Simplicial Categories

Def: A simplicial category $\mathcal{C}$ is a category enriched over sSet, in the sense that for every $X, Y \in \mathcal{C}$ we have $\mathcal{C}(X, Y) \in$ sSet and a unital and associative composition

$$\circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

Rmk: Such categories can be identified with functors $\Delta^{op} \to \mathrm{Cat}$ (i.e. simplicial objects in Cat)

category of small categories

whose face and degeneracy morphisms are bijective on objects

- A simplicial functor $F : \mathcal{C} \to \mathcal{D}$ can be viewed as $F : ob(\mathcal{C}) \to ob(\mathcal{D})$ and $F : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ respecting Id and s.t. composition is map sSet
(or as a nat'l transf of functors $\Delta^{op} \to \mathrm{Cat}$)

- Let sCat denote category of simplicial cat ex: $\text{path}\,\underline{n} \in s\mathcal{C}at$ objects $= \underline{n} = \{0,1,\dots,n\}$

For $i,j \in \underline{n}$ define $P_{ij} = \big\{ \{ i = i_0 \leq i_1 \leq \cdots \leq i_n = j\} \subset \underline{n} \big\}$

Consider P_{ij} as a poset ordered by reverse inclusion

$$\text{path}\,\underline{n}\,(i,j) := N(P_{ij})$$

cf. HTT Def 1.1.5.1

$$\mathrm{Id}_i = \{i\} \in N(P_{ii})_0$$

Composition: $N(P_{ij}) \times N(P_{jk}) \longrightarrow N(P_{ik})$

$$(I, J) \longmapsto I \cup J$$

Ex $\text{Path}\,\underline{3}\,(0,3)$

different ways of composing

$$\begin{tikzcd}
 & \{0,1,2,3\} \arrow[dl] \arrow[dd] \arrow[dr] & \\
\{0,1,3\} \arrow[dr] & & \{0,2,3\} \arrow[dl] \\
 & \{0,3\} &
\end{tikzcd}$$

<u>Def/Prop</u>: For a simplicial category $\mathcal{C}$, the <u>homotopy coherent nerve</u> $N_H(\mathcal{C})$ is the ∞-cat whose n-simplices are

$$N_H(\mathcal{C})(\underline{n}) = s\mathrm{Cat}(\mathrm{Path}\,\underline{n}, \mathcal{C})$$

- It turns out that the homotopy coherent nerve is an ∞-category (at least when mapping spaces Kan ?)

$$N_H : s\mathrm{Cat} \longrightarrow \infty\text{-Cat}$$

<u>ex</u> $\sigma \in N_H(\mathcal{C})_2$

$$\begin{tikzcd} & X_1 \arrow[dr, "f_{1,2}"] & \\ X_0 \arrow[ur, "f_{0,1}"] \arrow[rr, bend left, "f_{12}\circ f_0"] \arrow[rr, bend right, "f_{02}"'] \arrow[rr, phantom, "\Downarrow"] & & X_2 \end{tikzcd}$$

<u>exercise</u>: $N_H(\mathcal{C})_3$

<u>Rmk</u>: Converse: every ∞-cat is equivalent to the homotopy coherent nerve of some topological category, which is essentially unique. So $\mathcal{C} \to N(\mathcal{C})$ determines an equivalence b/w the theory of top cats and the theory of ∞-cat

Examples of simplicial categories:

1) sSet $\qquad K, M \in sSet$

$sSet(K,M) \in sSet$ defined by

$sSet(K,M)(\underline{n}) = Hom(K \times \Delta^n, M)$

2) $Kan \subset sSet$ full subcategory whose objects are Kan complexes

$Kan(K,M) = sSet(K,M)$

3) $sCat_\infty$ "simplicial category of ∞-cats"

objects are sSets which are ∞-cat

$$sCat_\infty(\mathcal{C}, \mathcal{D}) = \text{core}(sSet(\mathcal{C}, \mathcal{D}))$$

largest Kan complex inside of sSet

We have

$$\mathrm{Kan} \longrightarrow s\mathrm{Cat}_\infty \longrightarrow s\mathrm{Set}$$

Apply N_H

$$\begin{array}{ccc} N_H(\mathrm{Kan}) & \longrightarrow & N_H(s\mathrm{Cat}_\infty) \\ \| \colon & & \| \colon \\ \mathcal{S} & & \mathrm{Cat}_\infty \end{array}$$

"infinity category of spaces" "infinity category of ∞-categories"

<u>Rmk</u>: Cat_∞ is also called "the homotopy theory of homotopy theories"

- A morphism of ∞-Cat $F: \mathcal{C} \to \mathcal{D}$ is a nat'l transf of functors $\Delta^{op} \to \mathrm{Set}$
- We want a notion of "categorical equivalence" for $\mathcal{C} \xrightarrow{F} \mathcal{D}$ ∞-cat or $K \to M$ simplicial set

$$\underline{n} \longmapsto \mathrm{Path}(\underline{n})$$

$$\begin{tikzcd} \Delta \arrow[r, "\mathrm{Path}"] \arrow[d, hook, "\text{Yoneda}"'] & s\mathrm{Cat} \\ s\mathrm{Set} \arrow[ur, dashed] & \end{tikzcd}$$

$\mathfrak{C}$ = "left Kan extension"

For $K \in s\mathrm{Set}$

$$K = \operatorname*{colim}_{(\underline{n},\, \Delta^n \to K)} \Delta^n$$

$$\mathfrak{C}(K) = \operatorname*{colim}_{(\underline{n},\, \Delta^n \to K)} \mathrm{Path}\, \underline{n}$$

For $K \in s\mathrm{Set}$, define $\mathrm{ho}K := \mathrm{ho}\, N_H \mathfrak{C}(K)$

- when K is an ∞-category, this will be equivalent to $\mathrm{ho}\, K$

<u>Def</u>: $F : K \to M$ map in $s\mathrm{Set}$ is said to be a "categorical equivalence" if $\mathfrak{C}(F) : \mathfrak{C}(K) \to \mathfrak{C}(M)$ is an equivalence of simplicial categories in the sence that it induces:

- htpy equivalences on mapping spaces

- an equivalence on htpy categories

Rmk: When K and $\mathcal{M}$ are ∞-categories, this is the same as

F is iso in $\mathrm{ho}\,\mathrm{Cat}_\infty$

Derired cat of a scheme as ∞-cat

Ex R ring

$\mathcal{D}^{\mathrm{Perf}}(R)$ is an ∞-cat

objects: bounded chain complexes of finitely generated projective R-modules

$$\cdots \to 0 \to 0 \to P_N \to P_{N-1} \to \cdots \to P_M \to 0 \to \cdots$$

morphisms: maps of chain complexes $f: P_\bullet \to Q_\bullet$

2-simplices: diagrams

$$\begin{tikzcd} & Q_\bullet \arrow[dr, "g"] & \\ P_\bullet \arrow[ur, "f"] \arrow[rr, "h"] & & R_\bullet \end{tikzcd}$$

with a chain homotopy from h to $g \circ f$

⋮

Rmk: To fill in dot dot dot:

Dold-Kan : $ch_{\geq 0}\mathcal{A} \simeq s\mathcal{A}$ for $\mathcal{A}$ abelian category

By taking chain complexes of modules, we are giving a simplicial direction.

$$Mor(P_\bullet, P_\bullet') \in sSet$$

$$Mor(P_\bullet, P_\bullet')(\underline{n}) = Mor_{s\mathcal{A}}(P_\bullet \otimes \Delta^n, P_\bullet')$$

$$\Delta^{op} \xrightarrow[(-,\underline{n})]{\Delta^n} sSet \xrightarrow[\text{replace } * \text{ by } A]{} s\mathcal{A}$$

So we can make a simplicial category $D^{perf}_{\substack{simp\\cat}}(R)$

ob: bounded chain complexes of finitely generated projective R-modules

$$\cdots \to 0 \to 0 \to P_N \to P_{N-1} \to \cdots \to P_M \to 0 \to \cdots$$

$$D^{perf}_{\substack{simp\\cat}}(R)(P_\bullet, P_\bullet') = Mor(P_\bullet, P_\bullet') \in sSet$$

Then $\mathcal{D}^{perf}(R) = N_H(D^{perf}_{\substack{simp\\cat}}(R))$ ∞-cat

- $\mathrm{ho}\,\mathcal{D}^{\mathrm{Perf}}(R)$ is classical derived cat, meaning there is an equivalence of Δ'd categories

<u>Issue</u>: The Δ'd cats $\mathrm{ho}\,\mathcal{D}^{\mathrm{Perf}}(R)$ do <u>not</u> satisfy Zariski descent

<u>Def</u>: X scheme

$$\mathcal{D}^{\mathrm{Perf}}(X) := \lim_{\substack{\mathrm{Spec}R \subset X \\ \text{open affine}}} \mathcal{D}^{\mathrm{Perf}}(R)$$

Take limit in Cat_∞

<u>References</u>

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