Simplicial Categories

Def: A simplicial category $\mathcal{C}$ is a category enriched over $sSet$, in the sense that for every $X, Y \in \mathcal{C}$ we have $\mathcal{C}(X, Y) \in sSet$ and a unital and associative composition

$$o: \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

Such categories can be identified with functors $\Delta^{op} \to \mathbf{Cat}$ (i.e. simplicial objects in $\mathbf{Cat}$)

whose face and degeneracy morphisms are bijective on objects

A simplicial functor $F: \mathcal{C} \to \mathcal{D}$ can be viewed as $F: \text{ob}(\mathcal{C}) \to \text{ob}(\mathcal{D})$ and $F: \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ respecting $\text{id}$ and s.t. composition is map $sSet$ (or as a natural transf of funcs $\Delta^{op} \to \mathbf{Cat}$)

Let $s\mathbf{Cat}$ denote category of simplicial cat
\[ \text{Ex: path } n \in \text{ sCat} \quad \text{objects } = n = \{0, 1, \ldots, n\} \]

For \( i, j \in n \), define \( P_{ij} = \mathcal{P} \) \( \exists i_0 \leq i_1 \leq \ldots \leq i_n = j \in \mathbb{N} \)

Consider \( P_{ij} \) as a poset ordered by reverse inclusion.

\[ \text{path } n \circ (i, j) := \mathcal{N}(P_{ij}) \]

\[ \text{Id}_i = \{i\} \in \mathcal{N}(P_{ii}) \]

**Composition:**
\[ \mathcal{N}(P_{ij}) \times \mathcal{N}(P_{jk}) \rightarrow \mathcal{N}(P_{ik}) \]

\( (I, J) \rightarrow I \circ J \)

\[ \text{Ex: Path } 3 \circ (0, 3) \]

different ways of composing

\[ \begin{array}{c}
  \{0, 1, 2, 3\} \\
  \downarrow \\
  \{0, 1, 3\} \\
  \downarrow \\
  \{0, 3\} \\
  \downarrow \\
  \{0\} \\
  \end{array} \]

\[ \begin{array}{c}
  \{0, 2, 3\} \\
  \downarrow \\
  \{0, 3\} \\
  \end{array} \]

\[ \begin{array}{c}
  \{0, 1, 2, 3\} \\
  \downarrow \\
  \{0, 1, 3\} \\
  \end{array} \]
**Def:** For a simplicial category $\mathcal{C}$, the homotopy coherent nerve $N^h(\mathcal{C})$ is the $\infty$-cat whose $n$-simplices are

$$N^h(\mathcal{C})(n) = s\text{Cat}(\text{Path}_n, \mathcal{C})$$

It turns out that the homotopy coherent nerve is an $\infty$-category (at least when mapping spaces Kan?)

$$N^h : s\text{Cat} \rightarrow \infty\text{-Cat}$$

**Exercise:** $N^h(\mathcal{C})_2$

**Remark:** Converse: every $\infty$-cat is equivalent to the homotopy coherent nerve of some topological category, which is essentially unique. So $\mathcal{C} \rightarrow N(\mathcal{C})$ determines an equivalence b/w the theory of top cats and the theory of $\infty$-cat.
Examples of simplicial categories:

1) $sSet^+$  \quad $K, M \in sSet$

$sSet(K, M) \in sSet^+$ defined by

$sSet(K, M)(n) = \text{Hom}(K \times \Delta^n, M)$

2) $\text{Kan} \subset sSet$ full subcategory whose objects are Kan complexes

$\text{Kan}(K, M) = sSet(K, M)$

3) $s\text{Cat}_\infty$ “simplicial category of $\infty$-cats”

objects are $s$sets which are $\infty$-cat

$s\text{Cat}_\infty(\mathcal{C}, \mathcal{D}) = \text{core}(s\text{Set}(\mathcal{C}, \mathcal{D}))$

\[\uparrow\]

largest Kan complex inside of $s\text{Set}$
We have

\[
\text{Kan} \rightarrow \text{sCat}_\infty \rightarrow \text{sSet}
\]

Apply \(N^\text{h}\)

\[
N^\text{h}(\text{Kan}) \rightarrow N^\text{h}(\text{sCat}_\infty)
\]

\text{``infinity category of spaces''} \hspace{1cm} \text{``infinity category of \(\infty\)-categories''}

\(\text{Rmk: Cat}_\infty \) is also called \text{``the homotopy theory of homotopy theories''}

A morphism of \(\infty\text{-Cat} \) \(F: \mathcal{C} \rightarrow \mathcal{D}\) is a natural transf of functors \(\Delta^{op} \rightarrow \text{Set}\)

we want a notion of \text{``categorical equivalence''}

for \(\mathcal{C} \rightarrow \mathcal{D} \) \(\infty\text{-Cat}\)

or \(K \rightarrow \mathcal{M} \) simplicial set
\[ \begin{array}{ccc}
\Delta & \xrightarrow{\text{path}} & \mathcal{S} \mathcal{C}^+ \\
\downarrow \text{Yoneda} & & \\
\mathcal{S} \mathcal{S} \mathcal{E}t^+ & & \\
\end{array} \]

\[ \mathcal{C} = \text{"left Kan extension"} \]

For \( K \in \mathcal{S} \mathcal{S} \mathcal{E}t^+ \)

\[ K = \colim \Delta^n \]
\[ (\n, \Delta^n \to K) \]

\[ \mathcal{C}(K) = \colim \text{Path } n \]
\[ (\n, \Delta^n \to K) \]

For \( K \in \mathcal{S} \mathcal{S} \mathcal{E}t^+ \), define \( \text{ho } K := \text{ho } N_\mathcal{H} \mathcal{C}(K) \)

When \( K \) is an \( \infty \)-category, this will be equivalent to \( \text{ho } K \)

**Def:** \( F: K \to M \) map in \( \mathcal{S} \mathcal{S} \mathcal{E}t^+ \) is said to be a "categorical equivalence" if \( \mathcal{C}(F): \mathcal{C}(K) \to \mathcal{C}(M) \) is an equivalence of simplicial categories in the sense that it induces:

- htpy equivalences on mapping spaces
an equivalence on htpy categories

\[ \text{Rmk: When } K \text{ and } M \text{ are } \infty \text{-categories, this is the same as} \]
\[ F \text{ is iso in } \text{ho } \text{Cato} \infty \]

Derived cat of a scheme as \( \infty \text{-cat} \)

\[ \text{Ex } R \text{ ring} \]
\[ D^{\text{perf}}(R) \text{ is an } \infty \text{-cat} \]

objects: bounded chain complexes of finitely generated projective \( R \)-modules

\[ \cdots \to 0 \to 0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to 0 \to \cdots \]

morphisms: maps of chain complexes \( f : P \to Q \).

2-simplices: diagrams

\[ \begin{array}{c}
\begin{array}{ccc}
P & \xrightarrow{f} & Q \\
\downarrow{g} & & \downarrow{h} \\
\_ & \xrightarrow{h} & R
\end{array}
\end{array} \]

with a chain homotopy from \( h \) to \( g \circ f \).

\[ \text{To fill in dot dot dot :} \]
Dold-Kan: \( ch_{\geq 0} A \simeq sA \) for an abelian category

By taking chain complexes of modules, we are giving a simplicial direction.

\[
\text{Mor}(P, P') \in SSet
\]

\[
\text{Mor}(P, P')(n) = \text{Mor}_{sA}(P \otimes \Delta^n, P')
\]

\[
\Delta^o \xrightarrow{\Delta^n} SSet \longrightarrow sA
\]

\[
(-, n) \quad \text{replace } \ast \text{ by } A
\]

So we can make a simplicial category \( \text{D}_{\text{perf}}^{\text{simp cat}}(R) \)

\( \text{ob: bounded chain complexes of} \)

\( \text{finitely generated projective } R\text{-modules} \)

\[
\ldots \to 0 \to 0 \to P_n \to P_{n-1} \to \ldots \to P_m \to 0 \to \ldots
\]

\[
\text{D}_{\text{perf}}^{\text{simp cat}}(R)(P, P') = \text{Mor}(P, P') \in SSet^+
\]

Then \( \text{D}_{\text{perf}}^{\text{simp cat}}(R) = N_{\mathbb{H}}(\text{D}_{\text{perf}}^{\text{simp cat}}(R)) \) \( \infty\text{-cat} \)
ho \( D^{\text{perf}}(R) \) is classical derived cat, meaning there is an equivalence of \( \Delta' \)d categories.

\underline{Issue}: The \( \Delta' \)d cats \( ho \ D^{\text{perf}}(R) \) do not satisfy Zariski descent.

\underline{Def}: \( X \) scheme

\[
D^{\text{perf}}(X) := \lim_{\text{Spec} R \in C, X \text{ open affine}} D^{\text{perf}}(R)
\]

Take limit in \( \text{Cat}^{\infty} \)

References

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