## Motivation

- Top — category (convenient) of topological spaces with cont maps
- HoTop — category (nice) of topological spaces with homotopy classes of cont maps

Many lovely invariants of Top pass thru HoTop

$$H^* : \mathrm{HoTop}^{op} \longrightarrow Ab$$

$$\chi = \sum (-1)^i \, rk \, H^i \in \mathbb{Z}$$

$$X \simeq Y \Rightarrow \chi(X) = \chi(Y)$$

On the other hand, HoTop does not have many (co)limits

Here is the definition of a colimit

$\mathcal{C}$ cat

$\mathcal{F} : I \to \mathcal{C}$ diagram in $\mathcal{C}$ e.g.

$$\begin{tikzcd} & C_1 \\ C_0 \arrow[r] & C_2 \arrow[u] \end{tikzcd}$$

<u>Def.</u> $\operatorname{colim} \mathcal{F} \in \mathcal{C}$ (if it exists) is $\iota_i : \mathcal{F}(i) \to \operatorname{colim} \mathcal{F} \ \forall i \in I$

s.t. $\operatorname{map}(\operatorname{colim} \mathcal{F}, C) = \{ (\alpha_i) \in \prod_{i \in I} \operatorname{map}(\mathcal{F}(i), C) \mid \forall\, i_1 \to i_2 \in I$

$$\begin{tikzcd} i_1 \arrow[r] \arrow[dr] & i_2 \arrow[d] \\ & C \end{tikzcd}$$

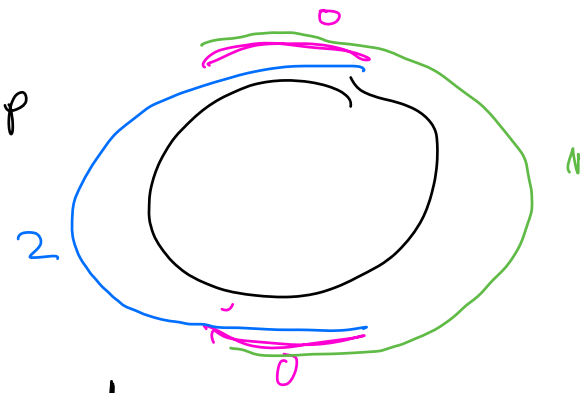
commutes $\}$

"universal property"

ex :

$$I = \begin{tikzcd} & 1 \\ 0 \arrow[r] \arrow[ur] & 2 \end{tikzcd}$$

$\mathcal{F} : I \to Top$

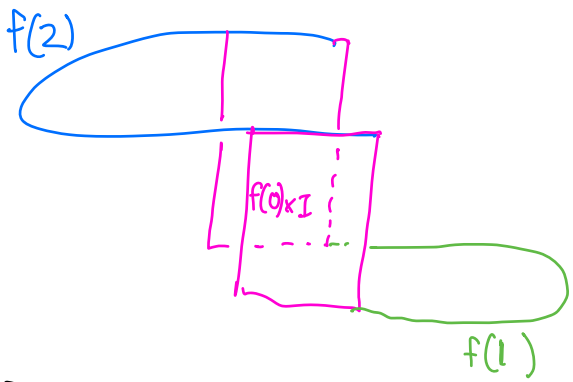



$$\text{Colim} = \bigcirc \simeq S^1$$

Non-ex: There is not a colim of the composite

$$\begin{tikzcd} I \arrow[r, "\mathcal{F}"] \arrow[rr, bend right, "f"'] & Top \arrow[r] & HoTop \end{tikzcd}$$

The natural guess for the colim is



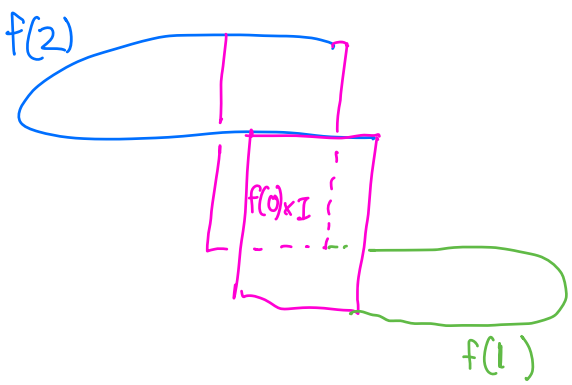

$$f(2) \cup f(0) \times I \cup f(1)$$

$x \sim x \times 1$, $x \in f(0)$ $\qquad$ $x \times 0 \sim x$, $x \in f(0)$

Given a commutative diagram in HoTop

$$\begin{tikzcd} F(0) \arrow[r, "i_1"] \arrow[d, "i_2"'] & f(2) \arrow[d, "g"] \\ F(1) \arrow[r, "h"'] & D \end{tikzcd}$$

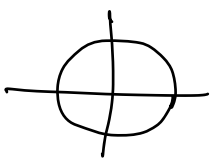
We can form




$$g \cup H \cup h \longrightarrow D$$

where $H : F(0) \times I \longrightarrow D$ is a homotopy between $g \circ i_1$ and $h \circ i_2$

So we do have induced maps to $D$. The problem is we do not have a unique choice of such a map to $D$. The induced map depends on the homotopy class of $H$ and there can be many

For example, let $D = S^1 =$ 

$$g(x) = (1,0) \quad \forall x$$
$$h(x) = (1,0) \quad \forall x$$

For any $n$, we may define $H(x,t) = (\cos 2\pi n t, \sin 2\pi n t)$ $\forall x$

$H$ is a htpy between $g$ and $H$. The degree of $g \cup H \cup h$ is $n$.

- Instead of considering diagrams which commute up to homotopy, we can consider homotopy coherent diagrams.

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<u>Definition of an $\infty$-category</u>

<u>Def</u>: $\Delta$ category with objects

$\underline{n}$ $\quad n = 0, 1, 2, \ldots \qquad \underline{n} = \{0, 1, \ldots, n\}$

$\mathrm{mor}(\underline{n}, \underline{m}) =$ set maps respecting $\leq$

<u>Def</u>: $\mathrm{sSet} = \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$

terminology: For $X \in \mathrm{sSet}$, "n-simplices" of $X$ is $X(\underline{n})$

<u>ex</u>: For a topological space $X$, define

$\mathrm{Sing}_\bullet X \in sSet$

standard n-simplex = convex hull of origin & standard basis in $\mathbb{R}^n$

by $\mathrm{Sing}_\bullet X(\underline{n}) = \mathrm{Mor}(\Delta^n, X)$

given $f : \underline{n} \to \underline{m}$ we have an induced map $f_* : \Delta^n \to \Delta^m$ by mapping vertices by $f$ and extending using convex combinations

Thus we have $\mathrm{Mor}(\Delta^m, X) \to \mathrm{Mor}(\Delta^n, X)$

<u>ex</u>: $\mathrm{mor}_\Delta(-, \underline{n}) \in sSet$ gives the sSet analogue of $\Delta^n$

<u>Def</u>: let $\mathcal{C}$ be a category. The <u>nerve</u> of $\mathcal{C}$ is $N(\mathcal{C}) \in sSet$ with n-simplices given by composable sequences of morphisms

$$C_0 \to C_1 \to \dots \to C_n$$

in $\mathcal{C}$

<u>Rmk</u>. $\mathcal{C}$ can be recovered from $N(\mathcal{C})$

- $\mathrm{ob}(\mathcal{C}) \overset{\text{bij}}{\longleftrightarrow}$ 0-simplices of $N(\mathcal{C})$

- For $X, Y \in \mathcal{C}$,

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \overset{\text{bij}}{\longleftrightarrow} \text{1-simplices of } N(\mathcal{C}) \text{ from } X \text{ to } Y$$

- Given $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ there is a unique 2-simplex of $N(\mathcal{C})$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"] & & Z \end{tikzcd}$$

$\Rightarrow$ <u>Prop</u>: The Nerve $\mathcal{C} \mapsto N(\mathcal{C})$ determines a fully faithful embedding from the category of (small) categories into the category of $sSet$.

- The essential image is given:

Let $S \in sSet$. $S$ is isomorphic to the Nerve of a category $\Leftrightarrow$

(☆) For every pair of integers $0 < i < n$, every map $f_0 : \Lambda^n_i \to S$ extends uniquely to an $n$-simplex $f : \Delta^n \to S$

$\Lambda^n_i$ ith horn: simplicial subset of $\Delta^n$ obtained by removing interior and face opposite ith vertex

ex: $i=1$, $n=2$ $(\star)$ says every pair of "composable" edges $(f,g)$ determine a unique 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, dashed] & & Z \end{tikzcd}$$

"$g \circ f$"

Def: $S \in SSet$ is a Kan complex if

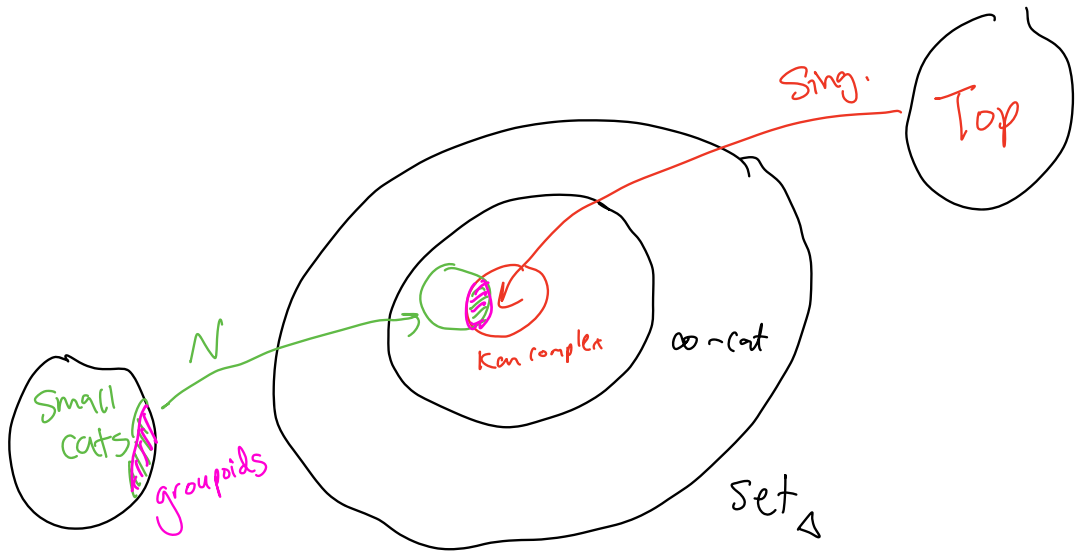
$(\star')$ For every pair of integers $0 \leq i \leq n$ every map $f_0 : \Lambda^n_i \longrightarrow S$ extends to an $n$-simplex $f : \Delta^n \longrightarrow S$

Rmk: The extension is not required to be unique

Common generalization of $(\star)$ and $(\star')$:

Def: An $\infty$-category is a simplicial set $\mathcal{C}$ satisfying

$(\star'')$ For every pair of integers $0 < i < n$, every map $f_0 : \Lambda^n_i \longrightarrow \mathcal{C}$ extends uniquely to an $n$-simplex $f : \Delta^n \longrightarrow \mathcal{C}$




<u>Def</u>: A <u>groupoid</u> is a category where every morphism is an isomorphism

<u>Rmk</u>: $\infty$-categories are also called $(\infty,1)$-categories, quasi-categories or weak Kan complexes

$\mathcal{C}$ $\infty$-category

0-simplices of $\mathcal{C}$ are called its <u>objects</u>

1-simplices of $\mathcal{C}$ are called its <u>morphisms</u>

<u>ex</u> ($\Delta''$) for $i=1$ $n=2$ is the codition that for every pair of "composable" morphisms

$f: X \to Y$ and $g: Y \to Z$, we can find a 2-simplex $\sigma$:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

Think of $h$ as a composition of $f$ and $g$ and write $h = g \circ f$

<u>Warning</u>: ($\star''$) does NOT require that $\sigma$ be unique, so there may be several choices for the composition $h$.

However, one can show that $h$ is unique up to a suitable notion of homotopy

<u>Def</u> (HTT p. 29) Let $f, g: X \to Y$ be morphisms of an $\infty$-category $\mathcal{C}$. Then $f \simeq g$ "$f$ is homotopic to $g$" if there is a 2-simplex

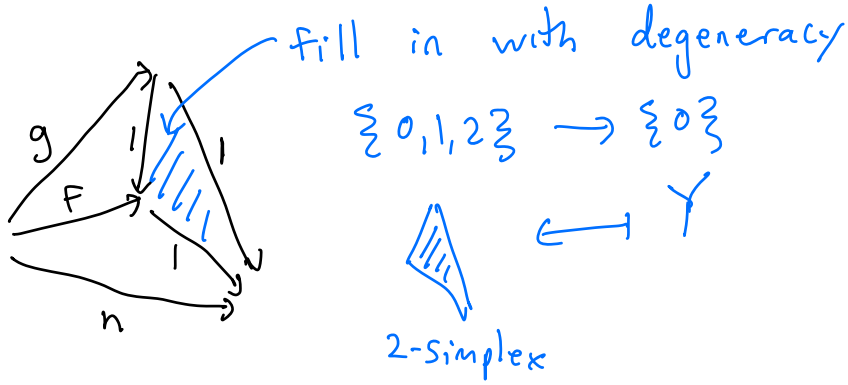
$$\begin{tikzcd} & Y \arrow[dr, "id"] \arrow[d, phantom, "\sigma"] & \\ X \arrow[ur, "f"] \arrow[rr, "g"'] & & Y \end{tikzcd}$$

(HTT Prop 1.2.3.5)

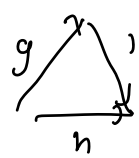
<u>claim</u>: $\simeq$ is an equivalence relation,

so $f \simeq g$ and $g \simeq h \Rightarrow f \simeq h$

<u>pf</u>:




Then fill in $\Lambda^3_2$ to obtain 2-simplex




$\square$

<u>Exer</u>:

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "(f,g,\cdot)"] \arrow[d, hook] & \mathcal{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

Show that $h$ is unique up to homotopy

<u>Def</u> Let $\mathcal{C}$ be an $\infty$-category. The associated homotopy category $h\mathcal{C}$ has objects 0-simplices of $\mathcal{C}$ and morphisms htpy classes of 1-simplices

of $\mathcal{C}$

<u>ex</u> 1) $\mathrm{ho}\,\mathrm{Sing}\,X = \Pi_{\leq 1} X$ ← fundamental groupoid

ob = pts $X$

morphisms = paths b/w points

2) $\mathrm{hoN}(\mathcal{C}) = \mathcal{C}$

## References


HTT J. Lurie Higher Topos Theory Annals of Math Studies

J. Lurie L2 Categorical background Lecture notes for L-theory course Harvard 2011

M Rohrbach L1 Introduction to $\infty$-categories Essen Motives Seminar Fall 2021