Motivation

- Top, category of topological spaces with cont maps (convenient)
- HoTop, category of topological spaces with homotopy classes of cont maps

Many lovely invariants of Top pass thru HoTop

\[ \text{Ho}^* : \text{HoTop} \rightarrow \text{Ab} \]

\[ \mathcal{K} = \sum (-1)^r \text{Ker} H^r \in \mathbb{Z} \]

\[ \mathcal{X} \simeq \gamma \quad \Rightarrow \quad \mathcal{X}(\mathcal{X}) = \mathcal{X}(\gamma) \]

On the other hand, HoTop does not have many

(\text{co})limits

Here is the definition of a colimit

\[ \text{Colim} \mathbf{F} \in \mathcal{C} \quad \text{is} \quad \zeta : \mathbf{F}(i) \rightarrow \text{colim} \mathbf{F} \quad \forall \ i \in I \]

s.t. \[ \text{map} (\text{colim} \mathbf{F}, C) = \left\{ (a_i)_{i \in I} \mid \text{map} (\mathbf{F}(i), C) \right\} \]

"universal property" \[ \forall \ i_1, i_2 \in I \]

\[ i_1 \rightarrow i_2 \quad \text{commutes} \]
\[
\text{Colim} = S^1
\]

**Non-ex:** There is not a colim of the composite

\[
\begin{array}{c}
I \xrightarrow{\gamma} \text{Top} \xrightarrow{H \circ \text{Top}} \\
\xrightarrow{f}
\end{array}
\]

The natural guess for the colim is

\[
\text{f}(2) \cup \text{f}(0) \times I \cup \text{f}(1)
\]

\[
\begin{array}{c}
X \times X_1 \\
X_0 \times X_1
\end{array}
\]

\[
\begin{array}{c}
x \in f(0) \\
x \in f(0)
\end{array}
\]
Given a commutative diagram in HoTop

\[
\begin{array}{ccc}
F(0) & \xrightarrow{f_1} & f(2) \\
i_2 \downarrow & & \downarrow g \\
F(1) & \xrightarrow{h} & D \\
\end{array}
\]

We can form

\[
f(2)
\]

where \( H : F(0) \times I \rightarrow D \) is a homotopy between \( g \circ i_1 \) and \( h \circ i_2 \).

So we do have induced maps to \( D \). The problem is we do not have a unique choice of such a map to \( D \). The induced map depends on the homotopy class of \( M \) and there can be many. For example, let \( D = S^1 = \)
For any \( n \), we may define \( H(x,t) = (\cos \pi t, \sin \pi t) \)

\( H \) is a htpy between \( g \) and \( h \). The degree of \( g \) is \( n \).

Instead of considering diagrams which commute up to homotopy, we can consider homotopy coherent diagrams.

**Definition of an \( \infty \)-category**

**Def**: \( \Delta \) category with objects

\[
\Delta \quad n = 0, 1, 2, \ldots \\
\downarrow_{n = 0, 1, 2, \ldots , n} \quad \begin{array}{c}
\text{Mor}(n, m) = \text{set maps respecting } \leq \\
\text{Def}: \text{SSet } = \text{Fun } (\Delta^{op}, \text{Set})
\end{array}
\]

**terminology**: For \( X \in \text{SSet} \), “\( n \)-simplices” of \( X \) is \( X(\Delta^n) \)

**ex**: For a topological space \( X \), define
Sing. $X \in sSet$

by $\text{Sing. } X (n) = \text{Mor} (\Delta^n, X)$

given $f: n \to m$ we have an induced map $f_*: \Delta^n \to \Delta^m$ by mapping vertices by $f$ and extending using convex combinations.

Thus we have $\text{Mor} (\Delta^m, X) \to \text{Mor} (\Delta^n, X)$

$\text{ex}: \text{Mor} (-, n) \in sSet$ gives the $sSet$

analogue of $\Delta^n$

Def: let $\mathcal{C}$ be a category. The nerve of $\mathcal{C}$ is $\text{N}(\mathcal{C}) \in sSet$ with $n$-simplices given by composable sequences of morphisms

$$C_0 \to C_1 \to \ldots \to C_n$$

in $\mathcal{C}$

Rmk: $\mathcal{C}$ can be recovered from $\text{N}(\mathcal{C})$

$\text{ob}(\mathcal{C}) \leftrightarrow$ o-simplices of $\text{N}(\mathcal{C})$
For $X, Y \in \mathcal{C}$,

$\text{Map}_{\mathcal{C}}(X, Y) \leftrightarrow \text{1-simplices of } N(\mathcal{C})$

from $X$ to $Y$

- Given $f \in \text{Hom}_\mathcal{C}(X, Y)$ and $g \in \text{Hom}_\mathcal{C}(Y, Z)$

there is a unique 2-simplex of $N(\mathcal{C})$

$\xymatrix{ f \ar[r] & Y \\
X \ar[r] \ar[u]^{g \circ f} \ar[ru] & Z \ar[l]_g }$

$\therefore \text{Prop: } \text{The Nerve } \mathcal{C} \rightarrow N(\mathcal{C}) \text{ determines a fully faithful embedding from the category of small categories into the category of } SSet.$

- The essential image is given:

Let $S \in SSet$. $S$ is isomorphic to the Nerve of a category $\Rightarrow$

(\#) For every pair of integers $0 \leq i \leq n$, every map $f_0 : \Lambda^i_n \rightarrow S$ extends uniquely to an $n$-simplex $f : \Delta^n \rightarrow S$

$\Lambda^i_n$ is the simplicial subset of $\Delta^n$ obtained by removing $i$th face and face opposite $i$th vertex
ex: \( i=1, n=2 \) (\( \# \)) says every pair of “comparable” edges \((f, g)\) determine a unique 2-simplex

\[
f \Rightarrow g \\
X \rightarrow Y \rightarrow Z
\]

Def: \( S \in \text{SSet} \) is a **Kan complex** if

(\( \#' \)) For every pair of integers \( 0 \leq i \leq n \)

every map \( f_0 : \Lambda_i^n \rightarrow S \) extends to an \( n \)-simplex \( f : \Delta^n \rightarrow S \)

Rmk: The extension is not required to be unique.

Common generalization of (\( \# \)) and (\( \#' \)):

Def: An \( \infty \)-category is a simplicial set \( \mathcal{C} \) satisfying

(\( \#'' \)) For every pair of integers \( 0 < i < n \), every map \( f_0 : \Lambda_i^n \rightarrow \mathcal{C} \) extends uniquely to an \( n \)-simplex \( f : \Delta^n \rightarrow \mathcal{C} \)
Def: A groupoid is a category where every morphism is an isomorphism.

Rmk: $\infty$-categories are also called $(\infty, 1)$-categories, quasi-categories or weak Kan complexes.

$C$ is an $\infty$-category.

0-simplices of $C$ are called its objects.

1-simplices of $C$ are called its morphisms.

ex. $(\pi^i)$ for $i=1, n=2$ is the condition that for every pair of "composable" morphisms.
$F : X \to Y$ and $g : Y \to Z$, we can find a 2-simplex $\tau$:

![Diagram]

Think of $h$ as a composition of $f$ and $g$
and write $h = g \circ f$

**Warning:** (*$\star$*) does NOT require that $\tau$ be unique, so there may be several choices for the composition $h$.
However, one can show that $h$ is unique up to a suitable notion of homotopy

**Def (HTT p. 29)** let $F, g : X \to Y$ be morphisms of an $\infty$-category $\mathcal{C}$. Then $f \simeq g$ "$f$ is homotopic to $g$" if there is a 2-simplex

![Diagram]
(HTT Prop 1.2.3.5)

Claim: $\simeq$ is an equivalence relation, so $f \simeq g$ and $g \simeq h \Rightarrow f \simeq h$

\[ \begin{array}{c}
\text{pf:} \\
\end{array} \]

Then fill in \( \Lambda^3_2 \) to obtain 2-simplex \( g \xymatrix{ \Delta^3 \ar[r]_{\sim} & \Delta^2 } \)

Exer. \( \Lambda^2_1 \xymatrix{ \Delta^2 \ar[r] & \mathcal{C} } \) show that $h$ is unique up to homotopy

Def: Let $\mathcal{C}$ be an $\infty$-category. The associated homotopy category $\mathrm{h} \mathcal{C}$ has objects 0-simplices of $\mathcal{C}$ and morphisms htpy classes of 1-simplices.
1) $\text{ho } \text{Sing } X = \Pi_{\leq 1} X \leftrightarrow \text{fundamental groupoid}$
   $\text{ob} = \text{pts } X$
   $\text{morphisms} = \text{paths b/w points}$

2) $\text{hoN}(\mathbb{C}) = \mathbb{C}$

References

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