\mathbf{P}^1_{k(r)}$ that we also denote by r. Writing $f/g \otimes k[r]$ for f/g considered as a rational function with coefficients in the residue field $k[r]$ of r, we consider both its local degree $\deg_r^{\mathbf{A}^1}(f/g \otimes k[r]) \in \mathrm{GW}(k[r])$ and the local degree $\deg_r^{\mathbf{A}^1}(f/g) \in \mathrm{GW}(k)$. If $k[r]/k$ is a separable extension, then these degree are related by

$$\deg_r^{\mathbf{A}^1}(f/g) = \mathrm{Tr}_{k[r]/k}(\deg_{r\otimes 1}^{\mathbf{A}^1}(f/g \otimes k[r])). \tag{2}$$

Here $\mathrm{Tr}\colon \mathrm{GW}(L) \to \mathrm{GW}(k)$ is the trace function, which is defined by sending a bilinear pairing $\beta\colon V \times V \to L$ to the composition $\mathrm{Tr}_{L/k} \circ \beta$ of β with the field trace.

When f/g is étale at r (i.e. the derivative is nonzero), this is [KW16, Proposition 15], although the result was certainly known to experts. Below in Lemma 11 we prove (2) in general.

The expression of the global degree as a sum of local degrees is the following one. Suppose $f/g\colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ is a rational function with the property that the preimage of the origin is a union $\{r_1, \ldots, r_m\}$ of closed points with residue fields that are separable extensions of k. Then the global $\mathbf{A}^1$-degree satisfies

$$\deg^{\mathbf{A}^1}(f/g) = \mathrm{Tr}_{k[r_1]/k}(\deg^{\mathbf{A}^1}_{r_1}(f/g \otimes k[r_1])) + \cdots + \mathrm{Tr}_{k[r_m]/k}(\deg^{\mathbf{A}^1}_{r_m}(f/g \otimes k[r_m])). \tag{3}$$

This formula is [KW16, Proposition 14] combined with (2), although like Equation (2), Equation (3) was known to experts. Under additional hypotheses, the formula is described in [Mor04, Section 2], [Mor06, page 1036], [Lev08, Section 5.2].

The invariance property of the global degree we use is that

$$\deg^{\mathbf{A}^1}(f_0/g_0) = \deg^{\mathbf{A}^1}(f_1/g_1)$$

when there exists a morphism $H\colon \mathbf{P}^1_k \times_k \mathbf{A}^1_k \to \mathbf{P}^1_k$ such that $f_0/g_0(x) = H(x, 0)$, $f_1/g_1(x) = H(x, 1)$, and $H(\infty, t) = \infty$ (i.e. when there exist a naive $\mathbf{A}^1$-homotopy). The local analogue is the following lemma.

Lemma 4. *Let $r \in \mathbf{P}^1_k$ be a closed point and $f_0/g_0, f_1/g_1\colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ be rational functions satisfying $f_0(r) = f_1(r) = 0$. If there exists an open subscheme $U \subset \mathbf{A}^1 \times_k \mathbf{A}^1_k$ that contains $\{r\} \times_k \mathbf{A}^1_k$ and a morphism $H\colon U \to \mathbf{P}^1_k$ such that*

$$H(x, 0) = f_0/g_0(x), H(x, 1) = f_1/g_1(x), \textit{ and}$$
$$\{r\} \times_k \mathbf{A}^1_k \textit{ is a connected component of } H^{-1}(\{0\} \times \mathbf{A}^1_k),$$

then

$$\deg^{\mathbf{A}^1}_r(f_0/g_0) = \deg^{\mathbf{A}^1}_r(f_1/g_1).$$

Proof. The local degree $\deg^{\mathbf{A}^1}_r(f/g)$ of a rational function f/g with an isolated zero at r is defined using the purity theorem [MV99, Section 3, Lemma 2.1]. If $V \subset \mathbf{P}^1_k$ is the complement of the zeros distinct from r, then $f/g\colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ induces a morphism of quotient spaces

$$\frac{V}{V - \{r\}} \to \frac{\mathbf{P}^1_k}{\mathbf{P}^1_k - \{0\}}. \tag{4}$$

The purity theorem identifies the target quotient space with $\mathbf{P}^1_k$ and excision identifies the source quotient space with $\frac{\mathbf{P}^1_k}{\mathbf{P}^1_k - \{r\}}$. If we compose the resulting map $\frac{\mathbf{P}^1_k}{\mathbf{P}^1_k - \{r\}} \to \mathbf{P}^1_k$ with the natural map $\frac{\mathbf{P}^1_k}{\infty} \to \frac{\mathbf{P}^1_k}{\mathbf{P}^1_k - \{r\}}$ and identify $\frac{\mathbf{P}^1_k}{\infty}$ with $\mathbf{P}^1_k$, then we obtain from (4) a map $\mathbf{P}^1_k \to \mathbf{P}^1_k$. The global degree of this last map is defined to be the local degree $\deg^{\mathbf{A}^1}_r(f/g)$.

Now suppose that we are given H as in the statement of the lemma. We use H to construct a global naive $\mathbf{A}^1$-homotopy as follows. Let Z be the union of the connected components of $H^{-1}(\{0\} \times_k \mathbf{A}^1_k)$ that are distinct from $\{r\} \times_k \mathbf{A}^1_k$. We have the identification

$$\begin{aligned}\frac{U}{U - H^{-1}(0)} =& \frac{U}{U - (\{x = r\} \coprod Z)} \\ \simeq& \frac{U}{U - \{x = r\}} \vee \frac{U}{U - Z} \\ \simeq& \frac{\mathbf{P}^1_k}{\mathbf{P}^1_k - \{x = r\}} \vee \frac{U}{U - Z}\end{aligned}$$

Thus the morphism $\frac{U}{U-H^{-1}(0)} \to \frac{\mathbf{P}^1_k}{\mathbf{P}^1_k - 0}$ induced by H can be identified with a morphism $\frac{\mathbf{P}^1_k \times_k \mathbf{A}^1_k}{\mathbf{P}^1_k \times_k \mathbf{A}^1_k - \{x=r\}} \vee \frac{U}{U-Z} \to \frac{\mathbf{P}^1_k}{\mathbf{P}^1_k - 0}$. Pre-composing this morphism with the natural morphism $\frac{\mathbf{P}^1_k}{\mathbf{P}^1_k - \{r\}} \times_k \mathbf{A}^1_k \to \frac{\mathbf{P}^1_k \times_k \mathbf{A}^1_k}{\mathbf{P}^1_k \times_k \mathbf{A}^1_k - \{x=r\}} \to \frac{\mathbf{P}^1_k \times_k \mathbf{A}^1_k}{\mathbf{P}^1_k \times_k \mathbf{A}^1_k - \{x=r\}} \vee \frac{U}{U-Z}$, we obtain a naive $\mathbf{A}^1$-homotopy from the morphism induced by f_0/g_0 to the morphism induced by f_1/g_1.

□

We first use these properties to compute the degree of a power map.

Lemma 5. *For $A \in k^*$, we have*

$$\deg^{\mathbf{A}^1}(x^\mu/A) = \begin{cases} \langle A \rangle + \frac{\mu-1}{2} \cdot \langle 1, -1 \rangle & \mu \text{ odd;} \\ \frac{\mu}{2} \cdot \langle 1, -1 \rangle & \mu \text{ even.} \end{cases} \tag{5}$$

Proof. First, we prove the case where $A = 1$. In this case, the right-side of (5) can alternatively be described recursively as the class w_μ satisfying

$$\begin{aligned} w_1 &= \langle 1 \rangle \\ w_{\mu+1} &= w_\mu + \langle (-1)^\mu \rangle. \end{aligned}$$

We will show $\deg^{\mathbf{A}^1}(x^\mu)$ satisfies the same recursion.

The recursion is satisfied when $\mu = 1$ by construction, and for $\mu > 1$, we compute the degree of the auxiliary function $x^{\mu+1} + x^\mu \colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ in two different ways. First, the expression $x^{\mu+1} + tx^\mu$ defines a naive $\mathbf{A}^1$-homotopy from $x^{\mu+1} + x^\mu$ to $x^{\mu+1}$, so

$$\deg^{\mathbf{A}^1}(x^{\mu+1} + x^\mu) = \deg^{\mathbf{A}^1}(x^{\mu+1}). \tag{6}$$

Second, by Formula (3), we have that $\deg^{\mathbf{A}^1}(x^{\mu+1}+x^\mu) = \deg^{\mathbf{A}^1}_0(x^{\mu+1}+x^\mu)+\deg^{\mathbf{A}^1}_{-1}(x^{\mu+1}+x^\mu)$. To compute the local degrees, observe that $(x, t) \mapsto x^\mu(1 + tx)$ defines a naive local $\mathbf{A}^1$-homotopy in the sense of Lemma 4, so

$$\deg^{\mathbf{A}^1}_0(x^{\mu+1} + x^\mu) = \deg^{\mathbf{A}^1}_0(x^\mu).$$

A similar argument shows

$$\deg^{\mathbf{A}^1}_{-1}(x^{\mu+1} + x^\mu) = \deg^{\mathbf{A}^1}_{-1}((-1)^\mu(x + 1)).$$

We now compute

$$\begin{aligned}\deg^{\mathbf{A}^1}(x^{\mu+1}) &= \deg^{\mathbf{A}^1}(x^{\mu+1}+x^{\mu}) \\ &= \deg_0^{\mathbf{A}^1}(x^{\mu}) + \deg_{-1}^{\mathbf{A}^1}((-1)^{\mu}(x+1)) \\ &= \deg^{\mathbf{A}^1}(x^{\mu}) + \langle(-1)^{\mu}\rangle,\end{aligned}$$

verifying the recursion.

We complete the proof by observing that $\deg^{\mathbf{A}^1}(x^{\mu}/A) = \deg^{\mathbf{A}^1}(x/A)\cdot\deg^{\mathbf{A}^1}(x^{\mu})$ because $\deg^{\mathbf{A}^1}$ transforms composition into multiplication.

□

The next lemma identifies the elements of $\mathrm{GW}(k)$ appearing in Lemma 5 with explicit matrices.

Lemma 6. *If* $A(1), A(2), \ldots, A(\mu) \in k$ *are scalars with* $A(\mu)$ *nonzero, then the nondegenerate symmetric matrix*

$$\begin{pmatrix} A(1) & A(2) & \cdots & A(\mu-1) & A(\mu) \\ A(2) & A(3) & \cdots & A(\mu) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A(\mu-1) & A(\mu) & \cdots & 0 & 0 \\ A(\mu) & 0 & \cdots & 0 & 0 \end{pmatrix}$$

represents the Grothendieck–Witt class

$$w = \begin{cases} \langle A(\mu)\rangle + \frac{\mu-1}{2}\cdot\langle 1,-1\rangle & \mu \text{ odd;} \\ \frac{\mu}{2}\cdot\langle 1,-1\rangle & \mu \text{ even.}\end{cases}$$

Proof. Let $x_1, \ldots, x_\mu$ denote the basis dual to the standard basis on $k^{\oplus\mu}$, so the bilinear form β defined by the given matrix is $\sum A(i+j-1)x_i \otimes x_j$. Rewrite the bilinear form as

(7) $\quad x_1 \otimes \Psi_1 + \Psi_1 \otimes x_1 + x_2 \otimes \psi_2 + \psi_2 \otimes x_2 + \ldots x_{\mu/2} \otimes \Psi_{\mu/2} + \Psi_{\mu/2} \otimes x_{\mu/2}$ if μ even;

$x_1 \otimes \Psi_1 + \Psi_1 \otimes x_1 + \ldots x_{(\mu-1)/2} \otimes \Psi_{(\mu-1)/2} + \Psi_{(\mu-1)/2} \otimes x_{(\mu-1)/2} + A(\mu)x_{(\mu+1)/2} \otimes x_{(\mu-3)/2}$ if μ odd

for $\Psi_1, \Psi_2, \ldots$ the linear functions

$$\begin{aligned}\Psi_1 &= \frac{A(1)}{2}x_1 + A(2)x_2 + A(3)x_3 + \cdots + A(\mu-2)x_{\mu-2} + A(\mu-1)x_{\mu-1} + A(\mu)x_\mu, \\ \Psi_2 &= \frac{A(3)}{2}x_2 + A(4)x_3 + \cdots + A(\mu-1)x_{\mu-2} + A(\mu)x_{\mu-1}, \\ \Psi_3 &= \frac{A(5)}{2}x_3 + \cdots + A(\mu)x_{\mu-2}, \\ &\ldots\end{aligned}$$

The elements $x_1, \Psi_1, x_2, \Psi_2, \ldots$ form a dual basis, so (7) shows that β is the orthogonal sum of the hyperbolic planes $x_1 \otimes \Psi_1 + \Psi_1 \otimes x_1, x_2 \otimes \Psi_2 + \Psi_2 \otimes x_2, \ldots$ and, when μ is odd, the 1-dimensional space $A(\mu)x_{(\mu-1)/2} \otimes x_{(\mu-1)/2}$. □

We now introduce an auxiliary matrix, the Newton matrix $\mathrm{New}(f/g)$, which will be used in the proof.

Definition 7. *Given a root $r \in k$ of f of multiplicity μ_0, let*

$$g(x)/f(x) = \frac{A_r(\mu_0)}{(x-r)^{\mu_0}} + \frac{A_r(\mu_0-1)}{(x-r)^{\mu_0-1}} + \cdots + \frac{A_r(2)}{(x-r)^2} + \frac{A_r(1)}{(x-r)} + \text{higher order terms} \tag{8}$$

*be a partial fractions decomposition. Define the **local Newton matrix** by*

$$\mathrm{New}(f/g, r) := \begin{pmatrix} A_r(1) & A_r(2) & \cdots & A_r(\mu_0-1) & A_r(\mu_0) \\ A_r(2) & A_r(3) & \cdots & A_r(\mu_0) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_r(\mu_0-1) & A_r(\mu_0) & \cdots & 0 & 0 \\ A_r(\mu_0) & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{9}$$

*If k contains all the roots $\{r_1, \ldots, r_m\}$ of $f(x)$, then we define the **Newton matrix** by*

$$\mathrm{New}(f/g) := \begin{pmatrix} \mathrm{New}(f/g, r_1) & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{New}(f/g, r_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{New}(f/g, r_{m-1}) & 0 \\ 0 & 0 & \cdots & 0 & \mathrm{New}(f/g, r_m) \end{pmatrix}.$$

The previous two lemmas shows that $\deg^{\mathbf{A}^1}(f/g)$ equals the class of $\mathrm{New}(f/g)$ when $f/g = x^{\mu}/A$. The following lemma establishes equality in greater generality.

Corollary 8. *If $r \in k$ is a root of f, then $\mathrm{New}(f/g, r)$ represents $\deg_r^{\mathbf{A}^1}(f/g)$.*

Proof. Write $f(x) = (x-r)^{\mu_0}(A + (x-r)f_0(x))$ with $A \neq 0$. Then the expression

$$H_1(x,t) = \frac{(x-r)^{\mu_0}(A + t(x-r)f_0(x))}{g(x)}$$

defines a morphism $H_1 \colon U \to \mathbf{P}^1_k$ for $U = \{(x,t) \in \mathbf{P}^1_k \times_k \mathbf{A}^1_k \colon x \neq \infty, g(x) \neq 0\}$. We conclude from Lemma 4 that

$$\deg_r(f/g) = \deg_r(A(x-r)^{\mu_0}/g(x)).$$

Applying the lemma a second time to the morphism $\mathbf{P}^1_k \times_k \mathbf{A}^1_k \to \mathbf{P}^1_k$ defined by

$$H_2(x,t) = \frac{A(x-r)^{\mu_0}}{B + t(x-r)g_0(x)}$$

for $g_0(x)$ and B defined by $g(x) = B + (x-r)g_0(x)$, we conclude that $\deg_r(f/g) = \deg_r(A(x-r)^{\mu_0}/B)$. We now complete the proof using Lemmas 5 and 6. ☐

Corollary 9. *If k contains all the roots of $f(x)$, then $\mathrm{New}(f/g)$ represents the class of $\deg^{\mathbf{A}^1}(f/g)$.*

Proof. This is immediate from Corollary 8 because $\deg^{\mathbf{A}^1}(f/g)$ is the sum of the local degrees $\deg_{r_i}^{\mathbf{A}^1}(f/g)$ by Formula (3) and $\mathrm{New}(f/g)$ is the sum of the terms $\mathrm{New}(f/g, r_i)$ by definition. ☐

Lemma 10. *The matrix* $S(f/g)$ *is isomorphic to*

$$\mathrm{Tr}_{k[r_1]/k}(\mathrm{New}(f/g \otimes k[r_1], r_1)) \oplus \cdots \oplus \mathrm{Tr}_{k[r_m]/k}(\mathrm{New}(f/g \otimes k[r_m], r_m)). \tag{10}$$

Proof. First, assume k contains all the roots of $f(x)$, so Equation (10) is the bilinear form defined by $\mathrm{New}(f/g)$. We will show that $S(f/g)$ is conjugate to $\mathrm{New}(f/g)$ by manipulating the bilinear polynomial $\sum s_{i+j-1} a_i b_j$ defined by $S(f/g)$. Consider the expression

$$g(x)/f(x) \cdot \Theta(a;x)\Theta(b;x) \tag{11}$$

for Θ the polynomial

$$\Theta(a;x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{\mu-1} x^{\mu-1}.$$

The coefficient of $1/x$ in (11) is $\sum s_{i+j+1} a_i b_j$, as we see by substituting $s_1/x + s_2/x^2 + \ldots$ for $g(x)/f(x)$ and multiplying.

Alternatively the coefficient of $1/x$ is the sum of the residues at the roots of f. Here we take the residue at a root r algebraically as the coefficient of $1/(x-r)$ in a series expansion. If r is a root of $f(x)$ with multiplicity μ_0, write

$$\begin{aligned}\Theta(a;x) =& a_0 + a_1(x-r+r) + a_2(x-r+r)^2 + \ldots a_{\mu-1}(x-r+r)^{\mu-1} \\ =& (a_0 + a_1 r + a_2 r^2 + \cdots + a_{\mu-1} r^{\mu-1}) + (a_1 + 2r a_2 + \ldots)(x-r) + \cdots + a_{\mu-1}(x-r)^{\mu-1} \\ =& \Theta_0(a;r) + \Theta_1(a;r)(x-r) + \cdots + \Theta_{\mu-1}(a;r)(x-r)^{\mu-1}\end{aligned}$$

for some explicit polynomials $\Theta_0, \Theta_1, \ldots, \Theta_{\mu-1}$ that are linear in the a_i's. Substituting these expressions and $g(x)/f(x) = \frac{A_r(\mu_0)}{(x-r)^{\mu_0}} + \frac{A_r(\mu_0-1)}{(x-r)^{\mu_0-1}} + \ldots$ into (11), we get that the residue at $x = r$ is

$$\sum A_r(i+j+1)\Theta_i(a;r)\Theta_j(b;r).$$

Summing over all roots, we get that the coefficient of $1/x$ is

$$\sum A_{r_1}(i+j+1)\Theta_i(a;r_1)\Theta_j(b;r_1) + \cdots + \sum A_{r_m}(i+j+1)\Theta_i(a;r_m)\Theta_j(b;r_m).$$

As a bilinear polynomial in the Θ_i's, this is the bilinear polynomial associated to $\mathrm{New}(f/g)$. Thus if M is the change-of-basis matrix relating the a_i's to the $\Theta_i(a;r_j)$'s (i.e. a row of M is the vector built from the coefficients of a_i in $\Theta_i(a;r_j)$), then

$$S(f/g) = M^T \cdot \mathrm{New}(f/g) \cdot M. \tag{12}$$

The matrix M is invertible (M is a block diagonal matrix with blocks equal to confluence Vandermonde matrices).

When $f(x)$ has roots not contained in k, we can extend the above argument using descent theory. The theory of descent for symmetric bilinear forms is described in a high level of generality in [Knu91, Chapter III], but in the present context, we only need the basic formalism. An accessible exposition of the basic formalism for k-vector spaces with respect to a Galois extension L/k is [Con12]. Following loc. cit., suppose that L/k is a Galois extension with Galois group G and (V, β) is a finite dimensional L-vector space with nondegenerate symmetric bilinear pairing β. We define descent data (or G-structure) on (V, β) to be the data of a σ-linear map $r_\sigma \colon (V, \beta) \to (V, \beta)$ for every $\sigma \in G$ such that $r_{\sigma\tau} = r_\sigma \circ r_\tau$. Here σ-linear means $r_\sigma(av) = \sigma(a) r_\sigma(v)$ and $\beta(r_\sigma(v), r_\sigma(w)) = \sigma(\beta(v,w))$. A pair (V, β) that is obtained from a pair over k by extending scalars, i.e. $(V, \beta) = L \otimes (\overline{V}, \overline{\beta})$

for $(\overline{V}, \overline{\beta})$ a k-vector space with bilinear form, carries natural descent data, namely the descent data defined by $r_\sigma(a \otimes v) = \sigma(a) \otimes v$. For our purpose, the main result is that if there is an isomorphism $L \otimes_k (\overline{V}_1, \overline{\beta}_1) \cong L \otimes_k (\overline{V}_2, \overline{\beta}_2)$ respecting the natural descent data, then $(\overline{V}_1, \overline{\beta}_1) \cong (\overline{V}_2, \overline{\beta}_2)$. This follows from e.g. [Con12, 2.14] together with the observation that the isomorphism over k constructed in loc. cit. respects bilinear pairings.

We apply this result to $(L^{\oplus \mu}, S(f/g))$ and $(L^{\oplus \mu}, \mathrm{New}(f/g))$ to complete the proof. We can assume that the splitting field L of $f(x)$ is a Galois extension of k. Indeed, it is enough to verify the result after passing to the perfect closure $k^{p^{-\infty}}$ since extending scalars defines an injection $\mathrm{GW}(k) \to \mathrm{GW}(k^{p^{-\infty}})$. (We are assuming $\mathrm{char}\, k \neq 2$, so the map on Witt groups is injective by [Lam05, Chapter VII, Corollary 2.6] and this implies that the map on Grothendieck–Witt groups is injective since the map preserves the rank.)

In addition to assuming the splitting field is Galois, we can assume it contains the residue fields $k[r_1], \ldots, k[r_m]$. The closed points $r_1, \ldots, r_m$ then naturally correspond to roots of $f(x)$ that we will denote by the same symbols $r_1, \ldots, r_m$. After extending scalars from k to L, Equation (10) becomes isomorphic to $L^{\oplus \mu}$ with the bilinear form

$$\left(\oplus_{\sigma(r_1) \in G \cdot r_1} \mathrm{New}(f/g \otimes L, \sigma(r_1))\right) \oplus \cdots \oplus \left(\oplus_{\sigma(r_m) \in G \cdot r_m)} \mathrm{New}(f/g \otimes L, \sigma(r_m))\right) \tag{13}$$

$$= \mathrm{New}(f/g \otimes L). \tag{14}$$

by [Lam05, Chapter VII, Theorem 6.1]. The isomorphism identifies the natural descent data on (10) with the descent data on (13) associated to the permutation action of G on the roots of f. Explicitly this last descent data is the data defined by having r_σ map the factor $(L^{\oplus \mu_i}, \mathrm{New}(f/g \otimes L, \tau(r_i)))$ to $(L^{\oplus \mu_i}, \mathrm{New}(f/g \otimes L, \sigma\tau(r_i))$ by $(a_1, \ldots, a_{\mu_i}) \mapsto (\sigma(a_1), \ldots, \sigma(a_{\mu_i}))$.

The matrix M defines an isomorphism $(L^{\oplus \mu}, S(f/g)) \cong (L^{\oplus \mu}, \mathrm{New}(f/g))$ by Equation (12), and a computation shows that this isomorphism respects descent data. □

We now prove that the equation describing the behavior of the local degree with respect to field extensions holds.

Lemma 11. *Equation (2) holds. That is, if $r \in \mathbf{P}^1_k$ is a zero of the rational function f/g and $k[r]/k$ is a separable extension, then*

$$\deg_r^{\mathbf{A}^1}(f/g) = \mathrm{Tr}_{k[r]/k}(\deg_{r \otimes 1}^{\mathbf{A}^1}(f/g \otimes k[r])).$$

Proof. We will deduce the result from [KW16, Proposition 14] which is the lemma under the additional hypothesis that f/g is étale at r (i.e. r is a simple zero of $f(x)$). Consider first the case where $f/g \colon \mathbf{P}^1_k \to \mathbf{P}^1_k$ is separable and k is an infinite field. We begin by reducing to the case where f/g is étale at all zeros distinct from r. To make this reduction, write $f(x) = \pi^{\mu_0}(x) f_0(x)$ with $\pi(x)$ a prime polynomial defining r and $f_0(x)$ relatively prime to $\pi(x)$. We claim that there exists $A \in k$ such that

$$f_0(x) + A\pi(x)$$

has distinct roots. To show the claim, we introduce

$$Z := \{(x, A) \in \mathbf{A}^2_k \colon f_0(x) + A\pi(x) = f_0'(x) + A\pi'(x) = 0\}.$$

Given x, the existence of an A such that $(x, A) \in Z$ implies that $\begin{pmatrix} f_0(x) & \pi(x) \\ f_0'(x) & \pi'(x) \end{pmatrix}$ has determinant zero, so the image of Z under the first projection map is contained in $\{f_0(x)\pi'(x) - f_0'(x)\pi(x) = 0\}$. This last subscheme is a proper subscheme of $\mathbf{A}^1_k$ because it does not contain r, as π is separable by the assumption that $k[r]/k$ is separable. So the image of Z must be a finite collection of points, and therefore the same is true for Z since the projection map is finite-to-one. Now pick A to be an element not in the image of Z under the second projection map and not one of the at most finitely many points such that the derivative of $\pi^{\mu_0}(x)(f_0(x) + A\pi(x))/g(X)$ is identically zero.

By construction, the rational function $\pi^{\mu_0}(x)(f_0(x) + A\pi(x))/g(X)$ is a separable morphism and it is étale at every zero distinct from r. Furthermore, it has the same local degree at r as f/g. Indeed, the morphism

$$\mathbf{P}^1_k \times_k \mathbf{A}^1_k - \{(x, t) \colon g(x) = 0 \text{ or } x = \infty\} \to \mathbf{P}^1_k$$

defined by

$$H(x, t) = \pi(x)^{\mu_0}(f_0(x) + At\pi(x))/g(x).$$

satisfies the conditions of Lemma 4. We now replace f/g with $\pi^{\mu_0}(x)(f_0(x) + A\pi(x))/g(X)$ so that we can assume f/g is étale at every zero distinct from r.

We prove that the lemma by comparing the fiber over 0 with the fiber over a regular value. Thus pick a closed point $s \in \mathbf{P}^1_k$ with residue field k that has the property that f/g is étale at every point in the preimage of s (such a point exists because f/g is separable). Now consider the equations

$$\deg_r(f/g) + \deg_{r_1}(f/g) + \cdots + \deg_{r_m}(f/g) = \sum_{f(t)/g(t)=s} \deg_t(f/g) \text{ and} \tag{15}$$

$$\mathrm{Tr}_{k[r]/k}(\deg_r^{\mathbf{A}^1}(f/g \otimes k[r])) + \sum_i \mathrm{Tr}_{k[r_i]/k}(\deg_{r_i}^{\mathbf{A}^1}(f/g \otimes k[r_i])) = \sum_{f(t)/g(t)=s} \mathrm{Tr}_{k[t]/k}(\deg_t^{\mathbf{A}^1}(f/g \otimes k[t])) \tag{16}$$

Equation (15) holds by [KW16, Proposition 14] since both sides equal $\deg^{\mathbf{A}^1}(f/g)$. Equation (16) is a consequence of the fact that both sides of the equation are the specializations of a nonsingular bilinear form defined over $k[t]$, and any two such specializations are isomorphic [Lam06, Theorem 3.13]. Specifically, they are specializations of the bilinear form defined by $\left(s_{i+j-1}(t)\right)$ for $1/(f(x)/g(x) - t) = s_1(t)/x + s_2(t)/x^2 + s_3(t)/x^3 + \ldots$. The specialization of this matrix at $t = 0$ is the left-hand side of Equation (16) and the specialization at $t = s$ is the right-hand side by Lemma 10.

We use equations (15) and (16) to complete the proof. The right-hand side of (15) equals the right-hand side of (16) by [KW16, Proposition 14], so we conclude that the left-hand sides are also equal. Except for possibly the term $\deg_0(f/g)$ every term on the left-hand side of (15) equals corresponding term on the right-hand side, and by subtracting, we get $\deg_r^{\mathbf{A}^1}(f/g) = \mathrm{Tr}_{k[r]/k}(\deg_r^{\mathbf{A}^1}(f/g))$.

Consider now the case where k is a finite field (but we still assume f/g is separable) . In the argument just given, we used the hypothesis that k was infinite twice to assert that there exists an element of the field that avoids a finite set (we used it once when we picked a scalar $A \in k$ not in the image of Z and once when we picked a closed point $s \in \mathbf{A}^1_k$ with residue field k that avoided the critical values). This is no longer true over a finite field, but we can assert that an element exists after passing to an odd degree field extension L (chosen so that the cardinality is larger than the set to be avoided). Thus the lemma is true after passing to such an extension, and implies the lemma over k since $\mathrm{GW}(k) \to \mathrm{GW}(L)$ is an isomorphism (by an easy case of [Lam05, Chapter VII, Corollary 2.6]).

Finally, consider the case where f/g is inseparable. We can then write $f(x)/g(x) = f_0(x^{p^n})/g_0(x^{p^n})$ for some separable $f_0(x)/g_0(x)$ and $n > 0$. Since the local degree transforms composition of functions into multiplication, we have

$$\begin{aligned}
\deg^{\mathbf{A}^1}_r(f/g) &= \deg^{\mathbf{A}^1}_{r^{p^n}}(f_0/g_0) \cdot \deg_r(x^{p^n}) \\
&= \mathrm{Tr}_{k[r^{p^n}]/k}(\deg^{\mathbf{A}^1}_{r^{p^n}}(f_0/g_0 \otimes_k k[r^{p^n}])) \cdot \deg_r(x^{p^n}) \\
&= \mathrm{Tr}_{k[r^{p^n}]/k}(\deg^{\mathbf{A}^1}_{r^{p^n}}(f_0/g_0 \otimes_k k[r^{p^n}] \cdot \deg_r(x^{p^n} \otimes_k k[r^{p^n}])) \text{ by reciprocity} \\
&= \mathrm{Tr}_{k[r^{p^n}]/k}(\deg^{\mathbf{A}^1}_r(f/g).
\end{aligned}$$

Here the reciprocity referenced in the second-to-last line is Frobenius reciprocity [Lam05, Chapter VII, Theorem 1.3], which is the equality $\mathrm{Tr}_{k[r]/k}(w_1 \cdot (w_2 \otimes_k k[r])) = \mathrm{Tr}_{k[r]/k}(w_1) \cdot w_2$. ☐

Theorem 12. *The matrix* $S(f/g)$ *represents* $\deg^{\mathbf{A}^1}(f/g) \in \mathrm{GW}(k)$.

Proof. By Lemma 10, it is enough to show that

$$\mathrm{Tr}_{k[r_1]/k}(\mathrm{New}(f/g \otimes k[r_1], r_1)) \oplus \cdots \oplus \mathrm{Tr}_{k[r_m]/k}(\mathrm{New}(f/g \otimes k[r_m], r_m)).$$

equals $\deg^{\mathbf{A}^1}(f/g)$. This follows from Lemma 11 together with the fact that the global degree is the sum of the local degrees. ☐

Remark 13. *We deduce as a consequence of Corollary 9 that every nondegenerate symmetric bilinear form is equivalent to* $S(f/g)$ *for some rational function* f/g. *Indeed, suppose first that* k *is an infinite field. Every form is equivalent to a diagonal form* $\langle A_1 \rangle + \cdots + \langle A_\mu \rangle$, *and this form is the Newton matrix* $\mathrm{New}(f/g)$ *for* f/g *defined by* $g/f = \frac{A_1}{x-r_1} + \cdots + \frac{A_\mu}{x-r_\mu}$ *with* $r_1, \ldots, r_\mu$ *any sequence of distinct elements. By the corollary, the given form is equivalent to* $S(f/g)$.

When k *is finite, we instead define* f/g *by* $g/f = \frac{\pm 1}{x-1} + \frac{1}{x^{\mu-1}}$ *or* $g/f = \frac{\pm u}{x-1} + \frac{1}{x^{\mu-1}}$ *for* $u \in k^*$ *a nonsquare. (Over a finite field, two nondegenerate symmetric bilinear forms are equivalent if and only if they have the same rank and discriminant.)*

Corollary 14. *The matrix* $\mathrm{Béz}(f/g)$ *represents* $\deg^{\mathbf{A}^1}(f/g)$.

Proof. It is enough to show that $\text{Béz}(f/g)$ is congruent to $S(f/g)$. Writing $f(x) = x^{\mu} + a_1 x^{\mu-1} + \cdots + a_{\mu}$, we compute

$$
\begin{aligned}
\frac{f(x)g(y) - f(y)g(x)}{x-y} &= f(x)f(y)\frac{\frac{g(y)}{f(y)} - \frac{g(x)}{f(x)}}{x-y} \\
&= f(x)f(y)\frac{\sum_{i=1}^{\infty} s_i (y^{-i} - x^{-i})}{x-y} \\
&= f(x)f(y)\frac{\sum_{i=1}^{\infty} s_i (x^{i} - y^{i})x^{-i}y^{-i}}{x-y} \\
&= f(x)f(y)\sum_{i,j=1}^{\infty} s_{i+j-1} x^{-i} y^{-j} \\
&= \sum_{i,j=1}^{\infty} s_{i+j-1}(x^{\mu-i} + a_1 x^{\mu-i-1} + \cdots + a_{\mu} x^{-i})(y^{\mu-j} + a_1 y^{\mu-j-1} + \cdots + a_{\mu} y^{-j}). \qquad (17)
\end{aligned}
$$

The above expression equals $\sum b_{i,j} x^{i-1} y^{j-1}$, so the terms in (17) where x or y appears with negative degree must cancel out, and we deduce

$$\sum_{i,j=1}^{\mu} b_{i,j} x^{i-1} y^{j-1} = \sum_{i,j=1}^{\mu} s_{i+j-1}(x^{\mu-i} + a_1 x^{\mu-i-1} + \cdots + a_{\mu-i})(y^{\mu-j} + a_1 y^{\mu-j-1} + \cdots + a_{\mu-j}).$$

This shows that the relevant change-of-basis matrix conjugates $\text{Béz}(f/g)$ to $S(f/g)$. $\square$

Let us compare the argument just given with Kreĭn–Naĭmark's treatment of Hurwitz's result in [KN81]. We deduced the result that $\text{Béz}(f/g)$ represents $\deg^{\mathbf{A}^1}(f/g)$ (Corollary 14) from the analogous result about $S(f/g)$ (Theorem 12) in the same way that Kreĭn–Naĭmark do on [KN81, page 277]. Our proof of the result about $S(f/g)$ is similar to the argument on [KN81, pages 280–282] but with two significant differences. First, both we and Kreĭn–Naĭmark compute the class of $S(f/g)$ by passing to a splitting field L for $f(x)$ and working with $\text{New}(f/g \otimes L)$, but the details are different. Kreĭn–Naĭmark only consider $k = \mathbf{R}$, and they compute the signature of $S(f/g)$ from $\text{New}(f/g \otimes L)$ by observing that a complex conjugate pair of roots corresponds to a summand of $S(f/g)$ with signature zero. We allow k to be arbitrary (with $\operatorname{char} k \neq 2$), and in contrast to the case $k = \mathbf{R}$, a Galois orbit of roots can contribute to the class of $S(f/g)$ in a complicated way, so instead of directly computing we use descent theory.

The second difference occurs in our proof of Corollary 8, which identifies the local degree $\deg_r^{\mathbf{A}^1}(f/g)$ with the class of $\text{New}(f/g, r)$. The corollary is deduced from Lemmas 5 and 6. Lemma 6 is the computation of the class of $\text{New}(f/g, r)$, and the computation is the same as the one appearing on [KN81, page 281]. Lemma 5, however, does not have a direct analogue in [KN81]. That lemma computes the local degree as the expression in Lemma 6, and for Kreĭn–Naĭmark, the analogous fact does not need a proof since it is their definition of the local degree (or rather local Cauchy index; see Section 2.1 below).

As we explained in the introduction, the proof just given is different from Cazanave's result that $\text{Béz}(f/g)$ represents $\deg^{\mathbf{A}^1}(f/g)$. Furthermore, Cazanave proves the stronger result that $\mathrm{Béz}(f/g)$ represents the unstable homotopy class of f/g. This is not a consequence of our proof. The reason is clearly demonstrated by the special case where $f(x)$ has μ distinct roots defined over k. We deduced the result about $\mathrm{Béz}(f/g)$ from the analogous result about $\mathrm{New}(f/g)$, and $\mathrm{New}(f/g)$ does not represent the unstable homotopy class of f/g. Indeed, the unstable homotopy class is determined by $\deg^{\mathbf{A}^1}(f/g)$ and a scalar $d(f/g) \in k^*$ that represents the discriminant of $\deg^{\mathbf{A}^1}(f/g)$. The Newton matrix is a diagonal matrix with diagonal entries $g(r_1)/f'(r_1), \dots, g(r_\mu)/f'(r_\mu)$, so

$$\begin{aligned} \det(\mathrm{New}(f/g)) =& g(r_1)/f'(r_1) \cdots g(r_\mu)/f'(r_\mu) \text{ but} \\ \det(\mathrm{Béz}(f/g)) =& (-1)^{\mu(\mu-1)/2}\,\mathrm{Resultant}(f,g) \\ =& \mathrm{Disc}(f) \cdot g(r_1)/f'(r_1) \cdots g(r_\mu)/f'(r_\mu). \end{aligned} \tag{18}$$

Here $\mathrm{Resultant}$ is the resultant, and Disc the discriminant.

Equation (18) shows that, unlike $\deg^{\mathbf{A}^1}(f/g)$, $d(f/g)$ is not determined by the derivatives of $f(x)/g(x)$ at the roots. For example, both $f_1(x)/g_1(x) = x^2 - x$ and $f_2(x)/g_2(x) = (x^2-1)/2$ have two simple zeros at which the values of the derivative are $+1$ and -1 respectively, but the functions are not unstably homotopic since $d(f_1/g_1) = -1$ but $d(f_2/g_2) = -4$. This should not be surprising as the formula relating the global degree to a sum of local degrees is proven using stable techniques.

2.1. **References.** The authors used Kreĭn–Naĭmark's survey [KN81] as the primarily source on Hurwitz's theorem. The proof in loc. cit. is essentially the same as Hurwitz's proof in [Hur95, Section 3]. Other treatments of the result include [Gan64, page 210, Theorem 9], [RS02, Theorem 10.6.5], [BPR06, Theorem 9.4], and [Fuh12, Theorem 8.59]. All of these references do not discuss the topological degree and instead discuss the Cauchy index, an equivalent invariant. The local Cauchy index $\mathrm{ind}_r(f/g)$ at a pole $r \in \mathbf{R}$ of f/g (i.e. at a root of g) is

$$\mathrm{ind}_r(f/g) := \begin{cases} +1 & \text{if } f/g \text{ jumps from } -\infty \text{ to } +\infty \text{ at } r; \\ -1 & \text{if } f/g \text{ jumps from } +\infty \text{ to } -\infty \text{ at } r; \\ 0 & \text{otherwise.} \end{cases}$$

The global Cauchy index $\mathrm{ind}(f/g)$ is the sum of the local Cauchy indices. The local Cauchy index of f/g at r equals the local topological of the *reciprocal* function g/f and similarly with the global degree.

3. Connection with work of Eisenbud–Levine and Khimshiashvili

Here we explain how Cazanave's description of $\deg^{\mathbf{A}^1}(f/g)$ as the Bézout matrix is related to the signature formula of Eisenbud–Levine and Khimshiashvili, as well as results of Palamodov and the present authors.

The signature formula computes the local topological degree of the germ $h_0 \colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ of a C^∞-function as the signature of a bilinear form. The formula applies when h_0 has the property that the local algebra

$$Q_0(h_0) := C_0^\infty(\mathbf{R}^n)/I(h_0). \tag{19}$$

has finite length. Here $C_0^\infty(\mathbf{R})$ is the ring of germs of smooth real-valued functions on $\mathbf{R}^n$ based at 0 and $I(h_0)$ is the ideal generated by the components of h_0.

The bilinear form appearing in the signature formula is defined in terms of the Jacobian element $J := \det(\frac{\partial f_i}{\partial x_j}) \in Q_0(h_0)$. The formula states that if $\eta\colon Q_0(h_0) \to \mathbf{R}$ is a $\mathbf{R}$-linear function satisfying $\eta(J) > 0$, then the symmetric bilinear pairing $\beta_0(\eta)$ on $Q_0(h_0)$ defined by

$$\beta_0(\eta)(a, b) = \eta(ab)$$

satisfies

$$\deg_0^{\mathbf{R}}(h_0) = \text{signature of } \beta_0(\eta). \tag{20}$$

This is [EL77, Theorem 1.2] and [Khi77]; see [Khi01] and [AGZV12, Chapter 5] for recent expositions.

Earlier Palamodov [Pal67, Corollary 4] proved a parallel result that computes the topological degree $\deg^{\mathbf{C}}(h_0)$ of the complexification $h_0 \otimes \mathbf{C}\colon \mathbf{C}^n \to \mathbf{C}^n$ as

$$\deg^{\mathbf{C}}(h_0) = \text{rank of } \beta_0(\eta) \tag{21}$$

under the assumption that h_0 is analytic (so a complexification exists).

In the special case where $n = 1$ and h_0 is the germ of a rational function f/g, the signature formula is closely related to Hurwitz's theorem. To explain the connection, we need to introduce variants of the algebra $Q_0(h_0)$.

Definition 15. *If k is a field and $f(x)/g(x) \in \operatorname{Frac} k[x]$ is a nonzero rational function, then define the **algebra of** f/g to be $Q(f/g) := k[x, g^{-1}]/(fg^{-1})$. If $r \in \mathbf{P}^1_k$ is a closed point in the preimage of 0 under f/g, then we define the **local algebra** $Q_r(f/g)$ **of** f/g **at** r to be the localization of $Q(f/g)$ at the corresponding maximal ideal.*

The algebra $Q(f/g)$ is the same as the quotient ring $k[x]/(f)$, but we are writing it as the quotient of $k[x, g^{-1}]$ by the ideal (fg^{-1}) because the structure on $Q(f/g)$ that we will introduce depends on the presentation $Q(f/g)$ as a quotient of $k[x, g^{-1}]$ and not just the algebra itself.

The notation $Q_0(f/g)$ is chosen to suggest a relation with the algebra $Q_0(h_0)$ studied by Eisenbud et al. When $k = \mathbf{R}$ and h_0 is the restriction of a rational function f/g, the inclusion of the ring of rational functions in the ring of germs of smooth functions induces an isomorphism $Q_0(f/g) \cong Q_0(h_0)$.

The algebra $Q(f/g)$ admits a distinguished functional $\operatorname{Res}\colon Q(f/g) \to k$ called the residue functional (or generalized trace) that is constructed using local duality. The functional is easiest to describe when k is a subfield of the complex number $\mathbf{C}$. In this case, the functional is defined in terms of residues from complex analysis:

$$\operatorname{Res}(a) = \sum_{f(r)=0} \frac{1}{2\pi i} \oint_{\Gamma_r} g(z)/f(z) \cdot a(z)\, dz$$

for Γ_r a small positively oriented circle around r.

We are primarily interested in the bilinear pairing $\beta(\mathrm{Res})$ that the residue functional defines. The matrix $S(f/g)$ from the last section is nothing other than a Gram matrix for $\beta(\mathrm{Res})$. This should not be too surprising. We essentially showed this in the proof of Theorem 12. There we computed the bilinear polynomial associated to $S(f/g)$ as the coefficient of $1/x$ in $g(x)/f(x)\cdot\Theta(a;x)\Theta(b;x)$ or, in other words, as the sum of the residues of $g(x)/f(x)\cdot\Theta(a;x)\Theta(b;x)$. The most natural identification of $S(f/g)$ as a Gram matrix uses general results of Scheja–Storch, as we now explain.

Scheja–Storch described Res more generally over an arbitrary field using commutative algebra in [SS75]. In the present context, the key result is their description of the bilinear pairing $\beta(\mathrm{Res})$ in terms of the element

$$\Delta := \frac{f(x)/g(x) - f(y)/g(y)}{x-y}.$$

They show that if $\nu_1, \ldots, \nu_\mu$ is a k-basis for $Q(f/g)$ and

(22) $\qquad \Delta = \sum b_{i,j}\nu_i(x)\nu_j(y)$ modulo $f(x)/g(x)$ and $f(y)/g(y)$, then .

(23) $\qquad (b_{i,j}) =$ the Gram matrix of β with respect to the basis dual to $\nu_1, \ldots, \nu_\mu$.

This is not explicitly stated by Scheja–Storch, but it can be derived from the formula on [SS75, page 182, bottom]. That formula states that if $\nu_1^*, \ldots, \nu_\mu^*$ is the basis dual to $\nu_1, \ldots, \nu_\mu$ with respect to $\beta(\mathrm{Res})$, then $\Delta = \nu_1(x)\nu_1^*(y) + \cdots + \nu_\mu(x)\nu_\mu^*(y)$. The two bases are related by $\nu_i = \sum b_{i,j}\nu_j^*$, and the equality (23) follows from substituting $\sum b_{i,j}\nu_j^*(x)$ for $\nu_i(x)$ in $\sum \nu_i(x)\nu_i^*(y)$. (Strictly speaking Scheja–Storch only work with $Q(f/g)$ when $g = 1$ because they work with quotients of $k[x]$, but their argument applies to the localization $k[x, 1/g]$ with only notational changes.)

The symmetric matrices studied in the earlier sections are nothing other than Gram matrices for β with respect to distinguished bases that we now define.

Definition 16. *Define the **monomial basis** of* $Q(f/g)$ *to be* $1/g, x/g, \ldots, x^{\mu-1}/g$.

Write $f(x) = x^\mu + a_1x^{\mu-1} + \ldots a_{\mu-1}x + a_\mu$ *and define the **Horner basis** of* $Q(f/g)$ *to be* $(x^{\mu-1} + a_1x^{\mu-2} + \cdots + a_{\mu-1})/g$, $(x^{\mu-2} + a_1x^{\mu-3} + \cdots + a_{\mu-2})/g$, ..., $(x + a_1)/g$, $1/g$.

If f *factors into linear factors* $f(x) = (x-r_1)^{\mu_1} \ldots (x-r_m)^{\mu_m}$ *over* k*, then the define the **Newton basis** to be* $\frac{f(x)}{(x-r_1)g(x)}, \frac{f(x)}{(x-r_1)^2 g(x)}, \ldots, \frac{f(x)}{(x-r_1)^{\mu_1-1}g(x)}, \frac{f(x)}{(x-r_2)g(x)}, \ldots, \frac{f(x)}{(x-r_m)^{\mu_m-1}g(x)}$.

Table 1 identifies $\mathrm{Béz}(f/g)$, $S(f/g)$, and $\mathrm{New}(f/g)$ as Gram matrices. Indeed, the computations from Section 2 essentially show that the entries of the matrices satisfy (22). We get the identification of the Bézout matrix by dividing (1) by $g(x)g(y)$. If we instead divide (17), we get the identification of $S(f/g)$. Replacing the use of $g/f = \sum s_i/x^i$ with $g/f = \sum A_{r_i}(j)/(x-r_i)^j$ in (17) identifies $\mathrm{New}(f/g)$, and we explain this last identification in detail in the proof of Theorem 17 below.

We point out that this identification of $\mathrm{Béz}(f/g)$ as a Gram matrix is different from the identification that appears in [Caz12]. The relevant text is Lemma C.2 of the preprint version of loc. cit. that is available on the online repository the arXiv as arXiv:0912.2227v2.

TABLE 1. Gram Matrices

Gram matrix	Dual basis
$\mathrm{Béz}(f/g)$	Monomial basis
$S(f/g)$	Horner basis
$\mathrm{New}(f/g)$	Newton basis

(The lemma does not appear in the published version.) In the lemma, Cazanave identifies the inverse matrix $\mathrm{Béz}(f/g)^{-1}$, and hence $\mathrm{Béz}(f/g)$, as a Gram matrix in an important special case, namely the case where $f(x)$ and $g(x)$ have indeterminant coefficients and k is a suitable algebraic closure. His discussion, however, applies more generally to the case where $f(x)$ has only simple roots (so $Q(f/g)$ is étale). He shows that $\mathrm{Béz}(f/g)^{-1}$ is a Gram matrix for the bilinear pairing $\beta(\phi)$ on $Q(f/g)$ defined by functional

$$\phi(a) = \mathrm{Tr}\left(\frac{a}{f'(x)g(x)}\right). \tag{24}$$

Here $\mathrm{Tr}\colon Q(f/g) \to k$ is the algebra trace. Observe that the expression $\frac{s}{f'(x)g(x)}$ is well-defined because we assumed $f(x)$ has distinct roots (so $f'(x)$ is a unit in $Q(f/g)$).

The functional ϕ is not equal to Res. Indeed, when $f(x)$ has simple roots, Scheja–Storch show

$$\mathrm{Res}(a) = \mathrm{Tr}\left(\frac{ag(x)}{f'(x)}\right)$$

This is [SS75, (4.2) Satz] (specifically, it is equivalent to the equality $\mathrm{Sp} = J \cdot \eta$ in loc. cit.). The two functionals do, however, define isomorphic symmetric bilinear forms since multiplication by $1/g(x)$ transforms $\beta(\mathrm{Res})$ into $\beta(\phi)$.

Having explained the connection between the residue pairing $\beta(\mathrm{Res})$ and Hurwitz's theorem, we now explain the relation to the signature formula. Since $Q(f/g)$ is an artin algebra, $Q_0(f/g)$ is a direct summand of $Q(f/g)$. Scheja–Storch show that the restriction Res_0 of the residue functional satisfies $\mathrm{Res}_0(J) = \mathrm{length}\, Q_0(f/g)$, so in particular, the restriction $\beta_0(\mathrm{Res}_0)$ of $\beta(\mathrm{Res})$ to $Q_0(f/g)$ satisfies the hypothesis of the signature formula. We conclude that

$$\deg_0^{\mathbf{R}}(f/g) = \text{ the signature of } \beta_0(\mathrm{Res}_0).$$

In this manner, applied to a rational function f/g, the signature formula is the local analogue of Hurwitz's theorem.

Cazanave's theorem is an enrichment of Hurwitz's theorem, and it implies the following analogous enrichment of the signature formula:

Theorem 17 (The $n = 1$ case of the main result of [KW16]). *We have*

$$\deg_0^{\mathbf{A}^1}(f/g) = \beta_0(\mathrm{Res}_0) \text{ in } \mathrm{GW}(k). \tag{25}$$

Proof. The results we have proven so far show that both sides of (25) are represented by the Newton matrix $\mathrm{New}(f/g, 0)$. The Newton matrix represents the left-hand side by Corollary 8. To identify the right-hand side with the Newton matrix, it is enough to find a suitable expression for the image of Δ in $Q_0(f/g) \otimes_k Q_0(f/g)$. We derive this expression by substituting the partial fractions decomposition (8) for $g(x)/f(x)$ into the definition of Δ:

$$\begin{aligned}\frac{f(x)/g(x)-f(x)/g(y)}{x-y} =& \frac{g(y)/f(y)-g(y)/f(x)}{x-y}\cdot\frac{f(x)f(y)}{g(x)g(y)}\\ =&(A(\mu)\frac{1/y^{\mu}-1/x^{\mu}}{x-y}+A(\mu-1)\frac{1/y^{\mu-1}-1/x^{\mu-1}}{x-y}\\ &+\cdots+A(1)\frac{1/y-1/x}{x-y}+\text{higher order terms})\cdot\frac{f(x)f(y)}{g(x)g(y)}\\ =&(A(\mu)(x^{-1}y^{-\mu}+x^{-2}y^{-(\mu-1)}+\cdots+x^{-(\mu-1)}y^{-2}+x^{-\mu}y^{-1})\\ &+A(\mu-1)(x^{-1}y^{-(\mu-1)}+x^{-2}y^{-(\mu-2)}+\cdots+x^{-(\mu-2)}y^{-2}+x^{-(\mu-1)}y^{-1})\\ &+\cdots+A(1)x^{-1}y^{-1}+\text{higher order terms})\cdot\frac{f(x)f(y)}{g(x)g(y)}\\ =&\sum_{i,j=1}^{\mu}A(i+j-1)\frac{f(x)}{g(x)x^{i}}\frac{f(y)}{g(y)y^{j}}\text{ in }Q_0(f/g)\otimes_k Q_0(f/g).\end{aligned}$$

(The higher order terms can be omitted since they are multiples of $f(x)/g(x)$ which equals zero in $Q_0(f/g)\otimes_k Q_0(f/g)$.)

This last expression shows that the entries of the Newton matrix satisfy (22), so we conclude that it is the Gram matrix for the restriction of the residue pairing to $Q_0(f/g)$. □

Equality (25) answers a question of Eisenbud for polynomial maps of 1 variable. Eisenbud, in a survey article on his work with Levine, observed that $\beta_0(\mathrm{Res}_0)$ is defined for a polynomial map $f\colon \mathbf{A}^n_k \to \mathbf{A}^n_k$ with an isolated zero at the origin when k is an arbitrary field of odd characteristic. He then proposed β_0 as the definition of the degree and asked if this degree has an interpretation or usefulness, say in cohomology theory [Eis78, Some remaining questions (3)].

In this paper we have shown that, in odd characteristic, the stable isomorphism class of $\beta_0(\mathrm{Res}_0)$ is the local degree in $\mathbf{A}^1$-homotopy theory when f is a polynomial map of 1 variable. In the companion paper [KW16], the present authors answer Eisenbud's question in full generality: the main theorem of that paper (Theorem 3) states that (25) remains valid when f/g is replaced by a polynomial map $f\colon \mathbf{A}^n_k \to \mathbf{A}^n_k$ in any number of variables that has an isolated zero at the origin.

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