CARTIER'S FIRST THEOREM FOR WITT VECTORS ON $\mathbb{Z}^n_{\geq 0} - 0$

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ABSTRACT. We show that the dual of the Witt vectors on $\mathbb{Z}^n_{\geq 0} - 0$ as defined by Angeltveit, Gerhardt, Hill, and Lindenstrauss represent the functor taking a commutative formal group G to the maps of formal schemes $\hat{\mathbb{A}}^n \to G$, and that the Witt vectors are self-dual for $\mathbb{Q}$-algebras or when $n = 1$.

1. INTRODUCTION

Hesselholt and Madsen computed the relative K-theory of $k[x]/\langle x^a \rangle$ for k a perfect field of positive characteristic in [HM], and give the answer in terms of the Witt vectors of k. In the analogous computation for the ring $A = k[x_1, \ldots, x_n]/\langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle$, Angeltveit, Gerhardt, Hill, and Lindenstrauss define an n-dimensional version of the Witt vectors, which they use to express the relative K-theory and topological cyclic homology of A [AGHL].

We show that the Cartier dual of the additive group underlying their Witt vectors on the truncation set $\mathbb{Z}^n_{\geq 0} - 0$, denoted $\mathbb{W}_{\mathbb{Z}^n_{\geq 0} - 0}$, represents the functor taking a commutative formal group G to the pointed maps of formal schemes $\hat{\mathbb{A}}^n \to G$ (Theorem 2.2). We also show that the additive group of $\mathbb{W}_{\mathbb{Z}^n_{\geq 0} - 0}$ is self dual (Lemma 2.4) when $n = 1$ or R is a $\mathbb{Q}$-algebra. Combining these results implies that the additive formal group of $\mathbb{W}_{\mathbb{Z}^n_{\geq 0} - 0}$ represents the functor sending G to the group of maps $\hat{\mathbb{A}}^n \to G$ when when $n = 1$ or R is a $\mathbb{Q}$-algebra. The case of $n = 1$ is Cartier's first theorem [C] [H, Th. 27.1.14] on the classical Witt vectors.

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2. CARTIER'S FIRST THEOREM FOR WITT VECTORS ON $\mathbb{Z}^n_{\geq 0} - 0$

Here is Angeltveit, Gerhardt, Hill, and Lindenstrauss's n-dimensional version of the Witt vectors, defined in Section 2 of [AGHL]: a set $S \subseteq \mathbb{Z}^n_{\geq 0} - 0$ is a *truncation set* if

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$(kj_1, kj_2, \dots, kj_n)$ in S for $k \in \mathbb{N} = \mathbb{Z}_{>0}$ implies that $(j_1, j_2, \dots, j_n)$ is in S. For $\vec{J} = (j_1, \dots, j_n)$ in $\mathbb{Z}^n_{\geq 0} - 0$, let $\gcd(\vec{J})$ denote the greatest common divisor of the non-zero j_i. Given a ring R and a truncation set S, let the Witt vectors $\mathbb{W}_S(R)$ be the ring with underlying set R^S and addition and multiplication defined so that the ghost map

$$\mathbb{W}_S(R) \to R^S$$

that takes $\{r_{\vec{I}} : \vec{I} \in S\}$ to $\{w_{\vec{I}} : \vec{I} \in S\}$ where

$$w_{\vec{I}} = \sum_{k\vec{J}=\vec{I}} \gcd(\vec{J}) r_{\vec{J}}^k$$

is a ring homomorphism, where in the above sum, k ranges over $\mathbb{N}$ and $\vec{J}$ is in S. In [AGHL], one requires S to be a subset of $\mathbb{N}^n$, but the same proof that there is a unique functorial way to define such a ring structure [AGHL, Lem 2.1] holds for $S \subseteq \mathbb{Z}^n_{\geq 0} - 0$. Note that

$$\mathbb{W}_S(R) = \prod_{Z \subsetneq \{1,\dots n\}} \mathbb{W}_{S_Z}(R)$$

where S_Z is defined $S_Z = \{(j_1, \dots, j_n) \in S : j_i = 0 \text{ if and only if } i \in Z\}$, and that for $S = \mathbb{Z}^n_{\geq 0} - 0$, we have $\mathbb{W}_{S_Z}(R) \cong \mathbb{W}_{\mathbb{N}^m}(R)$ with $m = n - |Z|$.

Let R be a ring. For any truncation set S, the additive group underlying the ring $\mathbb{W}_S(R)$ determines a commutative group scheme and formal group over R.

Let $\hat{\mathbb{A}}^n = \operatorname{Spf} R[[t_1, t_2, \dots, t_n]]$ be formal affine n-space and consider $\hat{\mathbb{A}}^n$ as a pointed formal scheme, equipped with the point $\operatorname{Spf} R \to \hat{\mathbb{A}}^n$ corresponding to the ideal $\langle t_1, \dots t_n \rangle$. Let $\mathrm{Mor}_{fs}(\hat{\mathbb{A}}^n, G)$ denote the morphisms of pointed formal schemes over R from $\hat{\mathbb{A}}^n$ to a pointed formal R-scheme G. The identity of a formal group G gives G the structure of a pointed formal scheme.

For commutative formal groups G_1 and G_2 over R, let $\mathrm{Mor}_{fg}(G_1, G_2)$ denote the corresponding morphisms.

2.1. *Theorem. — Suppose R is a $\mathbb{Q}$-algebra or $n = 1$. The additive formal group of $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$ represents the functor*

$$G \mapsto \mathit{Mor}_{fs}(\hat{\mathbb{A}}^n, G)$$

from commutative formal groups over R to groups, i.e. there is a natural identification

$$\mathit{Mor}_{fg}(\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R), G) \cong \mathit{Mor}_{fs}(\hat{\mathbb{A}}^n, G)$$

for commutative formal groups G over R.

Theorem 2.1 is proven by combining Theorem 2.2 and Lemma 2.6 below.

Cartier duality gives a contravariant equivalence between certain topological R-algebras and R-coalgebras [H, Prop 37.2.7]. For such a topological R-algebra (respectively coalgebra) B, let B^* denote its Cartier dual

$$B^* = \mathrm{Mor}_R(B, R)$$

where $\mathrm{Mor}_R(B, R)$ denotes the continuous R-module homomorphisms from B to R (respectively the R-module homomorphisms from B to R). Say that an algebra or coalgebra is *augmented* if it is equipped with a splitting of the unit or counit map. It is straightforward to see that Cartier duality induces an equivalence between augmented topological R-algebras satisfying the conditions of [H, 37.2.4] and augmented R-coalgebras satisfying the conditions of [H, 37.2.5]. Denote the morphisms in the former category by $\mathrm{Mor}_{\text{top alg}}(-,-)$ and the morphisms in the latter category by $\mathrm{Mor}_{\text{coalg}}(-,-)$.

The commutative group scheme determined by the additive group underlying $\mathbb{W}_S(R)$ has a Cartier dual $\mathbb{W}_S(R)^\star$ which is a topological Hopf algebra or formal group.

2.2. *Theorem. — The Cartier dual of the additive group scheme of* $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$ *represents the functor*

$$G \mapsto \mathit{Mor}_{fs}(\hat{\mathbb{A}}^n, G)$$

from commutative formal groups over R *to groups, i.e. there is a natural identification*

$$\mathit{Mor}_{fg}(\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)^\star, G) \cong \mathit{Mor}_{fs}(\hat{\mathbb{A}}^n, G)$$

for commutative formal groups G *over* R.

Proof. First assume that the formal group G is affine. Let A denote the functions of G, so A is a Hopf algebra and $G = \operatorname{Spf} A$.

$$\mathrm{Mor}_{\text{fs}}(\hat{\mathbb{A}}^n, G) = \mathrm{Mor}_{\text{top alg}}(A, R[[t_1, t_2, \dots, t_n]]).$$

By Cartier duality,

$$\mathrm{Mor}_{\text{top alg}}(A, R[[t_1, t_2, \dots, t_n]]) = \mathrm{Mor}_{\text{coalg}}(R[[t_1, t_2, \dots, t_n]]^\star, A^\star).$$

Let F denote the left adjoint to the functor taking a Hopf algebra (as defined [H, 37.1.7]) to its underlying augmented coalgebra. Since A is a Hopf algebra, so is $A^\star$. Therefore,

$$\mathrm{Mor}_{\text{coalg}}(R[[t_1, t_2, \dots, t_n]]^\star, A^\star) = \mathrm{Mor}_{\text{Hopf alg}}(F(R[[t_1, t_2, \dots, t_n]]^\star), A^\star)$$
$$= \mathrm{Mor}_{\text{top Hopf alg}}(A, F(R[[t_1, t_2, \dots, t_n]]^\star)^\star) = \mathrm{Mor}_{\text{fg}}(\operatorname{Spf} F(R[[t_1, t_2, \dots, t_n]]^\star)^\star, G),$$

where $\mathrm{Mor}_{\text{top Hopf alg}}(-,-)$ denotes morphisms of topological Hopf algebras whose underlying topological R-algebra is as before.

By Lemma 2.3 proven below, the formal group $\operatorname{Spf} F(R[[t_1, t_2, \dots, t_n]]^\star)^\star$ is isomorphic to the Cartier dual of the additive group scheme of $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$.

Thus $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)^\star$ represents the functor $G \mapsto \mathrm{Mor}_{\text{fs}}(\hat{\mathbb{A}}^n, G)$ restricted to affine commutative formal groups G. Since $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)^\star$ is an affine formal group, the identity morphism determines an element of $\mathrm{Mor}_{\text{fs}}(\hat{\mathbb{A}}^n, \mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)^\star)$, which in turn defines a natural transformation

$$\eta : \mathrm{Mor}_{\text{fg}}(\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)^\star, -) \to \mathrm{Mor}_{\text{fs}}(\hat{\mathbb{A}}^n, -).$$

For any formal group G, the sets $\mathrm{Mor}_{\text{fg}}(\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)^\star, G)$ and $\mathrm{Mor}_{\text{fs}}(\hat{\mathbb{A}}^n, G)$ extend to sheaves on $\operatorname{Spf} R$. Since locally on $\operatorname{Spf} R$, every formal group G is affine, η is a natural isomorphism.

$\square$

2.3. *Lemma.* — *The group scheme determined by the Hopf algebra* $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ *is isomorphic to the additive group scheme of* $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$.

Proof. For notational convenience, given $\vec{I} = (i_1, i_2, \ldots, i_n)$ and $\vec{J} = (j_1, \ldots, j_n)$ in $\mathbb{Z}^n_{\geq 0}$, let $t^{\vec{I}} = t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$, and write $\vec{I} \leq \vec{J}$ when $i_k \leq j_k$ for all k.

$R[[t_1, t_2, \ldots, t_n]]^\star$ is a free R-module on the basis $\{b_{\vec{I}} : \vec{I} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}\}$ where $b_{\vec{I}}$ is dual to $t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$. The R-coalgebra structure is given by the comultiplication

$$b_{\vec{I}} \mapsto \sum_{0 \leq \vec{J} \leq \vec{I}} b_{\vec{J}} \otimes b_{\vec{I}-\vec{J}}, \tag{1}$$

and the augmentation $R \to R[[t_1, t_2, \ldots, t_n]]^\star$ sends r to $r b_{\vec{0}}$.

It follows that $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ is the polynomial algebra

$$R[b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0}]/\langle b_{\vec{0}} - 1 \rangle$$

with comultiplication equal to the R-algebra morphism determined by (1). Thus, for any R-algebra B

$$\mathrm{Mor}_{\mathrm{alg}}(F(R[[t_1, t_2, \ldots, t_n]]^\star), B)$$

is the group under multiplication of power series in n variables $t_1, t_2, \ldots, t_n$ with leading coefficient 1 and coefficients in B

$$\{1 + \sum_{\vec{I} \in \mathbb{Z}^n_{\geq 0} - 0} b_{\vec{I}} t^{\vec{I}} : b_{\vec{I}} \in B\}. \tag{2}$$

Any such power series can be written uniquely in the form

$$\prod_{\vec{I} \in \mathbb{Z}^n_{\geq 0} - 0} (1 - a_{\vec{I}} t^{\vec{I}}) \tag{3}$$

with $a_{\vec{I}} \in B$. It follows that $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ is isomorphic as a Hopf algebra to the polynomial algebra $R[a_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0} - 0]$ with comultiplication determined by multiplication of power series of the form (3). By the definition of the Witt vectors, it suffices to show that the Witt polynomials $\sum_{k\vec{J}=\vec{I}} \gcd(\vec{J}) a_{\vec{J}}^k$ are primitives for this comultiplication for all $\vec{I}$ in $\mathbb{Z}^n_{\geq 0} - 0$. To show this, we may assume that R is a free ring, since every ring is a quotient of a free ring. Then R embeds into its field of fractions, so we may further assume that k

is invertible for all $k \in \mathbb{Z}_{>0}$. Note that

$$\begin{aligned}
\log \prod_{\vec{I}\in\mathbb{Z}^n_{\geq 0}-0} (1-a_{\vec{I}}t^{\vec{I}}) &= -\sum_{\vec{I}\in\mathbb{Z}^n_{\geq 0}-0}\sum_{k\in\mathbb{N}} \frac{a^k_{\vec{I}}}{k} t^{k\vec{I}} \\
&= -\sum_{\vec{I}\in\mathbb{Z}^n_{\geq 0}-0}\sum_{k\vec{J}=\vec{I}} \frac{a^k_{\vec{J}}}{k} t^{\vec{I}} \\
&= \sum_{\vec{I}\in\mathbb{Z}^n_{\geq 0}-0} \Big(\sum_{k\vec{J}=\vec{I}} \gcd(\vec{J}) a^k_{\vec{J}}\Big) \frac{-t^{\vec{I}}}{\gcd(\vec{I})}.
\end{aligned}$$

Thus the group under multiplication with elements (3) is isomorphic to the group with elements $\{a_{\vec{I}} \in B\}$ and whose group operation is such that $\big(\sum_{k\vec{J}=\vec{I}} \gcd(\vec{J}) a^k_{\vec{J}}\big)$ is an additive homomorphism, i.e. the Witt polynomials $\sum_{k\vec{J}=\vec{I}} \gcd(\vec{J}) a^k_{\vec{J}}$ are indeed primitives as desired. □

The additive group scheme of $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$ corresponds to a *graded* Hopf algebra, meaning that there is a grading on the underlying R-module such that the structure maps are maps of graded R-modules. This grading can be defined by giving $a_{\vec{J}}$ as in Lemma 2.3 degree $j_1 + j_2 + \ldots + j_n$. A graded Hopf algebra B whose underlying graded R-module is free and finite rank in each degree has a graded Hopf algebra dual B^* which we define to have mth graded piece $\mathrm{Gr}_m B^* = \mathrm{Hom}_R(\mathrm{Gr}_m B, R)$ and

$$B^* = \oplus_m \mathrm{Gr}_m B^*.$$

Note the difference with the Cartier dual

$$B^\star = \prod_m \mathrm{Gr}_m B^*.$$

Say that a graded Hopf algebra B is *self dual* if there is an isomorphism $B \cong B^*$. An affine group scheme corresponding to a graded Hopf algebra will be called *self dual* if its corresponding graded Hopf algebra is self dual.

2.4. *Lemma. — The graded additive group scheme of* $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$ *is self dual if* R *is a* $\mathbb{Q}$*-algebra or if* $n = 1$.

Proof. We give an isomorphism of graded Hopf algebras

$$F(R[[t_1, t_2, \ldots, t_n]]^\star) \cong F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$$

which is equivalent to the claim by Lemma 2.3.

We saw above that $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ is the polynomial algebra

$$R[b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0}]/\langle b_{\vec{0}} - 1\rangle$$

with comultiplication determined by (1). Thus, an R-basis for $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ is given by the collection of monomials $b^{m_1}_{\vec{I}_1} b^{m_2}_{\vec{I}_2} b^{m_3}_{\vec{I}_3} \cdots b^{m_k}_{\vec{I}_k}$ in the variables $\{b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0} - 0\}$. Let

$\mathcal{C} = \{c_{\vec{I}_1^{m_1}\vec{I}_2^{m_2}\vec{I}_3^{m_3}\cdots\vec{I}_k^{m_k}} : m_j > 0, \vec{I}_j \in \mathbb{Z}^n_{\geq 0} - 0\}$ denote the dual basis of $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$. For notational convenience, we will also write $c_{\vec{I}_1^{m_1}\vec{I}_2^{m_2}\vec{I}_3^{m_3}\cdots\vec{I}_k^{m_k}}$ even when some of the m_j are 0; it is to be understood that such an expression is identified with the corresponding expression with the $\vec{I}_j^{m_j}$ terms with $m_j = 0$ removed.

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}^n$, so $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$ etc. For notational convenience, for $\vec{M} = (m_1, m_2, \ldots, m_n)$ in $\mathbb{Z}^n_{\geq 0} - 0$, let $C_{\vec{M}}$ abbreviate $c_{e_1^{m_1}e_2^{m_2}\cdots e_n^{m_n}}$.

Note that

$$\mu(C_{\vec{M}}) = \sum_{0 \leq \vec{J} \leq \vec{M}} C_{\vec{J}} \otimes C_{\vec{M}-\vec{J}}$$

where μ denotes the comultiplication of $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$.

Sending $b_{\vec{I}}$ to $C_{\vec{I}}$ thus defines a morphism of Hopf algebras

$$F(R[[t_1, t_2, \ldots, t_n]]^\star) \to F(R[[t_1, t_2, \ldots, t_n]]^\star)^*,$$

and to prove the lemma it suffices to see that the $C_{\vec{I}}$ are free R-algebra generators of $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$ when either $n = 1$ or $\mathbb{Q} \subseteq R$.

We first show that the $C_{\vec{I}}$ generate $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$ as an R-algebra in both cases:

First assume that $n = 1$. We show that the $C_m = c_{e_1^m}$ for $m = 1, 2, 3, \ldots$ generate $F(R[[t_1]]^\star)^*$ as an R-algebra. An arbitrary element c of $\mathcal{C}$ is of the form $c_{i_1, i_2, \cdots, i_k}$ with the i_k not necessarily distinct in $\mathbb{Z}_{>0}$. Define the degree of c to be $d = \sum_{j=1}^k i_j$. Assume by induction that any element of $\mathcal{C}$ of degree less than d is in the subalgebra generated by the C_m. Define the length of c to be k. The length of c must be less than or equal to d. If the length of c equals d, then $c_{i_1, i_2, \cdots, i_k} = C_k$ and c is in the subalgebra. So we may assume by induction that any element of $\mathcal{C}$ of degree d and length greater than k is in the subalgebra. The multiplication on $F(R[[t_1]]^\star)^*$ is dual to

$$b_{i_1} b_{i_2} b_{i_3} \cdots b_{i_k} \mapsto \prod_{j=1}^{k} (\sum_{0 \leq J \leq i_j} b_J \otimes b_{i_j - J}).$$

Thus the difference

$$c - c_{i_1-1, i_2-1, \cdots, i_k-1} C_k$$

is a sum of terms of degree d and length greater than k. It follows by induction that the $C_m = c_{e_1^m}$ generate $F(R[[t_1]]^\star)^*$ as claimed.

Now let n be arbitrary. Consider the map $f : R[[t]] \to R[[t_1, \ldots, t_n]]$ defined by

$$f(t) = t_1 + t_2 + \ldots + t_n.$$

There is an induced map

$$f : F(R[[t_1, \ldots, t_n]]^\star) \cong R[b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0} - 0] \to R[b_m : m \in \mathbb{Z}_{>0}] \cong F(R[[t]]^\star)$$

which is determined by the following calculation of $f(b_{\vec{I}})$ for $\vec{I} = (i_1, i_2, \ldots, i_n)$.

$$f(b_{\vec{I}})(t^m) = b_{\vec{I}}(f(t^m)) = b_{\vec{I}}(t_1 + \ldots + t_n)^m$$
$$= b_{\vec{I}}\Big(\sum_{a_1,\ldots a_n \geq 0} \binom{m}{a_1 a_2 \cdots a_n} t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} \Big),$$

where the sum runs over non-negative a_i whose sum is m and where

$$\binom{m}{a_1 a_2 \cdots a_n} = \frac{m!}{a_1! a_2! \cdots a_n!}.$$

Thus

$$f(b_{\vec{I}}) = \binom{d}{i_1 i_2 \cdots i_n} b_d,$$

where $d = \sum_{j=1}^{n} i_j$. There is likewise an induced map

$$f : F(R[[t]]^\star)^* \cong R[c_{i_1,i_2,\cdots,i_k} : i_j \in \mathbb{Z}_{>0}] \to R[c_{\vec{I}_1 \vec{I}_2 \cdots \vec{I}_k} : \vec{I}_j \in \mathbb{Z}^n_{\geq 0} - 0] \cong F(R[[t_1, t_2, \ldots, t_n]]^\star)^*.$$

By calculation as above, this map satisfies

$$f(C_m) = \sum_{\text{degree } \vec{I} = m} C_{\vec{I}}$$

where the sum runs over $\vec{I} \in \mathbb{Z}^n_{\geq 0}$ of degree m, and

$$f(c_m) = \sum_{\text{degree } \vec{I} = m} \binom{m}{\vec{I}} c_{\vec{I}},$$

where

$$\binom{m}{\vec{I}} c_{\vec{I}} = \binom{m}{i_1 i_2 \ldots i_n}$$

when $\vec{I} = (i_1, i_2, \ldots, i_n)$. By the $n = 1$ case, $f(c_m)$ is in the R-subalgebra generated by the $f(C_m)$. Since $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$ is a $\mathbb{Z}^n$-graded Hopf algebra, it follows that the homogenous pieces of $f(c_m)$ are in the R-subalgebra generated by the homogeneous pieces of $f(C_m)$. Thus $\binom{m}{\vec{I}} c_{\vec{I}}$ is in the R-subalgebra generated by the $C_{\vec{I}}$. Since $\binom{m}{\vec{I}}$ is invertible in R, it follows that $c_{\vec{I}}$ is in this subalgebra.

An arbitrary element c of $\mathcal{C}$ is of the form $c_{\vec{I}_1 \vec{I}_2 \cdots \vec{I}_k}$. The multiplication on $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$ is dual to

$$b_{\vec{I}_1} b_{\vec{I}_2} b_{\vec{I}_3} \cdots b_{\vec{I}_k} \mapsto \prod_{j=1}^{k} \Big(\sum_{0 \leq \vec{J} \leq \vec{I}_j} b_{\vec{J}} \otimes b_{\vec{I}_j - \vec{J}} \Big).$$

It follows that the difference $c - c_{\vec{I}_1 \vec{I}_2 \cdots \vec{I}_{k-1}} c_{\vec{I}_k}$ is a linear combination of elements of $\mathcal{C}$ of length less than k. Thus c is in the R-subalgebra generated by the $C_{\vec{I}}$ by induction on the length k.

We now show that there are no relations among the $C_{\vec{I}}$, i.e. that the distinct monomials $C_{\vec{I}_1} C_{\vec{I}_2} \cdots C_{\vec{I}_k}$ form an R-linearly independent subset of $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$:

Fix $\vec{M}$ in $\mathbb{Z}^n_{\geq 0} - 0$. Let $\mathcal{I}$ denote the set of finite sets $\{\vec{I}_1, \vec{I}_2, \ldots, \vec{I}_k\}$ with $\vec{I}_j$ in $\mathbb{Z}^n_{\geq 0} - 0$ and $\sum_{j=1}^{k} \vec{I}_j = \vec{M}$. For S in $\mathcal{I}$ with $S = \{\vec{I}_1, \vec{I}_2, \ldots, \vec{I}_k\}$, let $C_S = \prod_{j=1}^{k} C_{\vec{I}_j}$ in $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$

and let $c_S = c_{\vec{I}_1\vec{I}_2\cdots\vec{I}_k}$ in $\mathcal{C}$. Note that for all S in $\mathcal{I}$, C_S is in the sub-R-module $\mathcal{F}_{\vec{M}}$ spanned by $\{c_S : S \in \mathcal{I}\}$. By the above, $\{C_S : S \in \mathcal{I}\}$ spans $\mathcal{F}_{\vec{M}}$. Since $\mathcal{F}_{\vec{M}}$ is isomorphic to R^N where N is the (finite) cardinality of $\mathcal{I}$, any spanning set of size N is also a basis [AM, Ch 3 Exercise 15]. In particular $\{C_S : S \in \mathcal{I}\}$ is an R-linearly independent set. Since any monomial in the $C_{\vec{I}}$ is of the form C_S for some $\vec{M}$, it follows that the distinct monomials in the $C_{\vec{I}}$ form a linearly independent set. □

2.5. *Remark.* The $C_{\vec{I}}$ do not generate $F(R[[t_1, t_2, t_3]]^\star)^*$ when 2 is not invertible in R as can be checked by computing that the homogenous degree-$(1, 1, 1)$ component of the R-subalgebra generated by the $C_{\vec{I}}$ is the span of the following five vectors

$$\begin{aligned}
C_{e_1}C_{e_2}C_{e_3} &= c_{(1,1,1)} + c_{(1,1,0)(0,0,1)} + c_{(1,0,1)(0,1,0)} + c_{(0,1,1)(1,0,0)} + c_{e_1e_2e_3},\\
C_{(0,1,1)}C_{e_1} &= c_{(1,1,0)(0,0,1)} + c_{(1,0,1)(0,1,0)} + c_{e_1e_2e_3}\\
C_{(1,0,1)}C_{e_2} &= c_{(0,1,1)(1,0,0)} + c_{(1,1,0)(0,0,1)} + c_{e_1e_2e_3}\\
C_{(1,1,0)}C_{e_3} &= c_{(1,0,1)(0,1,0)} + c_{(0,1,1)(1,0,0)} + c_{e_1e_2e_3}\\
C_{(1,1,1)} &= c_{e_1e_2e_3}.
\end{aligned}$$

2.6. *Lemma. — If* R *is a* $\mathbb{Q}$*-algebra or if* $n = 1$*, the Cartier dual of the additive group scheme of* $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$ *is the formal group associated to the additive group of* $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$.

Proof. By Lemma 2.3, the claim is equivalent to showing that the topological Hopf algebra $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$ is the ring of functions of the formal group associated to the additive group of $\mathbb{W}_{\mathbb{Z}^n_{\geq 0}-0}(R)$.

The Cartier dual $F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$ of the Hopf algebra $F(R[[t_1, t_2, \ldots, t_n]]^\star)$ is the product

$$F(R[[t_1, t_2, \ldots, t_n]]^\star)^* \cong \prod_{m=0}^{\infty} \mathrm{Gr}_m\, F(R[[t_1, t_2, \ldots, t_n]]^\star)^*$$

over m of the mth graded pieces of the graded Hopf algebra dual. By Lemma 2.4,

$$F(R[[t_1, t_2, \ldots, t_n]]^\star)^* \cong F(R[[t_1, t_2, \ldots, t_n]]^\star) \cong R[b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0}]/\langle b_{\vec{0}} - 1\rangle,$$

with comultiplication determined by (1). So

$$\prod_{m=0}^{\infty} \mathrm{Gr}_m\, F(R[[t_1, t_2, \ldots, t_n]]^\star)^* \cong R[[b_{\vec{I}} : \vec{I} \in \mathbb{Z}^n_{\geq 0}]]/\langle b_{\vec{0}} - 1\rangle,$$

and applying Lemma 2.3 completes the proof.

□

References

[AGHL] Angeltveit, V., Gerhardt, T., Hill, M.A. and Lindenstrauss, A. *On the Algebraic K-Theory of Truncated Polynomial Algebras in Several Variables*, arXiv:1206.0247, 2012.

[AM] Atiyah, M. F. and Macdonald, I. G., *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont.,1969.

[C] Cartier, P., *Modules associés à un groupe formel commutatif. Courbes typiques*, C. R. Acad. Sci. Paris Sér. A-B, vol. 265, 1967, A129–A132.

[H] Hazewinkel, M., *Formal groups and applications*, Pure and Applied Mathematics, vol. 78, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York,1978.

[HM] Hesselholt, L. and Madsen, I., *Cyclic polytopes and the* K*-theory of truncated polynomial algebras*, Inventiones Mathematicae, vol. 130, 1997, number 1, pg. 73–97.

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