

Simplicial Categories

Def: A simplicial category \mathcal{C} is a category enriched over $s\text{Set}$, in the sense that for every $X, Y \in \mathcal{C}$ we have $\mathcal{C}(X, Y) \in s\text{Set}$ and a unital and associative composition

$$\circ: \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$$

Rmk: Such categories can be identified with functors

$$\Delta^{\text{op}} \rightarrow \text{Cat} \quad (\text{i.e. simplicial objects in Cat})$$

category
of small
categories

whose face and degeneracy morphisms are bijective on objects

• A simplicial functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ can be viewed as $\mathcal{F}: \text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{D})$ and $\mathcal{F}: \mathcal{C}(X, Y) \rightarrow \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$ respecting Id and $s.t.$ composition is map $s\text{Set}$ (or as a nat'l transf of functors $\Delta^{\text{op}} \rightarrow \text{Cat}$)

• Let $s\text{Cat}$ denote category of simplicial cat

ex: path $\underline{n} \in \text{SCat}$ objects = $\underline{n} = \{0, 1, \dots, n\}$

For $i, j \in \underline{n}$ define $P_{ij} = \{ \{ i = i_0 \leq i_1 \leq \dots \leq i_n = j \} \in \underline{n} \}$

Consider P_{ij} as a poset ordered by reverse inclusion

$$\text{path } \underline{n} (i, j) := \mathcal{N}(P_{ij})$$

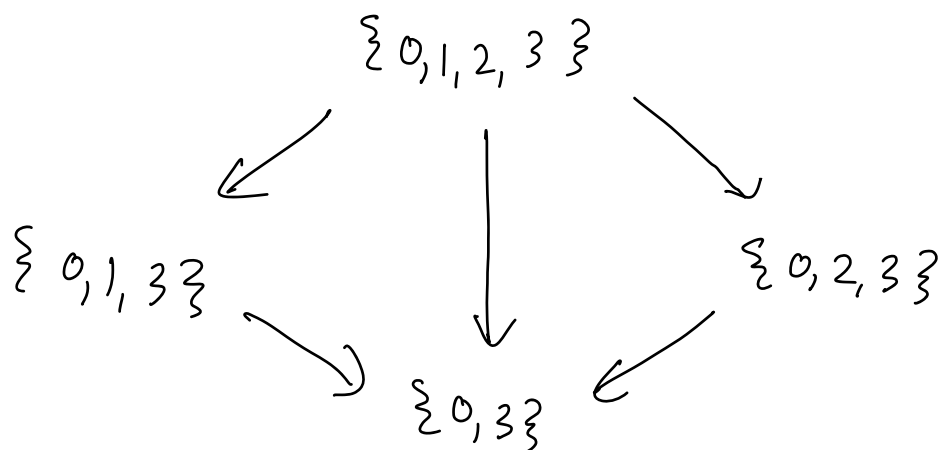
cf. HTT
Def 1.1.5

$$\text{Id}_i = \{i\} \in \mathcal{N}(P_{ii})_0$$

Composition: $\mathcal{N}(P_{ij}) \times \mathcal{N}(P_{jk}) \rightarrow \mathcal{N}(P_{ik})$

$$(I, J) \mapsto I \cup J$$

Ex Path $\underline{3} (0, 3)$



different
ways of
composing

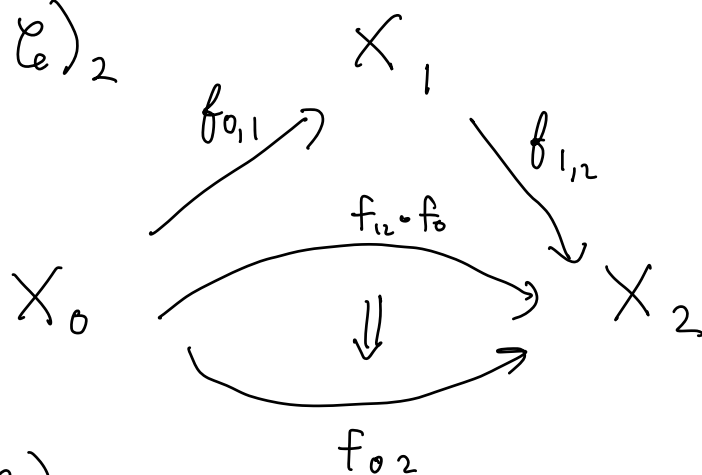
Def/Prop: For a simplicial category \mathcal{C} , the homotopy coherent nerve $N_H(\mathcal{C})$ is the ∞ -cat whose n -simplices are

$$N_H(\mathcal{C})(n) = \text{sCat}(\text{Path } \underline{n}, \mathcal{C})$$

- It turns out that the homotopy coherent nerve is an ∞ -category (at least when mapping spaces Kan?)

$$N_H : \text{sCat} \longrightarrow \infty\text{-Cat}$$

ex $\nabla \in N_H(\mathcal{C})_2$



exercise: $N_H(\mathcal{C})_3$

Rmk: Converse: every ∞ -cat is equivalent to the homotopy coherent nerve of some topological category which is essentially unique. So $\mathcal{C} \rightarrow N(\mathcal{C})$ determines an equivalence b/w the theory of top cats and the theory of ∞ -cat

Examples of simplicial categories:

1) $sSet$ $K, M \in sSet$

$sSet(K, M) \in sSet$ defined by

$$sSet(K, M)(\underline{n}) = \text{Hom}(K \times \Delta^n, M)$$

2) $Kan \subset sSet$ full subcategory whose objects are Kan complexes

$$Kan(K, M) = sSet(K, M)$$

3) $sCat_\infty$ "simplicial category of ∞ -cats"

objects are $sSets$ which are ∞ -cat

$$sCat_\infty(\mathcal{C}, \mathcal{D}) = \text{core}(sSet(\mathcal{C}, \mathcal{D}))$$

largest Kan complex inside of $sSet$

We have

$$\text{Kan} \longrightarrow \text{sCat}_\infty \longrightarrow \text{sSet}$$

Apply N_H

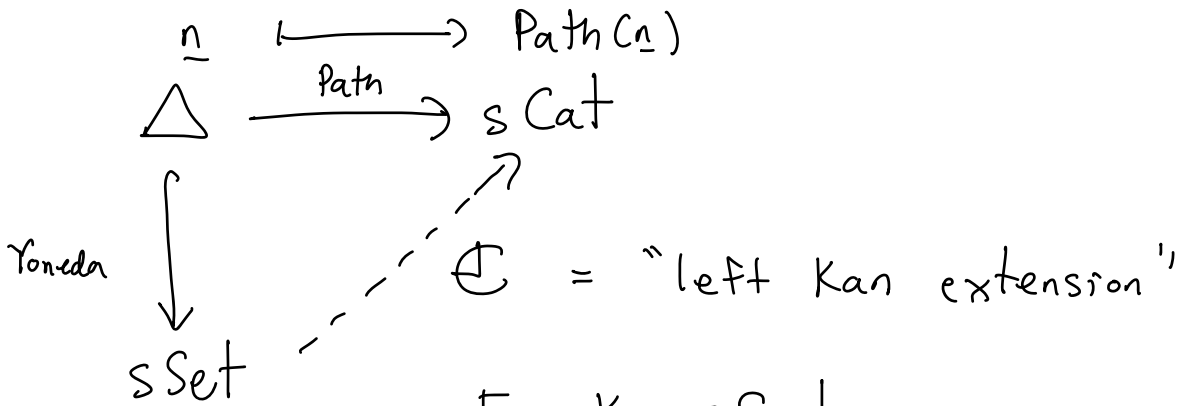
$$\begin{array}{ccc} N_H(\text{Kan}) & \longrightarrow & N_H(\text{sCat}_\infty) \\ \Downarrow & & \Downarrow \\ \mathcal{S} & & \text{Cat}_\infty \end{array}$$

"infinity category
of spaces"

"infinity category
of ∞ -categories"

Rmk: Cat_∞ is also called "the homotopy theory of homotopy theories"

- A morphism of ∞ -Cat $F: \mathcal{C} \rightarrow \mathcal{D}$ is a nat'l transf of functors $\Delta^{op} \rightarrow \text{Set}$
- we want a notion of "categorical equivalence"
for $\mathcal{C} \xrightarrow{F} \mathcal{D}$ ∞ -cat
or $K \rightarrow M$ simplicial set



For $K \in \text{sSet}$

$$K = \text{colim}_{(\underline{n}, \Delta^n \rightarrow K)} \Delta^n$$

$$\mathbb{E}(K) = \text{colim}_{(\underline{n}, \Delta^n \rightarrow K)} \text{Path } \underline{n}$$

For $K \in \text{sSet}$, define $\text{ho} K := \text{ho} N_H \mathbb{E}(K)$

• When K is an ∞ -category, this will be equivalent to $\text{ho} K$

Def: $F: K \rightarrow M$ map in sSet is

said to be a "categorical equivalence"

if $\mathbb{E}(F): \mathbb{E}(K) \rightarrow \mathbb{E}(M)$ is an equivalence of simplicial categories in the sense that it induces:

• htpy equivalences on mapping spaces

- an equivalence on htpy categories

Rmk: When K and M are ∞ -categories, this is the same as

F is iso in $h\text{Cat}_\infty$

Derived cat of a scheme as ∞ -cat

Ex R ring
 $\mathcal{D}^{\text{perf}}(R)$ is an ∞ -cat

objects: bounded chain complexes of finitely generated projective R -modules

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow P_N \rightarrow P_{N-1} \rightarrow \dots \rightarrow P_M \rightarrow 0 \rightarrow \dots$$

morphisms: maps of chain complexes $f: P_\bullet \rightarrow Q_\bullet$

2-simplices: diagrams

$$\begin{array}{ccc} & Q_\bullet & \\ f \nearrow & & \searrow g \\ P_\bullet & \xrightarrow{h} & R_\bullet \end{array}$$

with a chain homotopy from h to $g \circ f$

⋮

Rmk: To fill in dot dot dot:

Dold-Kan : $ch_{\geq 0} A \cong sA$ for A abelian category

By taking chain complexes of modules, we are giving a simplicial direction.

$$\text{Mor}(P_0, P_0') \in s\text{Set}$$

$$\text{Mor}(P_i, P_i')(n) = \text{Mor}_{sA}(P_i \otimes \Delta^n, P_i')$$

$$\Delta^{\text{op}} \xrightarrow{\Delta^n} s\text{Set} \xrightarrow{\text{replace } * \text{ by } A} sA$$

(-, n)

So we can make a simplicial category $D_{\text{simp cat}}^{\text{perf}}(R)$
 ob: bounded chain complexes of finitely generated projective R -modules

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow P_N \rightarrow P_{N-1} \rightarrow \dots \rightarrow P_M \rightarrow 0 \rightarrow \dots$$

$$D_{\text{simp cat}}^{\text{perf}}(R)(P_i, P_i') = \text{Mor}(P_i, P_i') \in s\text{Set}$$

$$\text{Then } D^{\text{perf}}(R) = \mathcal{N}_H(D_{\text{simp cat}}^{\text{perf}}(R)) \quad \infty\text{-cat}$$

- $\mathrm{ho} \mathcal{D}^{\mathrm{Perf}}(R)$ is classical derived cat,
meaning there is an equivalence of Δ 'd categories

Issue: The Δ 'd cats $\mathrm{ho} \mathcal{D}^{\mathrm{Perf}}(R)$ do
not satisfy Zariski descent

Def: X scheme

$$\mathcal{D}^{\mathrm{Perf}}(X) := \lim_{\substack{\mathrm{Spec} R \subset X \\ \text{open affine}}} \mathcal{D}^{\mathrm{Perf}}(R)$$

Take limit in Cat_{∞}

References

M Levine

L2 Presentable ∞ -cat

Essen Motives

Seminar Fall 2021

HTT J. Lurie

Higher Topos Theory

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Lecture notes for
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Essen Motives
Seminar Fall 2021