Comment on geometric surgery and surgery of Poincaré objects:

\[ S^n \times D^{m-n} \times I \xrightarrow{\sim} M \times I \]

\[ D^{n+1} \times D^{m-n} \twoheadrightarrow W \cong M \times I \cup D^{n+1} \times D^{m-n} \]

\[ S^n \times D^{m-n} \times I \xrightarrow{\sim} S^n \]

\[ M \times I \cong M \]

\[ D^{n+1} \times D^{m-n} \cong \ast \]

\[ C_\ast (S^n; \mathbb{Z}) \rightarrow C_\ast (M; \mathbb{Z}) \rightarrow C_\ast (W; \mathbb{Z}) \]

is a cofiber sequence

\[ B(P, Q) = \text{Mor} (P \otimes Q, \mathbb{Z}) \]

\[ D = \text{Hom} (\ast, \mathbb{Z}) \]
\( Q = P_i \rightarrow B(CP, P) \)

Apply \( \mathbb{D} \). Obtain fiber sequence

\[
\begin{align*}
C^* (S^n; \mathbb{Z}) & \rightarrow C^* (M; \mathbb{Z}) \rightarrow C^* (W; \mathbb{Z}) \\
\text{Geometric surgery data: } S^n \times D^{n-n} & \rightarrow M
\end{align*}
\]

Algebraic surgery data:

\[
C_\ast (S^n; \mathbb{Z}) \rightarrow C_\ast (M; \mathbb{Z})
\]

And nullhomotopy of pullback of intersection form from trivialization of the normal bundle surgery:

\[
\begin{align*}
C_\ast (S^n; \mathbb{Z}) & \rightarrow C_\ast (M; \mathbb{Z}) \\
\mathbb{D} C_\ast (M; \mathbb{Z}) & \rightarrow \mathbb{D} C_\ast (S^n; \mathbb{Z})
\end{align*}
\]

\[\xymatrix{C_\ast (S^n; \mathbb{Z}) \ar[r] & C_\ast (M; \mathbb{Z}) \ar[r]^-{\text{Id}} \ar[l]^-{\text{Id} & \text{Id} C_\ast (M; \mathbb{Z})} & \text{Id} C_\ast (S^n; \mathbb{Z}) \ar[l]^-{\text{Id} & \text{Id} C_\ast (M; \mathbb{Z})} \}
\]

Last time we saw:

Have cobordism \( \nu \) \( (C_\ast (M; \mathbb{Z}), i) \) and \(( C_\ast (M; \mathbb{Z}), i') \) of Poincaré objects

\[
C_\ast (M; \mathbb{Z}) \leftarrow \text{fib} (B) \rightarrow C_\ast (M; \mathbb{Z})'
\]
We have a dual triangle $D(*)$.

$c_{ fib \alpha} \simeq D \ fib \ D \alpha$

$\Rightarrow \ fib \ D \alpha \simeq C^*(W;\mathbb{Z})$

$fib \ D \alpha$ gives cobordism of the Poincaré objects dual to $C^*(M;\mathbb{Z})$ and $C^*(M;\mathbb{Z})'$

Thus $C^*(W;\mathbb{Z})$ gives cobordism btw the Poincaré objects $C^*(M;\mathbb{Z})$ and $D \ C^*(M;\mathbb{Z})'$

This is consistent with the notion that $C^*(M;\mathbb{Z})' \simeq C^*(M';\mathbb{Z})$ where $M'$ is the geometric surgery.

Hirzebruch Signature theorem
Orientations

Let \( H \in \text{Sp} \). So \( H \) represents a cohomology theory
\[ H^n(X) = [X, \Sigma^n H] \quad H_n(X) = \Omega^n(X \times H) \]

Ex: \( H = \mathbb{H} \) ordinary singular cohomology with \( \mathbb{Z} \) coefficients
\[ H = \mathbb{Q} \] ordinary singular cohomology with \( \mathbb{Q} \) coefficients

\( MU \) cobordism
\[ \mathbb{L} = \mathbb{L}(\text{Def}^s(\mathbb{Z}), \mathbb{Q}^s_{\mathbb{Z}}) \] a spectrum with \( \text{Th} \) assuming.

When \( H \) has a ring structure, have \( \mathbb{S} = S^0 \to H \) which we will assume.

Def: A rank \( r \) virtual vector bundle \( V \to X \) is said to be oriented with respect to \( H \) if

- There is a Thom class \( m \in H^r(\text{Th}(V)) \)
- S.t. \( A \times X \ni \gamma \in \mathbb{H}_X^r(\text{Th}(V_x)) \Leftrightarrow H^0 \)
  is a unit
We are given an equivalence
\[ \text{Th}(V) \wedge H \cong \Sigma_X \wedge H \]

**Def.:** A smooth manifold \( M \) is **oriented** if \( TM \) is oriented with respect to \( H \).

\( M \) is a manifold of dimension \( m \), compact. An orientation of \( TM \) with respect to \( H \) gives rise to a fundamental class \( [M] \in H_m(M) \) s.t.
\[ H^i(M) \cong H_{m-i}(M) \]

**Construction:** cf. 24.11

\[ M \hookrightarrow \mathbb{R}^k \text{ embedding} \]

**Thom collapse**
\[ S^k = \mathbb{R}^k \cup \Sigma^0 3 \overset{c}{\longrightarrow} \text{Th}(N_m \mathbb{R}^k) \]

In \( Sp \), \( S \Rightarrow S^0 \overset{n}{\longrightarrow} \Sigma^k \text{Th}(N_m \mathbb{R}^k) \cong \text{Th}(-TM) \)
(As before
\[ TM + N_m \mathbb{R}^k = \mathbb{I}^k \]
\[ \Rightarrow \quad \sum_{-k}^{k} Th N_m \mathbb{R}^k \sim Th (-TM) \quad \)

apply $\$ \rightarrow H$

$\$ \rightarrow H \wedge Th (-TM)$

\textbf{Remark:} TM oriented with respect to $H \quad \Rightarrow$

$-TM$ oriented with respect to $H$

\textbf{Proof:} $N_m \mathbb{R}^k \rightarrow \mathbb{I}^k \rightarrow TM$

The orientation of 2 out of 3 vector bundles in a short exact sequence gives an orientation of the third

$\Rightarrow N_m \mathbb{R}^k$ oriented

On the other hand $-TM = \mathbb{I}^{-k} + N_m \mathbb{R}^k$, so $-TM$ is oriented
Thus we have
\[ S \rightarrow \sum_{-m} H \Lambda M \]
equivalently \[ S^m \rightarrow H \Lambda M \]
equivalently \[ [M] \in H_m [M] \]

use \( \mathbb{D} M \cong M^{-TM} \) to obtain iso
\[ H^i(M) \cong H_{m-i}(M) \]
as in 12

- Two fundamental classes must differ by a unit in \( \text{Ho}(CM) \)

- Thus two different orientations of \( TM \) with respect to a cohomology theory \( H \) give a “Signature Formula”
\[ [CM]_1 \neq f(TM)[CM]_2 \]
for some unit \( F(\text{TM}) \in \text{Ho}(\mathcal{M}) \)

a characteristic class

Hirzebruch Signature formula:

We will have two canonical orientations of \( \text{TM} \) with respect to the cohomology theory

\[ \| \wedge M \otimes \mathcal{O} \]

for a smooth, compact oriented manifold \( M \)

Lurie has piecewise linear

First orientation: Since \( M \) is oriented, we have a given equivalence

\[ H \otimes \wedge \text{Th}(\text{TM}) \simeq \sum H \mathbb{Z} \wedge M \]

Smashing \( \wedge H \mathbb{Z} \wedge (H \mathbb{Q} \wedge \mathcal{O}) \) we have
our first orientation

\[(\mathcal{H} \otimes \mathcal{L}) \wedge \text{Th}(TM) = \Sigma^m(\mathcal{H} \otimes \mathcal{L}) \wedge M\]

Second orientation: we will construct an orientation

\[\mathcal{L} \wedge \text{Th}(TM) = \Sigma^m \mathcal{L} \wedge M\]

and then \(\mathcal{L} \wedge (\mathcal{L} \wedge \mathcal{H} \otimes)\) produces the claimed second orientation.

The canonical \(\mathcal{L}\)-theory orientation of \(M\):

Analogy: The \(K\)-groups of \(X\) are given by

\[K^i(X) = [X, \Sigma^i K]\]

\[K_i(X) = \pi_i(C_X \wedge K)\]

where \(K \in \text{Sp}\)

They are also given by forming a \(K\)-theory spectrum/space from a space of vector
bundles

• Similarly the $L$-groups

$$L_i(M) \cong \pi_1 (M, p^{\text{fib}}_\ast \mathscr{D}(\mathcal{Z}), Q) \quad L_i(M) = \pi_i (M, p^{\text{fib}}_\ast \mathscr{D}(\mathcal{Z}), Q)$$

of $M$ are also $L$-groups of a stable $\infty$-category of sheaves

$$L_i(M) = L_i \left( \mathbf{Shv}_{\text{const}}(M, \mathcal{O}(\mathcal{Z})), Q^S \right)$$

where $\mathbf{Shv}_{\text{const}}(M, \mathcal{O}(\mathcal{Z}))$ is a stable $\infty$-category of sheaves of objects in $\mathcal{O}(\mathcal{Z})$ and

$$Q^S : \mathbf{Shv}_{\text{const}}(M, \mathcal{O}(\mathcal{Z})) \overset{Q}{\to} \mathbf{Sp}$$

is a quadratic functor

• For Thom spaces $V \to M$

$$L_i(\text{Th}(V)) \cong L_i \left( \mathbf{Shv}_{\text{const}}(M, \mathcal{O}(\mathcal{Z})), Q^S_V \right)$$

where $Q^S_V$ is a "twist" of $Q^S$ by $V$
references:

Lurie L 11, L23