

# $A_{\infty}$ -vs $\infty$ -categories

AGUST TALK BY HIRSH

10:15 AM

## Basics:

- A ring  $R$  is an ab. gr

w/  
 $R$ ,

$$R \times R \xrightarrow{m} R$$

bilinear, i.e.,

$$R \otimes R \xrightarrow{m} R.$$

w/ assoc., (unitality).

2 directions of generalization:

- homotopical  $\rightsquigarrow$  chain cplxs.  
 $\rightsquigarrow$   $A_{\infty}$ -algebras.

- categorical!  $\rightsquigarrow$  cat. enriched M  
 abelian gpΣ

Set.  
v

rmk Chain cplxs are "homotopical"  
 b/c morphisms btwn ch cplxs  
 can have homotopies.

$\implies$  dg-algebra.

DEFN. A dg-algebra  $R$   
 is a ch cplx w/  $\xrightarrow{m}$  a map of ch  
 cplxs.

$$R \otimes R \xrightarrow{m} R$$

satisfying associativity. (unitality)

we can make articulate

a more homotopical notion  
 of associativity.

**$A_{\infty}$ -algebra**

{ eq:  $\frac{m(m \otimes 1)}{m(1 \otimes m)}(a \otimes b \otimes c) = m(m(a \otimes b) \otimes c)$   
 ask for a htpy,  
 and higher htpys.

Defn A cat enriched in abelian groups  
is data of:

- A class of objects

ob  $\mathcal{C}$

- $\nexists$  pair  $x, y \in \text{ob } \mathcal{C}$   
on ab gp

$$\text{hom}_{\mathcal{C}}(x, y) = \mathcal{C}(x, y)$$

- $\nexists$  triplet  $x, y, z \in \text{ob } \mathcal{C}$   
a bilinear map

$$\circ : \text{hom}(y, z) \otimes \text{hom}(x, y) \rightarrow \text{hom}(x, z)$$

s.t.  $\circ$  is associative.  
(unitality?)

$$m(d_{R \otimes R}) = d_R \circ m$$

$$= m(d_R \otimes 1 + 1 \otimes d_R)$$

$$\frac{\text{LHS}}{m(d_R \otimes 1 + 1 \otimes d_R)(a \otimes b)}$$

$$= m(d_R(a) \otimes b + (-1)^{|a| \cdot 1} a \otimes d_R(b))$$

$$= d_R(a) \cdot b + (-1)^{|a|} a \cdot d_R(b)$$

$$\frac{\text{RHS}}{(d_R \circ m)(a \otimes b)} = d_R(a \cdot b)$$

Rmk

$\mathcal{C} \Leftrightarrow$  Matrix algebra  
w/ entries  
indexed by

$$\text{ob } \mathcal{C} \times \text{ob } \mathcal{C}$$

$$\bigoplus_{(x,y) \in \text{ob } \mathcal{C} \times \text{ob } \mathcal{C}} \underline{\text{hom}(x,y)}$$

"A category is a ring w/  
many objects"  $\xrightarrow{\text{Ans-category}}$

Making  $\text{hom}(x,y)$  into chain complexes,  
and supplying htprs, rendering  
 $m$  - ie,  $\circ$  - associative

Some concrete formulas:

How to check that  $m$  is  
assoc. up to htpr?

$$m(m_2 \otimes 1) \sim m_2(1 \otimes m_2)$$

A htpr "m<sub>3</sub>" btwn these two operatv  
 $R \otimes R \otimes R \xrightarrow{\quad} R$ .      . TS a dg -1 map  
 (cohom. grady)

$$m_3 : R \otimes R \otimes R \longrightarrow R$$

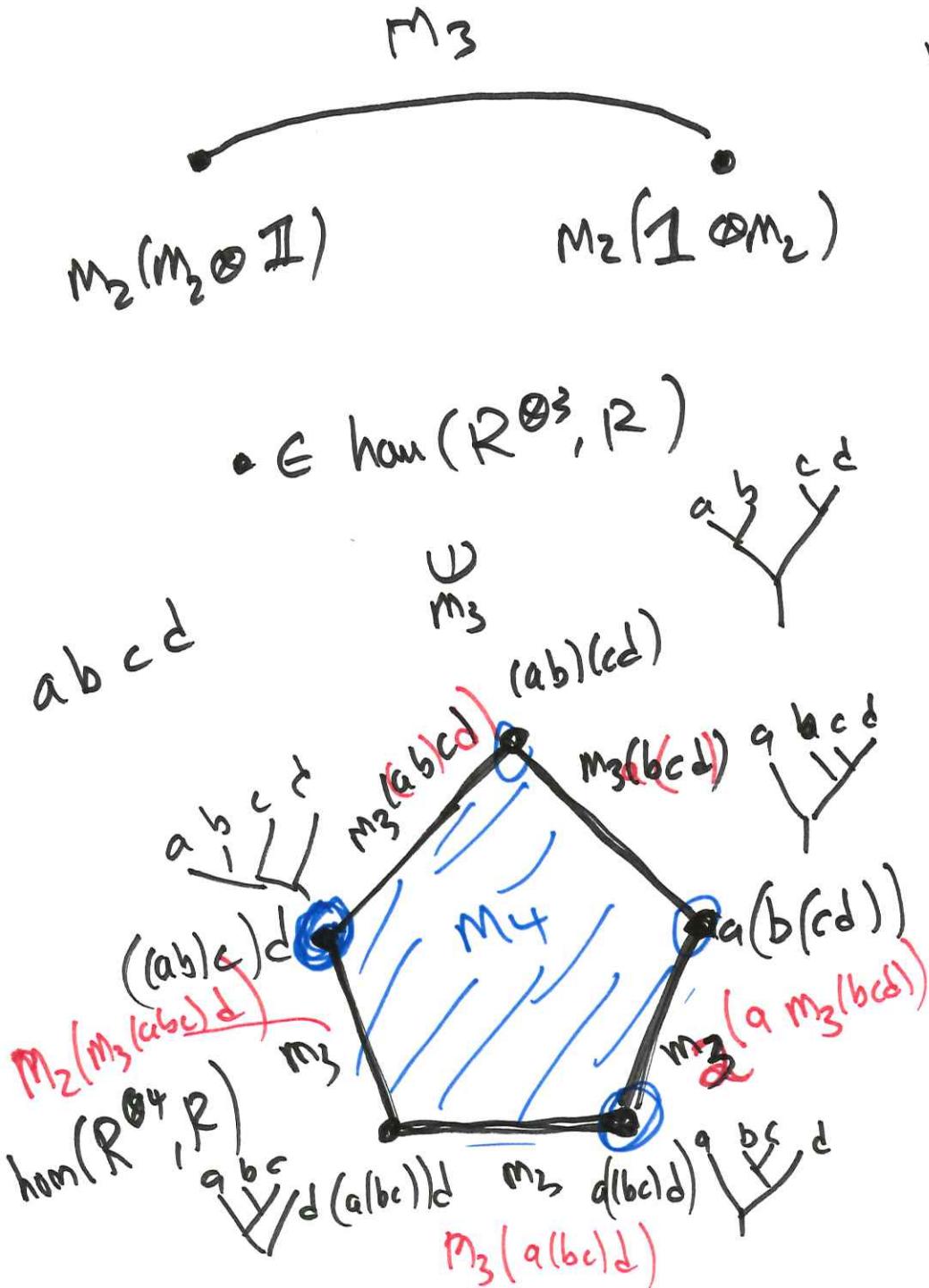
s.t.

$$d_R m_3 + m_3 d_{R \otimes R} = m(m_2 \otimes 1) - m_2(1 \otimes m_2)$$

i.e., *This is the first "Ans-relation".*

$$m, m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_2 \otimes 1 + 1 \otimes 1 \otimes m_2) \\ = m_2(m_2 \otimes 1) - m_2(1 \otimes m_2).$$

Rmk:



In general, by "higher tensors"  
we mean the data of

$$R \otimes \dots \otimes R \xrightarrow{M_k} R^{[2-k]}$$

i.e.,  $M_k$  is a  
deg  $2-k$  map.

satisfying:

$$\sum_{\substack{a+b+c=k \\ a, c \geq 0 \\ b \geq 1}} (-1)^m (I^{\otimes a} \otimes M_b \otimes I^{\otimes c}) = \circ$$

Defn: An  $A_\infty$ -alg  $B$  a gr. ab.

gp  $R$  w/  $\{M_k\}_{k \geq 1}$   
satisfying  $A_\infty$ -relations.  $\circ$

The key difference b/w

A $\infty$ -cats and  $\infty$ -cats

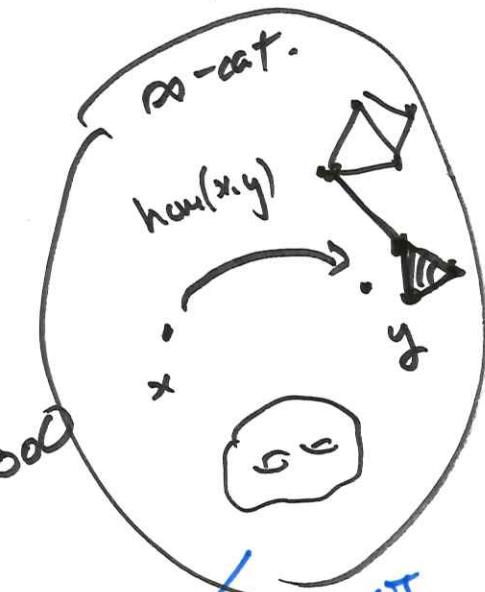
is the difference b/w

ch. cplxs and

simplicial sets  
(spaces)



composition is defined  
(M2) as are  
high n hypers  
for associativity



Composition is NOT  
given, NOR  
the higher  
hypers.

Defn: An  $\infty$ -category is the data

of

• sets

$X_0, X_1, X_2, \dots$

set of  
objects

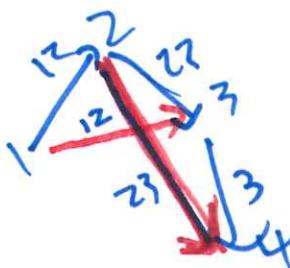
set of  
morphisms..

FACE MAPS

• maps

$d_i: X_k \rightarrow X_{k-1}$

$i = 0, 1, \dots, k$



$s_i: X_{k-1} \rightarrow X_k$

$i = 0, \dots, k-1$

DEGEN.  
MAPS

• satisfying some "simplicial  
relations"

NO Data of a map

$X_i \times X_j \rightarrow X_i$

• AND a  
weak KAN  
conditions.

## Relating the Two

Need a way to take  
chain complex style data

and make

Simplicial data.

Rmk Things are easier when (dg-cat.)

$$m_3 = m_4 = m_5 = \dots = 0.$$

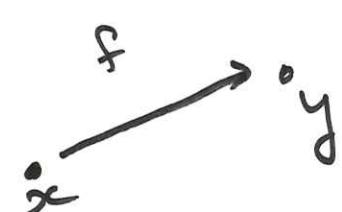
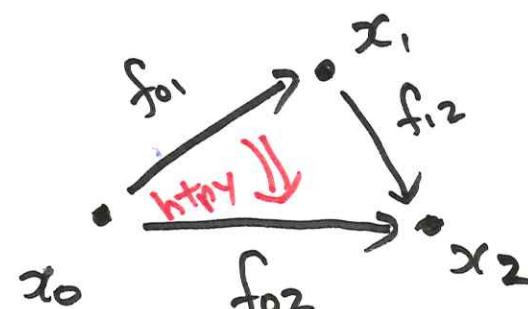
This construction is due to LURIE  
 (dg-nerve)

Fix dg-category  $\mathcal{C}$ .

$N_{dg}(\mathcal{C})$  is the following  
 simplicial cat:

$$N_{dg}(\mathcal{C})_0 := \text{ob } \mathcal{C}$$

$$N_{dg}(\mathcal{C})_1 := \coprod_{(x,y) \in \text{ob} \times \text{ob}} \begin{cases} \text{deg } 0, \\ \text{closed} \\ \text{elts} \end{cases} \text{ of } \text{hom}(x,y)$$



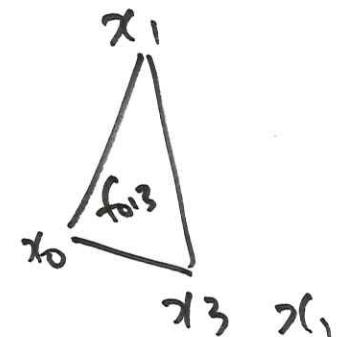
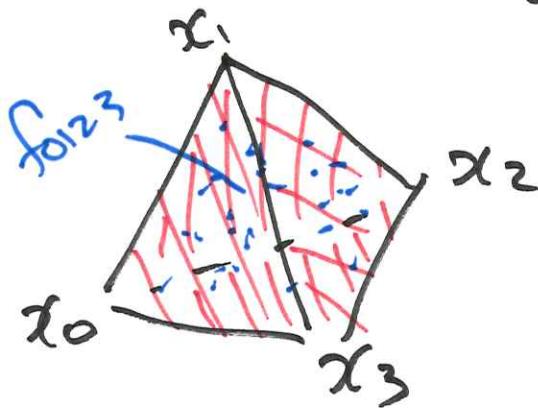
$$\text{if } f = 0, \quad \text{if } f = 0.$$

$$N_{dg}(\mathcal{C})_2 := \coprod_{(x_0, x_1, x_2)} \left\{ \left( \left\{ f_{ij} \right\}_{i < j}, f_{012} \right) \middle| \begin{array}{l} \text{st } f_{012} \in \text{hom}(x_0, x_2) \\ \text{of } \text{deg } -1 \text{ st } df_{012} = \\ f_{02} - f_{12} f_{01} \end{array} \right\}$$

$N_{dg}(\mathcal{C})_3 :=$

$\coprod \left\{ \left( \left\{ f_i \right\}, \left\{ f_{ij} \right\}_k \right), f_{0123} \right\}$   
 $(x_0, x_1, x_2, x_3)$  where

$f_{0123} \in \text{hom}(x_0, x_3)$   
of deg -2 s.t.



$$\begin{aligned} df_{0123} = & f_{013} \pm f_{023} \quad x_0 \swarrow x_2 \\ & \pm f_{23} f_{012} \quad x_0 \downarrow x_3 \\ & \pm f_{023} f_{01} \end{aligned}$$

$N_{dg}(\mathcal{C})_k :=$

Rmk There's an  $\infty$ -nerve as well.  
 $f_{0\dots k} \in \text{hom}(x_0, x_k)$  (Faonte, Tanaka).  
of deg  $-(k-1)$  s.t

$$df_{0\dots k} = \sum_{0 < i < k} \pm f_{0\dots k \setminus \{i\}}$$

$$+ \sum_{0 < i < k} \pm f_{i\dots k} \underbrace{f_{0\dots i}}_{\substack{\vdash \dots \dashv \\ M_{j\bar{j}}}}$$

Lemma / Thm For a unital dg-cut,

$N_{dg}(\mathcal{C})$  is an  $\infty$ -category.

## Properties of $N_{dg}, N_{A\infty}$ :

$H(x,y) \in \text{ob } \mathcal{C} \times \text{ob } \mathcal{C}$

$H^i \hom_{\mathcal{C}}(x,y)$

$$\cong \prod_{-i} \hom_{N_{dg}(\mathcal{C})}(x,y)$$

where  $i \leq 0$ .

$N_{dg}$  is a functor:

$$\begin{aligned} \mathcal{C} &\xrightarrow{F} \mathcal{D} \rightsquigarrow N_{dg}(\mathcal{C}) \\ &\quad \rightarrow N_{dg}(\mathcal{D}) \end{aligned}$$

## Relation to stable $\infty$ -category

Recall An  $\infty$ -cat. is stable if (Q)  $\mathcal{C}$  has a zero obj

(1)  $\mathcal{C}$  has all pushouts and pullbacks

(2)  $\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ O & \longrightarrow & Z \end{array}$  is a p.b.  
iff it's a p.o.

Rmk (2)  $\Leftrightarrow$  (2') where

$\begin{array}{ccc} X & \xrightarrow{\quad} & O \\ \downarrow & & \downarrow \\ O & \xrightarrow{\quad} & X[1] \end{array}$   $X \mapsto X[1]$  is  
an equivalence.

Ex:  $\mathcal{C} = \text{Chain}_R$ .

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{\text{pr.}} & \text{Cone}(f) \end{array}$$

$$\begin{array}{ccc} X & \rightarrow & 0 \\ \downarrow & \text{pr.} \downarrow & \\ 0 & \rightarrow & \text{Cone}(0) = X[-1] \end{array}$$

Rmk Fix  $A$  an  $\infty$ -cat.

$$N_{dg}(A)$$

need NOT be stable.

However, any  $\infty$ -cat  $A$  has

a "triangulated copletion":

$$A \hookrightarrow \text{Fun}(A^{op}, \text{Chain})$$

$\infty$

*Seidel's book,  
Chapter I*

$$\longrightarrow \text{Tw}(A)$$

$\square$  twisted  
(complexes).

$N_{\infty}(\text{Tw}(A))$  IS  
stable.

SLOGAN (Triangulated)  $\infty$ -cats/R ARE  
(Stable)  $\infty$ -cats that are R-linear.