Quadratically enriched binomial coefficients over a finite field

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Abstract. We compute an analogue of Pascal's triangle enriched in bilinear forms over a finite field. This gives an arithmetically meaningful count of the ways to choose j embeddings into an algebraic closure from an étale extension of degree n. We also compute a quadratic twist. These (twisted) enriched binomial coefficients are defined in joint work of Brugallé and the second-named author, building on work of Garibaldi, Merkurjev, and Serre. Such binomial coefficients support curve counting results over non-algebraically closed fields, using $\mathbb{A}^1$-homotopy theory.

1. Introduction

We consider combinatorics enriched in bilinear forms, in the sense that an integer n is replaced by the class of a symmetric, non-degenerate bilinear form on a vector space of dimension n. The resulting binomial coefficients arose in [**1**] in the context of curve counting over non-algebraically closed fields: to count curves on surfaces, one is lead to certain degeneration formulas in which curves of lower degrees are glued together. To perform such glueings, one chooses closed points. However, over non-algebraically closed fields, the points have different residue fields. To obtain a count retaining arithmetic information, it is effective to replace integer valued counts with ones enriched in bilinear forms. The counts now take values in the Grothendieck–Witt group of the base field, defined to be the group completion of isomorphism classes of symmetric, non-degenerate bilinear forms. See for example [**6**] [**10**] [**14**] [**13**] [**12**] [**3**]. In [**1**], Erwan Brugallé and the second named author obtained a wall-crossing formula for $\mathbb{A}^1$ Gromov–Witten invariants using combinatorial identities enriched in bilinear forms. Here we systematically compute the analogue of Pascal's triangle over a finite field of odd characteristic and a quadratic twist, arising in the enumerative context of [**1**],[**7**].

Let $k = \mathbb{F}_q$ be a finite field with odd characteristic. Let $\mathrm{GW}(k)$ denote the Grothendieck–Witt group of k and let $u \in \mathrm{GW}(\mathbb{F}_q)$ denote the class of the bilinear form $\mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q$ sending (x, y) to μxy, where μ is a nonsquare in $\mathbb{F}_q^*$. There is a ring structure on $\mathrm{GW}(k)$ induced from the tensor product of forms, and the unit, denoted by 1, is represented by the bilinear form $\mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q$ sending (x, y) to xy. For an étale k-algebra L, the paper [**1**] defines $\binom{L/k}{j}$ in $\mathrm{GW}(k)$ to be the trace form of the étale k-algebra associated via the Galois correspondence to the set of subsets of size j of the set of k-maps of L into $\overline{k}$. Here, $\overline{k}$ denotes the algebraic closure of k.

For a quadratic extension Q of k and $[L : k] = 2j$, this set has a twisted action, where $\mathrm{Gal}(Q/k)$ acts by taking the complement of the subset. This defines a quadratically twisted binomial coefficient $\binom{L[Q]/k}{j}$ in $\mathrm{GW}(k)$. We give these definitions in detail in Section 2.

In this paper, we show the following closed formula for the quadratically enriched binomial coefficients over $\mathbb{F}_q$.

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THEOREM 1.1. *Let q be an odd prime power and let j be a non-negative integer. Let L/k be the finite extension of $\mathbb{F}_q$ of degree n. Then*

$$\binom{L/k}{j} = \binom{n}{j} - (1-u) \cdot \binom{\frac{n-2}{2}}{\frac{j-1}{2}} \in \mathrm{GW}(\mathbb{F}_q),$$

where u is the non-square class in $\mathrm{GW}(k)$ and our convention is that $\binom{a}{b} := 0$, if either a or b is not in $\mathbb{Z}$.

We also compute the quadratically twisted enriched binomial coefficients over a finite field:

THEOREM 1.2. *Let q be an odd prime power and let j be a non-negative integer. Let L/k be the finite field extension of $\mathbb{F}_q$ of degree $2j$, and let Q/k the degree 2 field extension of $\mathbb{F}_q$. Then*

$$\binom{L[Q]/k}{j} = \frac{1}{2}\binom{2j}{j} \cdot (1+u) \in \mathrm{GW}(\mathbb{F}_q).$$

1.1. Summary of the proof. The idea of the proofs is first to rephrase the problems into purely combinatorial forms. In Section 3.1, we show the proofs of these two theorems reduce to calculations of the parities of the number of orbits of necklaces of even cardinality under different cyclic group actions.

1.1.1. *Untwisted case.* The proof of Theorem 1.1 in the case when n is odd is very straightforward, as shown in Proposition 3.2. In Section 3.4, we prove Theorem 1.1 when j is odd. The proof involves a direct calculation using Möbius inversion and Lucas's theorem. The remaining task is to show that when both n and j are even, the enumeration is always even, which is established combinatorially.

In Section 3.5, we use a C_2-action, referred to as flipping, on the set of cyclic orbits of necklaces. Since we are concerned only with the parity of the enumeration, it suffices to count the C_2-fixed points. Next, in Section 3.6, we introduce the concept of symmetry axes for the orbits of necklaces, which allows us to rewrite the set of C_2-fixed (symmetric) orbits as a non-disjoint union of two subsets: the sets of symmetric orbits with a symmetry axis passing through two beads (we call this a symmetry axis of type 1) and those with a symmetry axis passing between two pairs of beads (type 2). The enumeration methods for these two subsets differ.

In Section 3.6, we also decompose each cyclic orbit of necklaces into two smaller necklace orbits. This decomposition exhibits special properties when the orbit is symmetric, as different types of symmetry axes yield different properties. This decomposition induces a map whose codomain is the symmetric product of the sets of two smaller cyclic orbits of necklaces. The enumeration is carried out by summing the fibers of this map over the codomain. At the end of Section 3.6, we provide the enumeration of symmetric orbits with a type 1 symmetry axis when both n and j are even.

Using this approach, in Section 3.7, we enumerate the symmetric orbits with a type 2 symmetry axis for the case where $n \equiv 2 \mod 4$ and j is even. Combining this result with the results in Section 3.6, we prove Theorem 1.1 for the case where $n \equiv 2 \mod 4$ and j is even by calculating the parity of the enumeration using Kummer's theorem.

In Section 3.8, we consider the case where $n \equiv 0 \mod 4$. For a symmetric orbit with a type 2 symmetry axis, we show how to reduce it to a symmetric orbit with a type 1 symmetry axis. Since $n-2 \equiv 0 \mod 4$, we can now utilize the results from Section 3.6 and Section 3.7 to conclude the proof of Theorem 1.1 for the case where $n \equiv 0 \mod 4$ and j is even. This completes the proof of the untwisted case.

1.1.2. *Twisted case.* We prove Theorem 1.2 by reducing to an untwisted enumeration that we have studied in detail previously. In Section 4.1, we first provide a description of the twisted orbits in terms of untwisted orbits. Next, we construct a C_2-action, called swapping, on the set of twisted orbits and reduce the problem to the enumeration of the swapping fixed points. Then, by studying the swapping fixed twisted orbits, we reduce the problem to an enumeration of untwisted orbits under certain conditions.

In Section 4.2, we discuss another way to encode the information of an untwisted cyclic orbit of necklaces. This is the marked cyclic equivalence class of partitions. The condition for the untwisted orbits we need to enumerate can be rephrased in terms of the properties of the partitions, which allows us to calculate the parity of the enumeration using partition theory. This concludes the proof of Theorem 1.2.

More explicit forms of the untwisted and twisted binomial coefficients over a finite field are presented in Section 3.9 and Section 4.4, respectively.

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2. Quadratically enriched binomial coefficients over a field

The paper [**1**] defined quadratically enriched (twisted) binomial coefficients over a base scheme. We recall the definition in the case where the base scheme is a field.

Let k be a field, and fix a separable closure k^s of k. For a finite separable field extension $k \subseteq L$, let $\mathrm{Emb}_k(L, k^s)$ denote the set of maps of fields of L into k^s over k

$$\mathrm{Emb}_k(L, k^s) = \left\{ f : \begin{array}{ccc} L & \xrightarrow{\quad f \quad} & k^s \\ & \nwarrow \quad \nearrow & \\ & k & \end{array} \right\}.$$

Recall that a finite étale k-algebra $k \to E$ has an associated trace map

$$\mathrm{Tr}_{E/k} : E \to k$$

which takes an element e to the trace of the matrix associated to multiplication by e, viewed as an endomorphism of the finite dimensional k-vector space E. This trace map determines an element of $\mathrm{GW}(k)$ denoted $\mathrm{Tr}_{E/k}\langle 1 \rangle$ and defined to be the class of the bilinear form

$$E \times E \to E \to k$$

where the first map is multiplication on E and the second is $\mathrm{Tr}_{E/k}$.

Recall that the Galois correspondence

$$E \mapsto \mathrm{Emb}_k(E, k^s)$$

gives an equivalence of categories from finite étale k-algebras to finite sets equipped with a $\mathrm{Gal}(k^s/k)$-action.

To enrich binomial coefficients, we introduce the following notation. Let $\mathrm{Emb}_k^j(L, k^s)$ denote the subset of $\mathrm{Emb}_k(L, k^s)$ consisting of the set of subsets of size j.

Definition 2.1. [**1**] Let

$$\binom{L/k}{j} \in \mathrm{GW}(k)$$

denote the class of the trace form of the finite étale k-algebra corresponding to the finite set $\mathrm{Emb}_k^j(L, k^s)$ with $\mathrm{Gal}(k^s/k)$-action induced from the canonical action on $\mathrm{Emb}_k(L, k^s)$.

For $j = 2, 3$, these forms appeared in [**4**, 30.12-30.14]. In [**1**], it was important to twist such quadratically enriched binomial coefficients by a degree 2-field extension $k \subset Q$ as follows. There is a canonical isomorphism $\mathrm{Gal}(Q/k) \cong C_2$, where C_2 denotes the cyclic group of order 2. Let

$$q_Q : \mathrm{Gal}(k^s/k) \to \mathrm{Gal}(Q/k) \cong C_2$$

denote the corresponding quotient map.

For a set S of size $2j$, the set of subsets of S of size j has an action of C_2 given by taking a subset to its complement. If a group G acts on S, the set of subsets of size j inherits an action of $G \times C_2$ because taking a set to its complement commutes with any automorphism of S.

DEFINITION 2.2. [**1**] For a finite étale k-algebra L, let

$$\binom{L[Q]/k}{j} \in \mathrm{GW}(k)$$

denote the class of the trace form of the étale k-algebra corresponding to $\mathrm{Emb}_k^j(L, k^s)$ with the $\mathrm{Gal}(k^s/k)$-action given by the homomorphism

$$\mathrm{Gal}(k^s/k) \overset{(1,q_Q)}{\to} \mathrm{Gal}(k^s/k) \times C_2$$

and the canonical action of $\mathrm{Gal}(k^s/k) \times C_2$ on $\mathrm{Emb}_k^j(L, k^s)$.

3. Untwisted case - Proof of Theorem 1.1

Let $k = \mathbb{F}_q$ be a finite field of odd characteristic. We have

$$\mathrm{GW}(\mathbb{F}_q) \cong \frac{\mathbb{Z}[u]}{(u^2-1, 2-2u)} \cong \frac{\mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot u}{2-2u} \cong \mathbb{Z} \times \mathbb{F}_q^*/(\mathbb{F}_q^*)^2,$$

where the first isomorphism is an isomorphism of rings, the second and third only respect group structures, and the third isomorphism is the product of the rank and the discriminant. Note that $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2 \cong \mathbb{Z}/2\mathbb{Z}$. Here as above, u denotes the class of the bilinear form $k \times k \to k$ given by $(x, y) \mapsto \mu xy$ where μ in $\mathbb{F}_q$ is a non-square. See for example [**9**, Theorem 3.5, Corollary 3.6]. In particular, the class of any element in $\mathrm{GW}(\mathbb{F}_q)$ is determined by the rank and the discriminant.

Let L/k be a finite extension of degree n. Then $\mathrm{Gal}(L/k) \cong C_n$, where C_n is the cyclic group of order n that is generated by the Frobenius automorphism $\varphi : x \to x^q$. It is a classical fact that for $[L : k] = n$, the class of the trace form of L over k in $\mathrm{GW}(k)$ is given:

$$\mathrm{Tr}_{L/k}\langle 1\rangle = \epsilon(n) := \begin{cases} n-1+u, & n \equiv 0 \mod 2 \\ n, & n \equiv 1 \mod 2 \end{cases} \tag{3.1}$$

Indeed, the rank of $\mathrm{Tr}_{L/k}\langle 1\rangle$ is n, so it suffices to compute the discriminant of $\mathrm{Tr}_{L/k}\langle 1\rangle$. The discriminant of the trace form is the discriminant of the minimal polynomial of a generator. Since the Fröbenius acts as a n cycle on the roots of a minimal polynomial for a generator of $k \subseteq L$, we have that $\mathrm{Tr}_{L/k}\langle 1\rangle$ has square discriminant if and only if an n cycle has even sign, which occurs if and only if n is odd. For more on trace forms, see [**2**]. A proof of the classical fact (3.1) can be found in Lemma 58 of the first ArXiv version of [**6**].

Let $E_k^j(L, k^s)$ denote the finite étale k-algebra that is associated to $\mathrm{Emb}_k^j(L, k^s)$ under the Galois correspondence. Here, $\mathrm{Emb}_k^j(L, k^s)$ denotes the set of subsets of size j of the set of embeddings of L into k^s over k, as in Definition 2.1. In particular, $\binom{L/k}{j} = \mathrm{Tr}_{E_k^j(L,k^s)/k}\langle 1\rangle$. We have $\dim_k E_k^j(L, k^s) = |\mathrm{Emb}_k^j(L, k^s)| = \binom{n}{j}$.

For a group G acting on a set S, let $\mathrm{Orb}(G, S)$ denote the set of orbits. The set $\mathrm{Emb}_k^j(L, k^s)$ can be decomposed into Galois orbits $\mathrm{Emb}_k^j(L, k^s) = \amalg_{i\in I}\mathfrak{O}_i$ with

$$I = \mathrm{Orb}(\mathrm{Gal}(k^s/k), \mathrm{Emb}_k^j(L, k^s))$$

being a finite set. Let $H_i \lhd \mathrm{Gal}(k^s/k)$ be the stabilizer subgroup of $\mathfrak{O}_i$ and let

$$d_i = [\mathrm{Gal}(k^s/k) : H_i] = |\mathfrak{O}_i|$$

be the number of elements in the orbit $\mathfrak{O}_i$. Then $E_k^j(L,k^s) = \prod_{i\in I}(k^s)^{H_i}$, and

$$\mathrm{Tr}_{E_k^j(L,k^s)/k}\langle 1\rangle = \sum_{i\in I}\mathrm{Tr}_{(k^s)^{H_i}/k}\langle 1\rangle = \sum_{i\in I}\epsilon(d_i).$$

For a group G acting on a set S, let $\mathrm{Orb}_{even}(G,S) \subseteq \mathrm{Orb}(G,S)$ denote the subset consisting of those orbits with an even cardinality. Define $\mathrm{Orb}_{odd}(G,S)$ similarly.

The set $\mathrm{Orb}(\mathrm{Gal}(k^s/k), \mathrm{Emb}_k^j(L,k^s))$ decomposes

$$\mathrm{Orb}(\mathrm{Gal}(k^s/k), \mathrm{Emb}_k^j(L,k^s)) = GO_{odd}(L/k,j) \amalg GO_{even}(L/k,j),$$

where we use the abbreviations

$$\begin{aligned} GO_{odd}(L/k,j) :=& \mathrm{Orb}_{odd}(\mathrm{Gal}(k^s/k), \mathrm{Emb}_k^j(L,k^s)),\\ GO_{even}(L/k,j) :=& \mathrm{Orb}_{even}(\mathrm{Gal}(k^s/k), \mathrm{Emb}_k^j(L,k^s)). \end{aligned}$$

Let $\Delta(n,j)$ denote the difference $\Delta(n,j) := \binom{L/k}{j} - \binom{n}{j}$. The following lemma follows immediately from (3.1).

LEMMA 3.1. $\Delta(n,j) = (u-1)\cdot|GO_{even}(L/k,j)|$.

Therefore, to determine the value of $\Delta(n,j)$, one only needs to determine the mod 2 residue class of $|GO_{even}(L/k,j)|$.

PROPOSITION 3.2 (Theorem 1.1: case for n odd). *$\binom{L/k}{j} = \binom{n}{j}$ for n odd.*

PROOF. As $(k^s)^{H_i}$ is a subfield of L, we have $d_i|n$. Thus, if n is odd, $GO_{even}(L/k,j) = \emptyset$. □

3.1. Necklace interpretation. Let $\mathrm{Neck}(n,j)$ denote the set of necklaces consisting of j blue beads and $n-j$ red beads, with one bead designated as the bead at the top of the necklace. One could alternately view this set as the set of necklaces with j blue beads and $n-j$ red beads, lying on the plane, beads equally spaced on the unit circle with the top bead at $(0,1)$. It will be useful to consider the natural action of the dihedral group $D_n = C_n \rtimes C_2 = \langle r, f : r^n = 1, f^2 = 1, rfr = f\rangle$ on $\mathrm{Neck}(n,j)$ in which r acts by rotating the necklace one bead counterclockwise and f acts by flipping the necklace over the vertical line through the top bead.

As above, let $k \subseteq L$ be the degree n field extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^n}$. Fix an embedding $p \in \mathrm{Emb}_k(L,k^s)$. Then

$$\mathrm{Emb}_k(L,k^s) = \{p, \varphi p, \varphi^2 p \ldots, \varphi^{n-1}p\},$$

where φ denotes the Fröbenius $\varphi\colon k^s \to k^s$, $\varphi(x) = x^q$ of k. It follows that we can identify $\mathrm{Emb}_k^j(L,k^s)$ with the set $\mathrm{Neck}(n,j)$ and the action of $\mathrm{Gal}(k^s/k)$ on $\mathrm{Emb}_k^j(L,k^s)$ is given by the homomorphism $\mathrm{Gal}(k^s/k) \to \mathrm{Gal}(L/k) \cong C_n \to D_n$ and the above action of D_n on $\mathrm{Neck}(n,j)$

$$\mathrm{Emb}_k^j(L,k^s) \cong \mathrm{Neck}(n,j).$$

In particular, $GO_{odd}(L/k,j)$ can be identified with the subset of orbits of $\mathrm{Neck}(n,j)$ under the rotation action of $C_n = \langle r : r^n = 1\rangle$ consisting of those orbits with odd cardinality. A similar statement holds for $GO_{even}(L/k,j)$ as well, giving a canonical bijection

$$GO_{even}(L/k,j) \cong \mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j)).$$

Moreover, Lemma 3.1 says that

$$\binom{L/k}{j} = \binom{n}{j} + (u-1)|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))|.$$

For the twisted binomial coefficients, we let $D_{2j}\times C_2$ act on $\mathrm{Neck}(2j,j)$ by defining the D_{2j} action to be as above and defining the (commuting) action of $C_2 = \langle e : e^2 = 1\rangle$ so that e exchanges the colors of all the beads, turning the red beads of a necklace to blue, and the blue beads of a necklace to red. Define the *twisted* action of C_{2j} on $\mathrm{Neck}(2j,j)$ by the map $C_{2j} \to D_{2j} \times C_2$ defined by

$$r \mapsto (r, e)$$

and the $D_{2j} \times C_2$ action just defined. To distinguish between the twisted and untwisted actions, when we view $\mathrm{Neck}(2j,j)$ with its twisted C_{2j}-action, we will write $\mathrm{Neck}(2j,j)^\tau$. Otherwise, $\mathrm{Neck}(2j,j)$ has the C_{2j}-action from the morphism $C_{2j} \to D_{2j}$, given $r \mapsto r$ as above. We have

$$\binom{L[Q]/k}{j} = \binom{2j}{j} + (u-1)|\mathrm{Orb}_{even}(C_{2j}, \mathrm{Neck}(2j,j)^\tau)|,$$

and we will denote

$$\Delta'(2j,j) := \binom{L[Q]/k}{j} - \binom{2j}{j} = (u-1)|\mathrm{Orb}_{even}(C_{2j}, \mathrm{Neck}(2j,j)^\tau)|. \tag{3.2}$$

3.2. Möbius inversion. Let $N(n,j)$ denote the cardinality of the set of necklaces in $\mathrm{Neck}(n,j)$ whose stabilizer under the C_n-action by rotation is trivial

$$N(n,j) := |\{[l] \in \mathrm{Neck}(n,j) : \mathbf{Stab}_{C_n}([l]) = \{1\}\}|.$$

By the necklace interpretation of Section 3.1, we have

$$N(n,j) = |\{[l] \in \mathrm{Emb}_k^j(L,k^s) : \mathbf{Stab}_{\mathrm{Gal}(L/k)}([l]) = \{1\}\}|.$$

Note that that the numbers $N(n,j)$ and $|\mathrm{Emb}_k^j(L,k^s)| = \binom{n}{j}$ only depend on n and j.

As above, let $I = \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))$ index the orbits of the action of C_n on $\mathrm{Neck}(n,j)$ and, for i in I, let d_i denote the cardinality of the corresponding orbit. Let $\rho := \frac{j}{n}$ denote the fraction of beads which are blue. Given an orbit $i \in I$ of cardinality d_i, we can take d_i adjacent beads in the necklace and form a new necklace with d_i beads, ρd of which are blue, and which has a trivial stabilizer under the rotation action of C_{d_i}. This process can be run in reverse, creating a necklace with n beads from one with d_i beads. It follows that

$$|\{i \in I, d_i = d\}| = N(d, \rho d).$$

By the necklace interpretation, I is in canonical bijection with an indexing set for the orbits of the $\mathrm{Gal}(L/k)$ action on $\mathrm{Emb}_k^j(L,k^s)$. Combining with the above, we have

$$\binom{n}{j} = |\mathrm{Emb}_k^j(L,k^s)| = \sum_{d|n} d\ |\{i \in I, d_i = d\}| = \sum_{d|n} d\ N(d,\rho d), \quad \rho := \frac{j}{n},$$

under the convention $N(d,\rho d) = 0$ if $\rho d \notin \mathbb{Z}$.

We can now apply the Möbius inversion formula [**5**, 16.4], which yields

$$|N(d,b)| = \frac{1}{d}\sum_{j|d} \mu(j) \binom{\frac{d}{j}}{\frac{b}{j}},$$

where $\mu(j)$ is the Möbius function

$$\mu(j) = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j \text{ has a squared prime factor,} \\ (-1)^v & \text{if } j \text{ is a product of } v \text{ distinct prime numbers.} \end{cases}$$

Therefore

$$|GO_{even}(L/k,j)| = \sum_{d|n, 2|d} \frac{1}{d} \sum_{j|d} \mu(j) \binom{\frac{d}{j}}{\frac{\rho d}{j}}. \tag{3.3}$$

3.3. Lucas and Kummer's theorems. For a prime p, let $\nu_p : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ denote the p-adic valuation map. Lucas' classical theorem calculates the mod p residue class for binomial coefficients.

THEOREM 3.3. *[Lucas's theorem* [**11**]*] For $x, y \in \mathbb{Z}_{\geq 0}$, and p a prime*

$$\binom{x}{y} \equiv \prod_i \binom{x_i}{y_i} \mod p$$

where x_i, y_i are the coefficients of the p-adic expansions of x and y, $x = \sum_i x_i p^i$, $y = \sum_i y_i p^i$.

Define the p-adic carrier function for an integer x as $S_p(x) = \sum_i x_i$, where $x = \sum_i x_i p^i$ is the p-adic expansion. Kummer's classical theorem calculates the p-adic valuation of binomial coefficients.

THEOREM 3.4 (Kummer's theorem [**8**]). *The p-adic valuation of $\binom{n}{m}$ is*

$$\nu_p \binom{n}{m} = \frac{S_p(m) + S_p(n-m) - S_p(n)}{p-1}.$$

Some easy but useful corollaries that are relevant to our purpose are the following.

COROLLARY 3.5. *For $n, j \in \mathbb{Z}$, we have $\nu_2 \binom{n}{j} = \nu_2 \binom{2n}{2j} = \nu_2 \binom{2n+1}{2j+1}$.*

COROLLARY 3.6. *For $n, j \in 2\mathbb{Z}$ we have*

$$\nu_2 \binom{n}{j} = \nu_2 \binom{n+1}{j}.$$

COROLLARY 3.7. *For $j \in \mathbb{Z}_{>0}$, we have $\nu_2 \binom{2j}{j} \geq 1$, and the equality holds if and only if $j = 2^m$, $m \in \mathbb{Z}_{>0}$.*

3.4. Proof of Theorem 1.1 when $\nu_2(n) \geq 1$ and $\nu_2(j) = 0$. Define a partial order $\prec$ on $\mathbb{Q}_{\geq 0}$ as follows.

DEFINITION 3.8. For $x, y \in \mathbb{Q}_{\geq 0}$ written as 2-adic numbers $x = \sum_i x_i \cdot 2^i$, $y = \sum_i y_i \cdot 2^i$, then $x \prec y$ if and only if $x, y \in \mathbb{Z}_{\geq 0}$ and $x_i \leq y_i$ for all i.

PROPOSITION 3.9. *For $\nu_2(n) \geq 1$ and $\nu_2(j) = 0$ we have*

$$\Delta(n, j) = \begin{cases} -1 + u, & \frac{j-1}{2} \prec \frac{n-2}{2} \\ 0, & \text{else.} \end{cases}$$

PROOF. As above, define $\rho = j/n$. From (3.3), we have

$$\begin{aligned} |GO_{even}(L/k, j)| &= \sum_{d|n, 2|d} \frac{1}{d} \sum_{l|d} \mu(l) \binom{\frac{d}{l}}{\frac{\rho d}{l}}. \\ &= \sum_{d|n, 2|d} \frac{1}{d} \sum_{l|d} \mu(l) \frac{1}{\rho} \binom{\frac{d}{l} - 1}{\frac{\rho d}{l} - 1}. \\ &= \frac{1}{j} \sum_{d|n, 2|d} \frac{n}{d} \sum_{l|d} \mu(l) \binom{\frac{d}{l} - 1}{\frac{\rho d}{l} - 1}. \end{aligned}$$

By Lemma 3.1, $\Delta(n, j)$ only depends on the mod 2 residue of $|GO_{even}(L/k, j)|$. As j is odd by our assumption,

$$|GO_{even}(L/k, j)| \equiv \sum_{d|n, 2|d} \frac{n}{d} \sum_{l|d} \mu(l) \binom{\frac{d}{l} - 1}{\frac{\rho d}{l} - 1} \mod 2.$$

Further, as $\mu(l) \equiv 1 \mod 2$ if and only if l is square free, we have

$$|GO_{even}(L/k,j)| \equiv \sum_{d|n, 2|d} \frac{n}{d} \sum_{\substack{l|d \\ l \text{ square free}}} \binom{\frac{d}{l}-1}{\frac{\rho d}{l}-1}$$

$$(3.4) \qquad \equiv \sum_{(2^{\nu_2(n)}d)|n} \sum_{\substack{l|(2^{\nu_2(n)}d) \\ l \text{ square free}}} \binom{\frac{2^{\nu_2(n)}d}{l}-1}{\frac{\rho 2^{\nu_2(n)}d}{l}-1} \mod 2.$$

By Corollary 3.5, we have mod 2 equalities

$$\binom{x-1}{y-1} \equiv \binom{2x-2}{2y-2} \equiv \binom{2x-1}{2y-1} \mod 2, \quad \forall x, y \in \mathbb{Z}_{\geq 0}.$$

Thus the mod 2 residue of each of the summands of (3.4) only depends on $\frac{d}{l}$ and $\frac{\rho d}{l}$.

Let P denote the set of pairs (d,l) with $2^{\nu_2(n)}d$ dividing n and l square free dividing $2^{\nu_2(n)}d$. Define an equivalence relation on P by declaring $(d,l) \sim (d',l')$ when $\frac{d}{l} = \frac{d'}{l'}$. Suppose (d,l) in P with l odd. Then we can divide both l and d by l, obtaining a new pair (d',l') with the same ratio $\frac{d}{l} = \frac{d'}{l'}$ and $l' = 1$. The number of pairs with this given ratio is then equal to 2^m where m is the number of distinct odd prime factors of $n/(d')$. It follows that all these equivalence classes have even cardinality, except for the class of $(n,1)$. Now suppose (d,l) in P with l even. Then we can divide both l and d by $l/2$, obtaining an equivalence pair $(d',2)$. The number of pairs equivalent to $(d',2)$ is then 2^m where m is the number of odd prime factors of n/d'. Thus all these equivalence classes have even cardinatlity, except for the class of $(n,2)$. However, for $(d,l) = (n,2)$, the binomial coefficient $\binom{\frac{2^{\nu_2(n)}d}{l}-1}{\frac{\rho 2^{\nu_2(n)}d}{l}-1} = \binom{\frac{n}{2}-1}{\frac{j}{2}-1}$, which is 0 because $\frac{j}{2}$ is not an integer and by convention (See Theorem 3.3), binomial coefficients with fractional lower terms are 0. Thus,

$$|GO_{even}(L/k,j)| \equiv \binom{n-1}{j-1} \mod 2.$$

By Lucas's theorem (Theorem 3.3), we have $|GO_{even}(L/k,j)|$ is odd if and only if $j-1 \prec n-1$. Finally, because n is even and j is odd, it follows that $j-1 \prec n-1$ if and only if $j-1 \prec n-2$. Then since both $j-1$ and $n-2$ are even, we have $j-1 \prec n-2$ if and only if $\frac{j-1}{2} \prec \frac{n-2}{2}$, which gives the desired formula.

□

3.5. Combinatorics of symmetric orbits. We will use the necklace interpretation for the rest of the paper. Note that the C_2 generated by the flip f acts on the set $\text{Orb}(C_n, Neck(n,j))$. For a set $S \subset \text{Orb}(C_n, \text{Neck}(n,j))$, let S^f denote the subset of S that is fixed by the flipping action. Note that if l is a necklace with orbit $[l] \in \text{Orb}(C_n, Neck(n,j))^f$, then $fl = r^m l$, which implies that l has an axis of symmetry: the axis rotated $\frac{m\pi}{n}$ from the vertical counterclockwise. Since we will decompose a necklace with a symmetry axis using such an axis, it will be important to consider orbits under rotation of pairs consisting of a necklace and a symmetry axis.

Consider the 2-dimensional faithful representation $\Phi : D_n \hookrightarrow O(2,\mathbb{R})$,

$$\Phi(r) = \begin{pmatrix} \cos(\frac{2\pi i}{n}) & -\sin(\frac{2\pi i}{n}) \\ \sin(\frac{2\pi i}{n}) & \cos(\frac{2\pi i}{n}) \end{pmatrix}, \quad \Phi(f) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $x \in \mathbb{R}^2, x = (0,1)$ be a fixed point, then $l \in \text{Neck}(n,j)$ can be represented by a subset of size j in $\Phi(D_n)\cdot x$. If $[l] \in \text{Orb}(C_n, \text{Neck}(n,j))^f$, then for every representative $l \in \text{Neck}(n,j)$ there exists $\sigma \in \mathbb{P}^1(\mathbb{R})$ such that l is invariant under the reflection $f_\sigma \in O(2,\mathbb{R})$ with respect to σ. Any such linear space σ is called a symmetry axis of l. A symmetry axis $[(l,\sigma)]$ for $[l] \in \text{Orb}(C_n, \text{Neck}(n,j))^f$ is the C_n-orbit of the pair (l,σ).

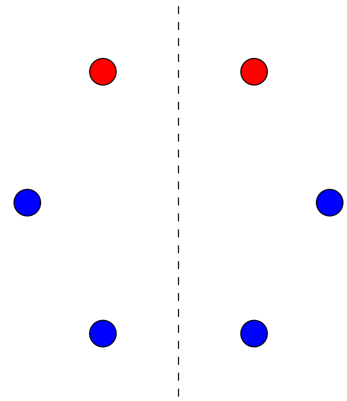

FIGURE 1. An element in $\mathrm{Orb}(C_6, \mathrm{Neck}(6,4))^f$ with a unique symmetry axis.

DEFINITION 3.10. For any $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))$, define the period of $[l]$ as $\pi([l]) := |C_n \cdot l|$, which is well defined as $|C_n \cdot l|$ is independent from the choice of representative of $[l]$.

DEFINITION 3.11. For two symmetry axes $[(l_1, \sigma_1)]$ and $[(l_2, \sigma_2)]$ of $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f$, define their distance as

$$d([(l_1,\sigma_1)],[(l_2,\sigma_2)]) = \min\{|m| : r^m \cdot \sigma_1 = \sigma_2, (l_i,\sigma_i) \in [(l_i,\sigma_i)], i = 1,2\},$$

where m is allowed to be half-integers in the sense that $\Phi(r)^{1/2}$ is the counterclockwise rotation by $\frac{2\pi i}{2n}$.

LEMMA 3.12. *Let $[l]$ be in $\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f$. If $\frac{n}{\pi([l])}$ is odd, then there is a unique symmetry axis. Otherwise, there are two distinct symmetry axes $[(l_1,\sigma_1)]$ and $[(l_2,\sigma_2)]$ with $d([(l_1,\sigma_1)],[(l_2,\sigma_2)]) = \frac{\pi([l])}{2}$.*

PROOF. Let $[(l_1,\sigma_1)]$ and $[(l_2,\sigma_2)]$ be two symmetry axes of $[l]$. As the composition $f_{\sigma_2} \circ f_{\sigma_1} \in \Phi(C_n)$ is r raised to the power of $2d([(l_1,\sigma_1)],[(l_2,\sigma_2)])$, we have

$$\pi([l]) \mid 2d([(l_1,\sigma_1)],[(l_2,\sigma_2)]).$$

On the other hand, by definition $d([(l_1,\sigma_1)],[(l_2,\sigma_2)]) < \pi([l])$. Thus $d([(l_1,\sigma_1)],[(l_2,\sigma_2)]) = \frac{\pi([l])}{2}$ or 0. Moreover, in the first case, we must have $\frac{n}{\pi([l])}$ is even. □

For example, Figure 2 shows an element $[l] \in \mathrm{Orb}(C_6, \mathrm{Neck}(6,4))^f$ with two different symmetry axes. In this example, $\pi([l]) = 3$, whence $\frac{6}{\pi([l])} = 2$, and the distance between the two axes of symmetry is $\frac{3}{2}$.

DEFINITION 3.13. A symmetry axis $[(l,\sigma)]$ for $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f$ is called *type 1* if σ does not intersect $\Phi(D_n) \cdot x$. It is called *type 2* if σ intersects at least one element of $\Phi(D_n) \cdot x$.

If n is odd, then the only symmetry axis for $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f$ is of type 2. When n is even, the symmetry axes for the example in Figure 2 have different types.

For $S \subset \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f$, denote S_1 as the subset of S_i that has a symmetry axis of type i, for $i = 1,2$. We have $\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f = \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f_1 \bigcup \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f_2$.

LEMMA 3.14. *For $\nu_2(n) \geq 1$,*

$$\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f_1 \cap \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f_2 = \mathrm{Orb}_{odd}(C_n, \mathrm{Neck}(n,j))^f.$$

In other words,

$$\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f_1 \cap \mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f_2 = \emptyset.$$

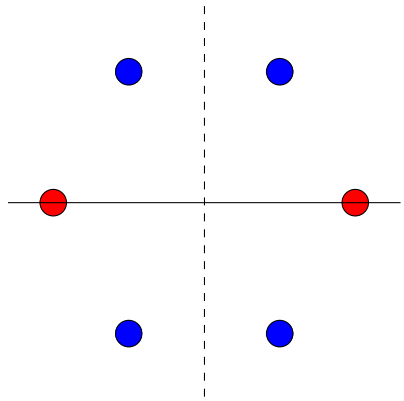

FIGURE 2. An element in $\mathrm{Orb}(C_6, \mathrm{Neck}(6,4))^f$ with two symmetry axes. The dashed line is a symmetry axis of type 1 and the solid line is of type 2.

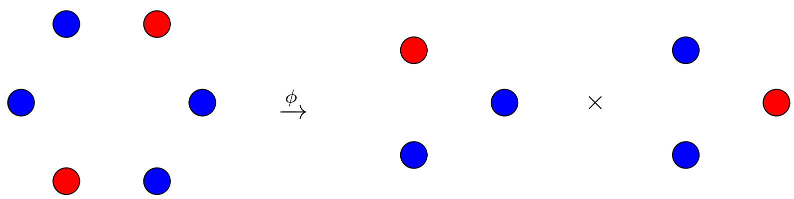

FIGURE 3. An element in $\mathrm{Orb}(C_6, \mathrm{Neck}(6,4))^f$ decomposes to a symmetric product in $\mathrm{Orb}(C_3, \mathrm{Neck}(3,2)^f$ under ϕ.

PROOF. If $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f$ has two axes $[(l_1,\sigma_1)], [(l_2,\sigma_2)]$ of different type, then $d([(l_1,\sigma_1)], [(l_2,\sigma_2)])$ is an half integer. By Lemma 3.12, we have $\pi([l]) = 2d([(l_1,\sigma_1)], [(l_2,\sigma_2)])$ is odd. $\square$

Consider again the necklace in Figure 2. This necklace has period 3, and is contained in both sides of the first equation in Lemma 3.14. An immediate corollary of Lemma 3.14 is the following.

COROLLARY 3.15. *For $\nu_2(n) \geq 1$, we have*

$$\begin{aligned}|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))| \equiv & |\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f| \mod 2 \\ = & |\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f_1| + |\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f_2|.\end{aligned}$$

3.6. Combinatorics of type 1 symmetric orbits. Now assume n is even. By choosing every other bead of a representative necklace l, any $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))$ can be uniquely decomposed into an unordered pair of necklace orbits (under the rotation action) with $\frac{n}{2}$ beads. This constructs a well-defined map

$$\begin{aligned}\phi : \mathrm{Orb}(C_n, \mathrm{Neck}(n,j)) \to & \bigcup_{j_1+j_2=j} \mathrm{Sym}(\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_1), \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_2)) \\ [l] \mapsto & ([l_1],[l_2]),\end{aligned} \tag{3.5}$$

where Sym denotes the symmetric product.

LEMMA 3.16. *For $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f_1$, we have $\phi([l]) = ([l_1], f \cdot [l_1])$, where $l_1 \in \mathrm{Neck}(\frac{n}{2}, \frac{j}{2})$.*

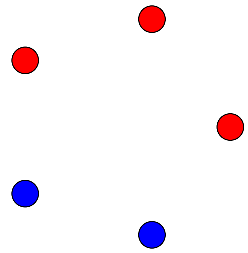

FIGURE 4. An element in $\mathrm{Orb}(C_5, \mathrm{Neck}(5,2))^f$.

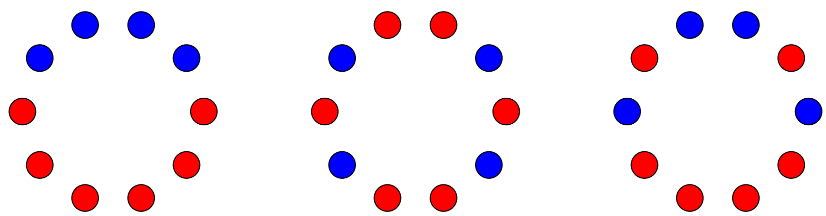

FIGURE 5. The three distinct elements in $\mathrm{Orb}(C_{10}, \mathrm{Neck}(10,4))^f$ that can be generated by the same element in $\mathrm{Orb}(C_5, \mathrm{Neck}(5,2))^f$.

PROOF. For $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_1^f$, let $[(l,\sigma)]$ denote a symmetry axis of type 1. The action of f on $\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_1^f$ can be induced by the reflection f_σ. Since f_σ exchanges the two smaller necklaces, we must have $[l_2] = f \cdot [l_1]$. □

Moreover, when restricted to $\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_1^f$, ϕ surjects to $\mathrm{Sym}(\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2})), f \cdot \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))$. Therefore, we have

$$\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_1^f = \bigcup_{([l], f\cdot[l]))\in \mathrm{Sym}(\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2})), f\cdot \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))} \phi^{-1}([l], f\cdot[l]).$$

LEMMA 3.17. *Consider the restriction of* ϕ *to* $\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_1^f$. *Then*

(1) $|\phi^{-1}([l], f\cdot[l])| = \pi([l])$, *if* $[l] \notin \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f$,
(2) $|\phi^{-1}([l], [l])| = \frac{\pi([l])+1}{2}$ *if* $[l] \in \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f$ *and* $\pi([l]) \equiv 1 \mod 2$,
(3) $|\phi^{-1}([l], [l])| = \frac{\pi([l])}{2}$ *if* $[l] \in \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f$ *and* $\pi([l]) \equiv 0 \mod 2$.

PROOF. $|\phi^{-1}([l], f\cdot[l])|$ is the number of elements in $\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_1^f$ that can be formed by interweaving $[l]$ and $f\cdot[l]$. The above statement can be seen by considering a fixed representative of $[l]$, and counting how many different ways one can insert $f \cdot l$. □

For example, the element shown in Figure 4 has period 5, which can generate 3 distinct elements by Lemma 3.17.(2), and these three elements are shown in Figure 5.

LEMMA 3.18. *Let* $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_1^f$ *be such that* $\phi([l]) = ([l_1], [l_1])$. *Then there is one such element* $[l]$ *such that* $[l_1]$ *in* $\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f$ *with* $\pi([l_1]) \equiv 1 \mod 2$, *which has* $\pi([l]) = \pi([l_1])$. *Otherwise* $\pi([l]) = 2\pi([l_1])$.

PROOF. The exceptional case is the following. View l_1 as a subset of $\Phi(D_n)\cdot x$. Then flip l_1 with respect to $\sigma_0 \in \mathbb{P}^1(\mathbb{R})$, where σ_0 is perpendicular to a symmetry axis of l_1. Then $[l]$ is the C_n-orbit of $l_1 \cup s_{\sigma_0} \cdot l_1$. □

For example, the three elements shown in Figure 5 have period $10, 5, 10$ respectively, where the middle one is the exceptional case in Lemma 3.18.

Therefore, for $[l] \in \mathrm{Orb}_{odd}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f$, we have

$$\phi^{-1}([l],[l]) \cap \mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j)) = \frac{\pi([l])-1}{2}. \tag{3.6}$$

PROPOSITION 3.19. *For $\nu_2(n) \geq 1$,*

$$|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f_1| = \frac{1}{2}\left(\binom{\frac{n}{2}}{\frac{j}{2}} - |\mathrm{Orb}_{odd}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f|\right).$$

PROOF. By Lemma 3.17 and Lemma 3.18, we have

$$\begin{aligned}
|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f_1| =& \frac{1}{2}\sum_{[l]\notin \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2},\frac{j}{2}))^f} \pi([l]) + \sum_{[l]\in \mathrm{Orb}_{even}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2},\frac{j}{2}))^f} \frac{\pi([l])}{2} \\
&+ \sum_{[l]\in \mathrm{Orb}_{odd}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2},\frac{j}{2}))^f} \frac{\pi([l])-1}{2} \\
=& \frac{1}{2}\left(\sum_{[l]\in \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2},\frac{j}{2}))} \pi([l]) - |\mathrm{Orb}_{odd}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f|\right) \\
=& \frac{1}{2}\left(\sum_{d|\frac{n}{2}} dN(d,\rho d) - |\mathrm{Orb}_{odd}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f|\right) \\
=& \frac{1}{2}\left(\binom{\frac{n}{2}}{\frac{j}{2}} - |\mathrm{Orb}_{odd}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f|\right).
\end{aligned}$$

□

LEMMA 3.20. *For any $n, j \in \mathbb{Z}_{\geq 0}$, we have*

$$|\mathrm{Orb}_{odd}(C_n, \mathrm{Neck}(n,j))^f| = \begin{cases} \binom{\frac{n}{2^{\nu_2(j)+1}} - \frac{1}{2}}{\frac{j}{2^{\nu_2(j)+1}}}, & \text{when } \nu_2(j) > \nu_2(n), \\ \binom{\frac{n}{2^{\nu_2(j)+1}} - \frac{1}{2}}{\frac{j}{2^{\nu_2(j)+1}} - \frac{1}{2}}, & \text{when } \nu_2(j) = \nu_2(n), \\ 0, & \text{when } \nu_2(j) < \nu_2(n). \end{cases}$$

PROOF. Notice that for any $n, j \in 2\mathbb{Z}$, we have

$$\mathrm{Orb}_{odd}(C_n, \mathrm{Neck}(n,j))^f = \mathrm{Orb}_{odd}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f.$$

Then, the problem reduces to the case when $\nu_2(n)\cdot\nu_2(j) = 0$. The counting in this case is clear. □

3.7. Proof of Theorem 1.1 when $\nu_2(n) = 1$ and $\nu_2(j) \geq 1$. Consider the restriction of the map ϕ in (3.5) to $\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))^f_2$. As $n \equiv 2 \mod 4$, the image is contained in

$$\bigcup_{j_1+j_2=j} \mathrm{Sym}(\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_1))^f, \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_2))^f).$$

For an example of $([l], \phi([l]))$ in this case, see Figure 6.

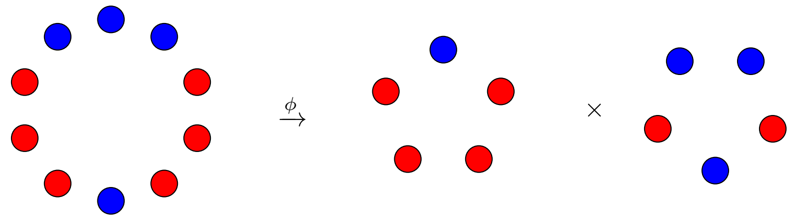

FIGURE 6. An element in $\mathrm{Orb}(C_{10}, \mathrm{Neck}(10,4))_2^f$ decomposes under ϕ to a symmetric product of two elements in $\mathrm{Orb}(C_5, \mathrm{Neck}(5,1))^f$ and $\mathrm{Orb}(C_5, \mathrm{Neck}(5,3))^f$, repsectively.

Moreover, since $\frac{n}{2}$ is odd, any symmetry axis for $[l] \in \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_1))^f$ is of type 2 and only passes through one single bead. It follows the restriction of ϕ is bijective, and we have

$$\begin{aligned} |\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_2^f| = & \sum_{j_1+j_2=j, j_1<j_2} |\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_1))^f| \cdot \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_2))^f \\ & + \frac{1}{2}|\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f| \cdot (|\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f| + 1). \end{aligned} \tag{3.7}$$

Next, we have

LEMMA 3.21. *Let* $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_2^f$. *Then*
(1) $\pi([l]) = \pi([l_1])$, *if* $\phi([l]) = ([l_1],[l_1])$,
(2) $\pi([l]) = 2\mathrm{lcm}(\pi([l_1]), \pi([l_2]))$, *if* $\phi([l]) = ([l_1],[l_2])$, $[l_1] \neq [l_2]$.

PROOF. l has a stabilizer r^k with $k \equiv 1 \mod 2$ under C_n-action if and only if $\phi([l]) = ([l_1],[l_1])$. □

Figure 6 provides an example of the second case. Since $\frac{n}{2}$ is odd, we have

$$|\mathrm{Orb}_{odd}(C_n, \mathrm{Neck}(n,j))^f| = |\mathrm{Orb}_{odd}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f|.$$

Thus

$$|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))_2^f| = |\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_2^f| - |\mathrm{Orb}_{odd}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f|. \tag{3.8}$$

PROPOSITION 3.22.

$$|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f| = \begin{cases} \binom{\frac{n}{2}}{\frac{j}{2}} - \binom{\frac{n-2}{4}}{\frac{j-2}{4}}, & j \equiv 2 \mod 4, \\ \binom{\frac{n}{2}}{\frac{j}{2}} - \binom{\frac{n-2}{4}}{\frac{j}{4}}, & j \equiv 0 \mod 4. \end{cases}$$

PROOF. Since $\frac{n}{2}$ is odd, we have $\mathrm{Orb}_{odd}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f = \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f$. Therefore, combining (3.7), (3.8), and Proposition 3.19, and noting Lemma 3.14, we have

$$\begin{aligned} |\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f| = & |\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))_1^f| + |\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))_2^f| \\ = & \frac{1}{2} \sum_{j_1+j_2=j, j_1<j_2} |\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_1))^f| \cdot \mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_2))^f \\ & + \frac{1}{2}\binom{\frac{n}{2}}{\frac{j}{2}} - |\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, \frac{j}{2}))^f|. \end{aligned}$$

Since j is even, we have $j_1 \equiv j_2 \mod 2$. As $\frac{n}{2}$ is odd, we further have $\nu_2(j_i) > \nu_2(n)$, $i = 1,2$ or $\nu_2(j_i) = \nu_2(n)$, $i = 1,2$. Thus by Lemma 3.20, we have

$$\frac{1}{2} \sum_{j_1+j_2=j, j_1<j_2} |\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_1))^f| \cdot |\mathrm{Orb}(C_{\frac{n}{2}}, \mathrm{Neck}(\frac{n}{2}, j_2))^f|$$

$$=\frac{1}{2}\sum_{i=0}^{\frac{j}{2}-1}\binom{\frac{n-2}{4}}{i}\binom{\frac{n-2}{4}}{\frac{j}{2}-1-i}+\frac{1}{2}\sum_{i=0}^{\frac{j}{2}}\binom{\frac{n-2}{4}}{i}\binom{\frac{n-2}{4}}{\frac{j}{2}-i}$$
$$=\frac{1}{2}\left(\binom{\frac{n}{2}-1}{\frac{j}{2}-1}+\binom{\frac{n}{2}-1}{\frac{j}{2}}\right)=\frac{1}{2}\binom{\frac{n}{2}}{\frac{j}{2}},$$

where in the last line we have used the Pascal identity. Thus

$$|\mathrm{Orb}_{even}(C_n,\mathrm{Neck}(n,j))^f| =\frac{1}{2}\left(\binom{\frac{n}{2}}{\frac{j}{2}}+\binom{\frac{n}{2}}{\frac{j}{2}}\right)-|\mathrm{Orb}(C_{\frac{n}{2}},\mathrm{Neck}(\frac{n}{2},\frac{j}{2}))^f|$$
$$=\begin{cases}\binom{\frac{n}{2}}{\frac{j}{2}}-\binom{\frac{n-2}{4}}{\frac{b-2}{4}}, & b\equiv 2 \mod 4,\\ \binom{\frac{n}{2}}{\frac{j}{2}}-\binom{\frac{n-2}{4}}{\frac{b}{4}}, & b\equiv 0 \mod 4.\end{cases}$$

□

Proposition 3.23. *For $\nu_2(n)=1$ and $\nu_2(j)\geq 1$, we have $\Delta(n,j)=0$.*

Proof. By the Corollary 3.5 of Kummer's theorem, for $j\equiv 2 \mod 4$, we have

$$\nu_2\binom{\frac{n-2}{4}}{\frac{j-2}{4}}=\nu_2\binom{\frac{n}{2}-1}{\frac{j}{2}-1}=\nu_2\binom{\frac{n}{2}}{\frac{j}{2}},$$

and for $b\equiv 0 \mod 4$,

$$\nu_2\binom{\frac{n-2}{4}}{\frac{j}{4}}=\nu_2\binom{\frac{n}{2}-1}{\frac{j}{2}}=\nu_2\binom{\frac{n}{2}}{\frac{j}{2}}.$$

By Proposition 3.22,

$$|\mathrm{Orb}_{even}(C_n,\mathrm{Neck}(n,j))^f|\equiv 0 \mod 2.$$

Finally, by Lemma 3.1,

$$\Delta(n,j)=(u-1)\cdot|\mathrm{Orb}_{even}(C_n,\mathrm{Neck}(n,j))^f|=0.$$

□

3.8. Proof of Theorem 1.1 when $\nu_2(n)\geq 2$ and $\nu_2(j)\geq 1$. Since both n and j are even, for any $[l]\in\mathrm{Orb}(C_n,\mathrm{Neck}(n,j))_2^f$ there are two beads on any type 2 symmetry axis, and they have to be of the same color. Denote $[l']$ as the subset formed by removing the two beads on a chosen type 2 symmetry axis for $l\in[l]$. If the two beads are blue, then

$$[l']\in\mathrm{Orb}(C_{n-2},\mathrm{Neck}(n-2,j-2))_1^f.$$

Otherwise the two beads are red and

$$[l']\in\mathrm{Orb}(C_{n-2},\mathrm{Neck}(n-2,j))_1^f.$$

Now, if we start with $[l']\in\mathrm{Orb}(C_{n-2},\mathrm{Neck}(n-2,j-2))_1^f$, then we can recover an element in $\mathrm{Orb}(C_n,\mathrm{Neck}(n,j))_2^f$ by picking a symmetry axis for $e\cdot[l]\in[l']$ and adding back two blue beads. We can also do a similar procedure for $[l']\in\mathrm{Orb}(C_{n-2},\mathrm{Neck}(n-2,j))_1^f$ by adding two red beads. However, the associated map

$$\psi:\mathrm{Orb}(C_{n-2},\mathrm{Neck}(n-2,j-2))_1^f\amalg\mathrm{Orb}(C_{n-2},\mathrm{Neck}(n-2,j))_1^f\to\mathrm{Orb}(C_n,\mathrm{Neck}(n,j))_2^f$$

is surjective but not injective[1]. In other words, if we count $\mathrm{Orb}(C_n,\mathrm{Neck}(n,j))_2^f$ via counting the domain, then there will be over-counting. Figure 7 and Figure 8 provide an example of the over-counting phenomenon. To understand the fibers of ψ, i.e., the over-counting, we first notice the following.

[1]For an example of how does the map ψ works, see Figure 7.

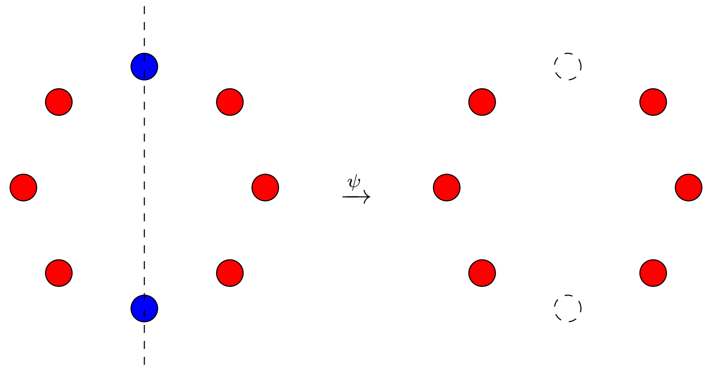

FIGURE 7. By removing the two beads on the symmetry axis, ψ maps an element in $\mathrm{Orb}(C_8, \mathrm{Neck}(8,2))_2^f$ to $\mathrm{Orb}(C_6, \mathrm{Neck}(6,0))_1^f$.

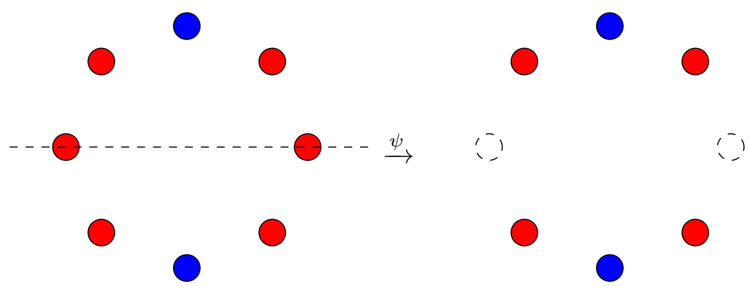

FIGURE 8. The element in Figure 7 has another symmetry axis, which maps to $\mathrm{Orb}(C_6, \mathrm{Neck}(6,2))_1^f$ under ψ.

LEMMA 3.24. *Every element in* $\mathrm{Orb}(C_{n-2}, \mathrm{Neck}(n-2, j-2))_1^f \amalg \mathrm{Orb}(C_{n-2}, \mathrm{Neck}(n-2, j))_1^f$ *has exactly one type 1 symmetry axis if* $\pi([l])$ *is even, and two symmetry axes of different type if* $\pi([l])$ *is odd.*

PROOF. Take $[l] \in \mathrm{Orb}(C_{n-2}, \mathrm{Neck}(n-2, j-2))_1^f \amalg \mathrm{Orb}(C_{n-2}, \mathrm{Neck}(n-2, j))_1^f]$. By Lemma 3.12, if $\frac{n-2}{\pi([l])}$ is odd, then $[l]$ has exactly one symmetry axis, which has to pass between beads as $\pi([l])$ is even. On the other hand, if $\frac{n-2}{\pi([l])}$ is even, since $n-2 \equiv 2 \mod 4$, $\pi([l])$ is odd, thus the two axes are of a different type, as their distance is a half-integer. □

The only elements $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_2^f$ that have $|\psi^{-1}([l])| = 2$ are the ones with two different axes both passing through two beads. Viewing on the domain, these are the elements in $\mathrm{Orb}(C_{n-2}, \mathrm{Neck}(n-2, j-2))_1^f$ and $\mathrm{Orb}(C_{n-2}, \mathrm{Neck}(n-2, j))_1^f$ that have two axes of different types. By Lemma 3.24, these are the exactly the elements that have odd period.

However, the over-counting is not simply

$$|\mathrm{Orb}_{odd}(C_{n-2}, \mathrm{Neck}(n-2, j-2))_1^f| + |\mathrm{Orb}_{odd}(C_{n-2}, \mathrm{Neck}(n-2, j))_1^f|$$

since there would be an over-counting of the over-counting. This happens in the following situation. Denote the two symmetry axes for $[l]$ in the domain with $n-2$ beads as $[(l, \sigma_1)]$, $[(l, \sigma_2)]$, which is of type 1 and type 2, respectively. After adding the two beads along $[(l, \sigma_1)]$, the two symmetry axes $[(l, \sigma_1)]$, $[(l, \sigma_2)]$ will both become type 2 symmetry axis in $\psi([l]) \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_2^f$. Therefore, the over-counting of over-counting will happen if $[(l, \sigma_1)]$, $[(l, \sigma_2)]$ become the same axis in $\psi([l])$. A little thought combining with Lemma 3.12 shows this will happen if and only if $\psi([l]) \in \mathrm{Orb}_{odd}(C_n, \mathrm{Neck}(n,j))^f$. For example, if one want to construct an element in $\mathrm{Orb}(C_8, \mathrm{Neck}(8,4))_2^f$

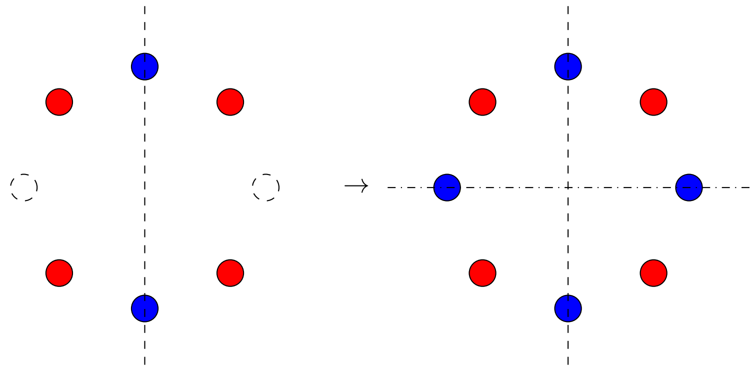

FIGURE 9. An example of the over-counting of over-counting.

from the right-hand side of Figure 8, then one needs to add two blue beads to the vacant spots. This becomes a new symmetry axis. However, as shown in Figure 9, this axis is the same as the dashed one from the left-hand side.

PROPOSITION 3.25. *For $\nu_2(n) \geq 2$, and $\nu_2(j) \geq 1$, we have*

$$|\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_2^f| = \frac{1}{2}\binom{\frac{n}{2}}{\frac{j}{2}} + \frac{1}{2}|\mathrm{Orb}_{odd}(C_n, \mathrm{Neck}(n,j))^f|.$$

PROOF. From the earlier argument in this section, we have

$$\begin{aligned}|\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_2^f| =& |\mathrm{Orb}(C_{n-2}, \mathrm{Neck}(n-2,j-2))_1^f| + |\mathrm{Orb}(C_{n-2}, \mathrm{Neck}(n-2,j))_1^f| \\ &- \frac{1}{2}\Big(|\mathrm{Orb}_{odd}(C_{n-2}, \mathrm{Neck}(n-2,j-2))_1^f| + |\mathrm{Orb}_{odd}(C_{n-2}, \mathrm{Neck}(n-2,j))_1^f| \\ &- |\mathrm{Orb}_{odd}(C_n, \mathrm{Neck}(n,j))^f|\Big).\end{aligned}$$

By Proposition 3.19, we get

$$\begin{aligned}|\mathrm{Orb}(C_n, \mathrm{Neck}(n,j))_2^f| =& \frac{1}{2}\left(\binom{\frac{n}{2}-1}{\frac{j}{2}-1} + \binom{\frac{n}{2}-1}{\frac{j}{2}} + |\mathrm{Orb}_{odd}(C_{\frac{n}{2}-1}, \mathrm{Neck}(\frac{n}{2}-1, \frac{j}{2}-1))_1^f|\right. \\ &+ |\mathrm{Orb}_{odd}(C_{\frac{n}{2}-1}, \mathrm{Neck}(\frac{n}{2}-1, \frac{j}{2}))_1^f - |\mathrm{Orb}_{odd}(C_{n-2}, \mathrm{Neck}(n-2,j-2))_1^f| \\ &\left.- |\mathrm{Orb}_{odd}(C_{n-2}, \mathrm{Neck}(n-2,j))_1^f| + |\mathrm{Orb}_{odd}(C_n, \mathrm{Neck}(n,j))^f|\right) \\ =& \frac{1}{2}\binom{\frac{n}{2}}{\frac{j}{2}} + \frac{1}{2}|\mathrm{Orb}_{odd}(C_n, \mathrm{Neck}(n,j))^f|\end{aligned}$$

where in the second identity, we have used the Pascal identity, and the same argument as in the first sentence of the proof of Lemma 3.20. □

PROPOSITION 3.26. *For $\nu_2(n) \geq 2$ and $\nu_2(j) \geq 1$, we have $\Delta(n,j) = 0$.*

PROOF. Combining Lemma 3.14, Proposition 3.19, and Proposition 3.25, we have

$$|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f| = \binom{\frac{n}{2}}{\frac{j}{2}} - |\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f|.$$

Now, apply Lemma 3.20, we have

(1) If $\nu_2(n) > \nu_2(j)$,

$$|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f| = \binom{\frac{n}{2}}{\frac{j}{2}}.$$

(2) If $\nu_2(n) = \nu_2(j)$,

$$|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f| = \binom{\frac{n}{2}}{\frac{j}{2}} - \binom{\frac{n}{2^{\nu_2(n)+1}} - \frac{1}{2}}{\frac{j}{2^{\nu_2(n)+1}} - \frac{1}{2}}.$$

(3) If $\nu_2(n) < \nu_2(j)$,

$$|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f|) = \binom{\frac{n}{2}}{\frac{j}{2}} - \binom{\frac{n}{2^{\nu_2(n)+1}} - \frac{1}{2}}{\frac{j}{2^{\nu_2(n)+1}}}.$$

In case (1), by Lucas's theorem, we have

$$|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f| \equiv 0 \mod 2.$$

In case (2), by Corollary 3.5 of Kummer's theorem,

$$\nu_2\binom{\frac{n}{2^{\nu_2(n)+1}} - \frac{1}{2}}{\frac{j}{2^{\nu_2(n)+1}} - \frac{1}{2}} = \nu_2\binom{\frac{n}{2^{\nu_2(n)}}}{\frac{j}{2^{\nu_2(n)}}} = \nu_2\binom{\frac{n}{2}}{\frac{j}{2}}.$$

Therefore, $|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f| \equiv 0 \mod 2$. In case (3), since $\frac{j}{2^{\nu_2(n)}}$ is even, and $\frac{n}{2^{\nu_2(n)}}$ is odd, then by Corollary 3.5 and 3.6 of Kummer's theorem

$$\nu_2\binom{\frac{n}{2^{\nu_2(n)}} - 1}{\frac{j}{2^{\nu_2(n)}}} = \nu_2\binom{\frac{n}{2^{\nu_2(n)}}}{\frac{j}{2^{\nu_2(n)}}} = \nu_2\binom{\frac{n}{2}}{\frac{j}{2}}.$$

Therefore, we have

$$|\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f| \equiv 0 \mod 2,$$

and by Lemma 3.1,

$$\Delta(n,j) = (u-1) \cdot |\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j))^f| = 0.$$

□

3.9. Conclusion of the proof of Theorem 1.1. Combining Proposition 3.2, Proposition 3.9, Proposition 3.23, and Proposition 3.26, then invoking Lemma 3.1, we have

$$\binom{L/k}{j} = \binom{n}{j} - (1-u)\cdot \delta(n,j),$$

where $\delta(n,j) = \begin{cases} 1, & \frac{j-1}{2} \prec \frac{n-2}{2}, \\ 0, & \text{else} \end{cases}$ and $\prec$ is as in Definition 3.8 in the beginning of Section 3.4. Finally, we can conclude the proof of Theorem 1.1 by Lucus's theorem. The first few rows of the untwisted Pascal triangle are the following:

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 1+u & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 3+u & & 6 & & 3+u & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 5+u & & 15 & & 20 & & 15 & & 5+u & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 7+u & & 28 & & 55+u & & 70 & & 55+u & & 28 & & 7+u & & 1
\end{array}.$$

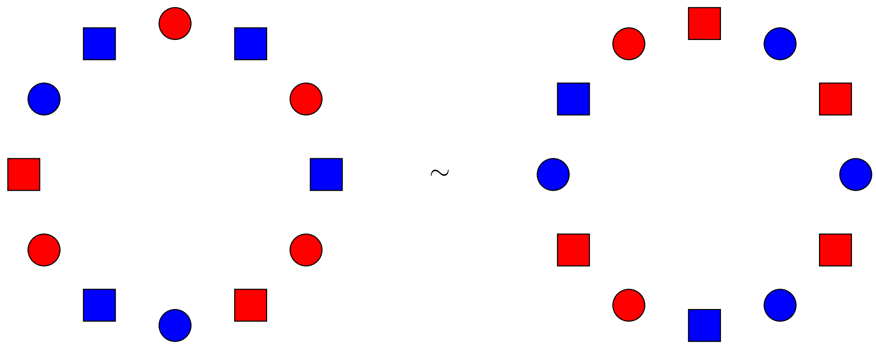

FIGURE 10. An example of elements in $\mathrm{Orb}(C_{12}, \mathrm{Neck}(12,6)^\tau)$.

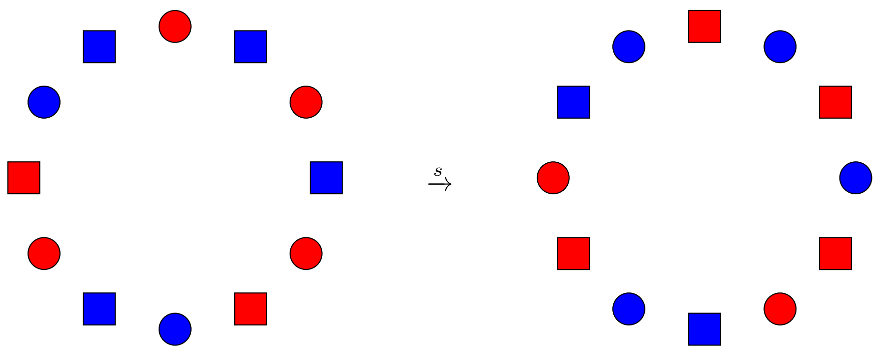

FIGURE 11. An example of swapping.

4. Twisted case - Proof of Theorem 1.2

The strategy for the proof of Theorem 1.2 is to rewrite $\mathrm{Orb}_{even}(C_n, \mathrm{Neck}(n,j)^\tau)$ in terms of the orbits for the untwisted action that we have studied in the previous sections.

4.1. Reduction to the untwisted case.

LEMMA 4.1. *The elements of* $\mathrm{Orb}(C_n, \mathrm{Neck}(n,j)^\tau)$ *can be interpreted as triples*

$$\mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j)^\tau) \cong \{([l],[l_1],[l_2]) : [l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j)), \phi([l]) = ([l_1],[l_2])\}/\sim,$$

where in $([l],[l_1],[l_2])$, $[l_1]$ *and* $[l_2]$ *are ordered. The equivalence relation is defined as* $([l],[l_1],[l_2]) \sim (e\cdot[l], e\cdot[l_2], e\cdot[l_1])$.

PROOF. An element in $\mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j)^\tau)$ is uniquely determined by an element $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))$ together with a chosen bead in $[l]$. On the other hand, two different chosen beads in $[l]$ will give rise to the same $\mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j)^\tau)$ orbits if and only if they both lie in the $[l_1]$ or $[l_2]$, where $\phi([l]) = ([l_1],[l_2])$. Therefore, an element in $\mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j)^\tau)$ can be denoted as $([l],[l_1],[l_2])$, where the second entry $[l_1]$ means the chosen bead lie in $[l_1]$. However, both $([l],[l_1],[l_2])$ and $(e\cdot[l], e\cdot[l_2], e\cdot[l_1])$ give rise to the same element in $\mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j)^\tau)$. By quotienting out this equivalence relation, we arrive at the desired isomorphism. □

One way to visualize the triple is shown in Figure 10. Denote the element as $([l],[l_1],[l_2]) \in \mathrm{Orb}(C_{12}, \mathrm{Neck}(12,6)^\tau)$. The circle-shaped beads form $[l_1] \in \mathrm{Orb}(C_6, \mathrm{Neck}(6,2))$, whereas the square-shaped beads form $[l_2] \in \mathrm{Orb}(C_6, \mathrm{Neck}(6,4))$.

From now on, we will use the triple $([l],[l_1],[l_2])$ to denote elements in $\mathrm{Orb}(C_{2j},\mathrm{Neck}(2j,j)^\tau)$. Define the twisted period as $\pi'([l],[l_1],[l_2]) := |([l],[l_1],[l_2])|$, i.e., the size of the twisted C_{2j}-orbit. Next, we have a C_2 action on $\mathrm{Orb}(C_{2j},\mathrm{Neck}(2j,j)^\tau)$, which acts by exchanging l_1, l_2. Combinatorically, this action interchanges the two potentially different twisted orbits that could come from the same underlining untwisted orbit $[l]$. We will call this action swapping, and an example is shown in Figure 11.

Lemma 4.2. *$\Delta'(2j,j) = (u-1)|\mathrm{Orb}(C_{2j},\mathrm{Neck}(2j,j)^\tau)^s|$, where $\mathrm{Orb}(C_{2j},\mathrm{Neck}(2j,j)^\tau)^s$ is the subset of $\mathrm{Orb}_{even}(C_{2j},\mathrm{Neck}(2j,j)^\tau)$ that is fixed by swapping.*

Proof. Since the swapping action does not change the twisted period, by (3.2) and $2(u-1)=0$, we got the desired formula. ☐

Lemma 4.3. *In $\mathrm{Orb}(C_n,\mathrm{Neck}(n,j)^\tau)$, $([l],[l_1],[l_2]) = ([l],[l_2],[l_1])$ if and only if either $[l_1]=[l_2]$ or $[l]=e\cdot[l]$.*

Proof. Definition chasing. ☐

Corollary 4.4. *$\Delta'(2j,j)=0$, for $\nu_2(j)=0$.*

Proof. Since j is odd, $[l_1]\neq[l_2]$ or $e\cdot[l_2]$ for all elements $([l],[l_1],[l_2]) \in \mathrm{Orb}(C_{2j},\mathrm{Neck}(2j,j)^\tau)$. Thus the C_2-action of swapping acts freely, so $|\mathrm{Orb}(C_{2j},\mathrm{Neck}(2j,j)^\tau)^s| \equiv 0 \mod 2$. ☐

From now on, we will assume $j\equiv 0 \mod 2$. Applying the above results to

$$|\mathrm{Orb}(C_{2j},\mathrm{Neck}(2j,j)^\tau)^s|,$$

we have

$$\begin{aligned}|\mathrm{Orb}(C_{2j},\mathrm{Neck}(2j,j)^\tau)^s| =& |([l],[l_1],[l_1]) \in \mathrm{Orb}_{even}(C_{2j},\mathrm{Neck}(2j,j)^\tau), [l]\neq e\cdot[l]| \\ &+ |([l],[l_1],[l_2]) \in \mathrm{Orb}_{even}(C_{2j},\mathrm{Neck}(2j,j)^\tau), [l]= e\cdot[l]|.\end{aligned}$$

For $([l],[l_1],[l_1]) \in \mathrm{Orb}_{even}(C_{2j},\mathrm{Neck}(2j,j)^\tau)$, $[l]\neq e\cdot[l] \Leftrightarrow [l_1]\neq e\cdot[l_1]$, and notice that if $[l_1]\neq e\cdot[l_1]$, then $2|\pi'([l],[l_1],[l_1])$. Therefore,

$$\begin{aligned}|\mathrm{Orb}(C_{2j},\mathrm{Neck}(2j,j)^\tau)^s| =& \sum_{[l]\in\mathrm{Orb}(C_j,\mathrm{Neck}(j,\frac{j}{2})),[l]\neq e\cdot[l]} \frac{|\phi^{-1}([l],[l])|}{2} \\ &+ |([l],[l_1],[l_2]) \in \mathrm{Orb}_{even}(C_{2j},\mathrm{Neck}(2j,j)^\tau), [l]= e\cdot[l]|\end{aligned}$$

To further simplify the second term, we notice the following facts.

Lemma 4.5. *$\pi([l])\equiv 0 \mod 2$, $[l]\in\mathrm{Orb}(C_{2j},\mathrm{Neck}(2j,j)^\tau)$.*

Proof. $\nu_2(2j)>\nu_2(j)$. ☐

Lemma 4.6. *If $[l]=e\cdot[l]$, then for all $l\in\mathrm{Neck}(2j,j)$, we have*

$$e\cdot l = r^{\frac{\pi([l])}{2}}\cdot l.$$

Proof. Apparently $e=r^m$ for some $m\in\mathbb{Z}/\pi([l])\mathbb{Z}$. Since e changes l, we can take $0<m<\pi([l])$. On the other hand, since $e^2=1$, one has $\pi([l])|2m$, thus $m=\frac{\pi([l])}{2}$. ☐

Proposition 4.7. *$\pi'([l],[l_1],[l_2])=\pi([l])$ unless $l=e\cdot[l]$, and $\nu_2(\pi([l]))=1$, in which case $\pi'([l],[l_1],[l_2])=\frac{\pi([l])}{2}\equiv 1 \mod 2$.*

Proof. Since $\pi([l])$ is even, $r^{\pi([l])}=e^{\pi([l])}$, and we always have $\pi'([l],[l_1],[l_2])\leq\pi([l])$. By the definition of the twisted action, if $\pi'([l],[l_1],[l_2])$ is even, we have $\pi'([l],[l_1],[l_2])=\pi([l])$. Therefore, the only possibility for $\pi'([l],[l_1],[l_2])<\pi([l])$ is when $\pi'([l],[l_1],[l_2])$ is odd. In this case, we have $2\pi'([l],[l_1],[l_2])=\pi([l])$, and thus $\nu_2([l])=1$. ☐

Therefore

$$
\begin{aligned}
|\mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j)^\tau)^s| = & \sum_{[l]\in \mathrm{Orb}(C_j, \mathrm{Neck}(j,\frac{j}{2})), [l]\neq e\cdot[l]} \frac{|\phi^{-1}([l],[l])|}{2} \\
& + |([l],[l_1],[l_2]) \in \mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j)^\tau), [l] = e\cdot[l], \nu_2(\pi([l])) > 1| \\
= & \sum_{[l]\in \mathrm{Orb}(C_j, \mathrm{Neck}(j,\frac{j}{2})), [l]\neq e\cdot[l]} \frac{|\phi^{-1}([l],[l])|}{2} \\
& + |l \in \mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j)), [l] = e\cdot[l], \nu_2(\pi([l])) > 1|
\end{aligned}
$$

where the last formula is an enumeration purely in terms of untwisted orbits.

4.2. Relations to partitions. Consider an element $[l] \in \mathrm{Orb}(C_n, \mathrm{Neck}(n,j))$. One can uniquely write $[l]$ as the cyclic orbit of a partition of n as follows. Call a maximal consecutive segment of beads of l of the same color a "cluster." By replacing each cluster of l by its number of beads, one obtains the corresponding partition on n. For example, fix $l \in [l]$, if the beads in positions 1 and 5 are red but everything in between is blue, we will replace the 3 blue beads with the number 3. If the cluster is red, we will mark it by adding an underline to distinguish it from being blue. We will use the notation $(\cdots)$ to denote the cyclic equivalent class of partitions. So $[l]$ can be uniquely written as the cyclic orbit of a partition of n as

$$[l] = (\underline{r_1}b_1\underline{r_2}b_2\cdots\underline{r_m}b_m), \quad r_i, b_i \in \mathbb{Z}_{>0},$$

with $\sum_i (r_i + b_i) = n, \sum_i b_i = j$. Under this notation, the action of e is simply

$$e\cdot(\underline{r_1}b_1\underline{r_2}b_2\cdots\underline{r_m}b_m) = (r_1\underline{b_1}r_2\underline{b_2}\cdots r_m\underline{b_m}).$$

Apparently, the length of the partition has to be even. For example, the three elements in Figure 5 can be written as $(\underline{6}4), (\underline{2}1\underline{1}1\underline{2}1\underline{1}1), (\underline{1}1\underline{4}1\underline{1}2)$, respectively.

Denote the set of even length partitions of n with an underlined marking for every other entry such that the sum of unmarked entries is j as $\mathrm{Part}(n,j,\underline{n-j})$. We further denote the set of its cyclic equivalent classes as $\mathrm{Orb}(C, \mathrm{Part}(n,j,\underline{n-j}))$. For an element $[p] \in \mathrm{Orb}(C, \mathrm{Part}(n,j,\underline{n-j}))$, we denote its length as $|p| := \mathrm{length}(p)$, and define its period as $\varpi([p]) = |C_{|p|}\cdot p|$[2].

Denote the set of partition of j as $\mathrm{Part}(j)$, and similarly denote the set of its cyclic equivalent class and orbit as in the marked case, then we have

Lemma 4.8.

$$\{[l] \in \mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j)), l = e\cdot[l]\} \cong \{[p] \in \mathrm{Orb}(C, \mathrm{Part}(j)), \nu_2(\varpi([p])) = 0\}$$

Proof. Let $[l] \in \mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j))$, and $[p] = (\underline{r_1}b_1\underline{r_2}b_2\cdots\underline{r_m}b_m)$ be its corresponding element in $\mathrm{Orb}(C, \mathrm{Part}(2j,j,\underline{j}))$. If $[l] = e\cdot[l]$, then

$$(\underline{r_1}b_1\underline{r_2}b_2\cdots\underline{r_m}b_m) = (r_1\underline{b_1}r_2\underline{b_2}\cdots r_m\underline{b_m}),$$

and apparently we need $(r_1\cdots r_m) = (b_1\cdots b_m)$. Moreover, by passing to the smaller partition that is formed by taking $\varpi([p])$ consecutive parts, without losing of generality we can assume the partition is non-periodic, i.e., $\varpi([p]) = 2m$. Then the only element in $\mathrm{Orb}(C, \mathrm{Part}(2j,j,\underline{j}))$ of the form $(\underline{r_1}, r_{i_1}, \underline{r_2}, r_{i_2}, \cdots, \underline{r_m}, r_{i_m})$ that satisfy $[p] = e\cdot[p]$ is

$$(\underline{r_1}r_{\frac{m+3}{2}}\underline{r_2}r_{\frac{m+5}{2}}\cdots r_m\underline{r_{\frac{m+1}{2}}}r_1\underline{r_{\frac{m+3}{2}}}\cdots\underline{r_m}r_{\frac{m+1}{2}}),$$

and m has to be odd. $\square$

[2]Note there is no direct relation between ϖ and π'.

Recall the total number of partitions of $j \in \mathbb{Z}_{>0}$ is 2^{j-1}. Then

$$2^{j-1} = \sum_d d|[p] \in \mathrm{Orb}(C, \mathrm{Part}(j)), \varpi([p]) = d|, \tag{4.1}$$

Lemma 4.9. *For $j > 1$, we have $|l \in \mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j, j)), [l] = e \cdot [l], \nu_2(\pi([l])) > 1| \equiv |l \in \mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j, j)), [l] = e \cdot [l], \nu_2(\pi([l])) = 1| \mod 2$.*

Proof. By (4.1), we have

$$\begin{aligned} |[p] \in \mathrm{Orb}(C, \mathrm{Part}(j)), \nu_2(\varpi([p])) = 0| &= \sum_{\nu_2(d)=0} d|[p] \in \mathrm{Orb}(C, \mathrm{Part}(j)), \varpi([p]) = d| \\ &\equiv \sum_d d|[p] \in \mathrm{Orb}(C, \mathrm{Part}(j)), \varpi([p]) = d| \mod 2 \\ &= 2^{j-1} \equiv 0 \mod 2. \end{aligned}$$

Then by Lemma 4.8, we have

$$|l \in \mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j, j)), [l] = e \cdot [l]| \equiv 0 \mod 2.$$

□

Therefore, we have

$$\begin{aligned} |\mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j, j)^\tau)^s| \equiv & \sum_{[l] \in \mathrm{Orb}(C_j, \mathrm{Neck}(j, \frac{j}{2})), [l] \neq e \cdot [l]} \frac{|\phi^{-1}([l], [l])|}{2} \\ & + |l \in \mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j, j)), [l] = e \cdot [l], \nu_2(\pi([l])) = 1| \mod 2. \end{aligned} \tag{4.2}$$

4.3. Conclusion of the proof of Theorem 1.2. First, we have:

Lemma 4.10. *For $[l] \in \mathrm{Orb}(C_j, \mathrm{Neck}(j, \frac{j}{2}))$, we have $|\phi^{-1}([l], [l])| = \frac{\pi([l])}{2}$.*

Proof. Similar to Lemma 3.17. □

Then

$$\begin{aligned} \sum_{[l] \in \mathrm{Orb}(C_j, \mathrm{Neck}(j, \frac{j}{2})), [l] \neq e \cdot [l]} \frac{|\phi^{-1}([l], [l])|}{2} &= \sum_{[l] \in \mathrm{Orb}(C_j, \mathrm{Neck}(j, \frac{j}{2})), [l] \neq e \cdot [l]} \frac{|\pi([l])|}{2} \\ &= \sum_{[l] \in \mathrm{Orb}(C_j, \mathrm{Neck}(j, \frac{j}{2}))} \frac{|\pi([l])|}{2} - \sum_{[l] \in \mathrm{Orb}(C_j, \mathrm{Neck}(j, \frac{j}{2})), [l] = e \cdot [l]} \frac{|\pi([l])|}{2} \end{aligned}$$

Since $\pi([l]) \equiv 0 \mod 2$, we have

$$\sum_{[l] \in \mathrm{Orb}(C_j, \mathrm{Neck}(j, \frac{j}{2})), [l] = e \cdot [l]} \frac{|\pi([l])|}{2} \equiv |[l] \in \mathrm{Orb}(C_j, \mathrm{Neck}(j, \frac{j}{2})), [l] = e \cdot [l], \nu_2(\pi(l)) = 1| \mod 2.$$

As

$$\{[l] \in \mathrm{Orb}(C_j, \mathrm{Neck}(j, \frac{j}{2})), [l] = e \cdot [l], \nu_2(\pi(l)) = 1\} = \{[l] \in \mathrm{Orb}(C_j, \mathrm{Neck}(2j, j)), [l] = e \cdot [l], \nu_2(\pi(l)) = 1\},$$

we can conclude

Proposition 4.11. $\Delta'(2j, j) = \frac{1}{2}\binom{2j}{j} \cdot (u - 1)$, *for $j \equiv 0 \mod 2$.*

PROOF. By (4.2), and the calculation in this subsection we have

$$|\mathrm{Orb}(C_{2j}, \mathrm{Neck}(2j,j)^{\tau})^{s}| \equiv \frac{1}{2}\binom{j}{\frac{j}{2}} \mod 2,$$

Then by Lemma 4.2 and Corollary 3.5 of Kummer's theorem, we have

$$\Delta'(2j,j) = \frac{1}{2}\binom{2j}{j} \cdot (u-1)$$

□

The proof of Theorem 1.2 is concluded by combining Proposition 4.11 and Corolarry 4.4.

4.4. Explicit value of twisted binomial coefficients. By Corollary 3.7 of Kummer's theorem, we have

$$\binom{L[Q]/k}{j} = \binom{2j}{j} + (u-1)\delta(j),$$

where $\delta(j) = \begin{cases} 1 & j = 2^m, m \in \mathbb{Z}_{>0} \\ 0 & \text{else} \end{cases}$.

The first few terms of the sequence $\binom{L[Q]/k}{j}$ are

$$2, 5+u, 20, 69+u, 252, 924, 3432, 12869+u, \ldots.$$

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