K field
more intuitive description Art invariant from:
Milnor conjecture filtration on \( W(K) \)

Metabolic forms have even dimension \( \Rightarrow \)

\[
d: W(K) \rightarrow \mathbb{Z}/2\mathbb{Z}
\]

\[
(P, b) \mapsto \dim P
\]

Let's use this

Fundamental ideal: \( I := \ker d \)

Let \((V, q) \in I\) \(\dim V\) even

\(\Rightarrow\) Clifford algebra

\[
\text{Cl}(V, q) = \frac{Tens(V)}{x^2 = q(x)}
\]

\(\mathbb{Z}/2\)-graded

\(\cong \text{Cl}(V, q) \oplus \text{Cl}(V, q)\)

Fact: Center \( \text{Cl}(V, q) \)

is a deg 2 étale extension of \( K \)

\(\text{Gal}(E/k) \rightarrow \mathbb{Z}/2\) homomorphism
\{ \text{homomorphisms} \ \text{Gal}(\bar{E}/k) \to \mathbb{Z}/2 \}^1

discriminant: \quad I \longrightarrow H^1(\text{Gal}(\bar{E}/k), \mathbb{Z}/2)
(CV, q) \mapsto I

\text{char } k \neq 2 \quad H^1(\text{Gal}(E/k), \mathbb{Z}/2) =
\text{coker} (\mathbb{F}^* \longrightarrow k^*) \cong k^*/(k^*)^2
x \mapsto x^2

\text{when char } k = 2 \quad H^1(\text{Gal}(E/k), \mathbb{Z}/2) =
\text{coker} (\mathbb{F} \longrightarrow k)
\quad x \mapsto x^2 - x

\text{when } k = \mathbb{F}^2 \quad H^1(\text{Gal}(E(k), \mathbb{Z}/2) = \mathbb{Z}/2

\text{and Arf invariant} = \text{discriminant}

\textit{Exercise:} prove this

Let \ J = \text{Ker discriminant} \n(CV, q) \in J \quad \Rightarrow \quad \text{Cl}_0(CV, q) = A_0 \times A_1
\quad A_i: \text{central simple algebras of order } 2
\text{in Brauer group

\text{Char } k \neq 2 \quad J \longrightarrow H^2(\text{Gal}(\bar{E}/k), \mathbb{Z}/2)
Thm "Milnor Conjecture" (Voevodsky, Orlov-Vishik-Voevodsky) - Rost

Char $K \neq 2 \quad \text{Im}^m/\text{Im}^{m+1} \cong H^m(\text{Gal}(E/K), \mathbb{Z}/2)

Def: $\Delta$ is the category with objects
$i = \{0, 1, 2, \ldots\}$
$\text{Mor}(i, j) = \text{maps of sets preserving } \leq$

A simplicial set is a functor
$X: \Delta^{op} \rightarrow \text{Set}$

Simplicial sets are a good substitute for topological spaces in many contexts.

$X_i = X(i) = \text{the set of } i \text{ simplices}$

Ex: $X$ top space \rightarrow

$\text{Sing} X: \Delta^{op} \rightarrow \text{Set}$

$\text{Sing}(i) = \text{Map}(i\text{-simplex}, X)$

is a corresponding simplicial set

$\text{Top} \rightarrow \text{S Set}$

$X \mapsto \text{Sing} X$
There is a function $1-1: \text{Set} \to \text{Top}$ defined by $X \mapsto \coprod_{\varphi: i \to j} \Delta^i \times X_i$ for all $i, j$, and $\varphi \in X_j$, $\rho \in \Delta^i$.

Natural weak equivalence:

$\text{Sing} X \to X$

**Def.** The nerve of a category $\mathcal{C}$ is the simplicial set $N(\mathcal{C})$ whose $n$-simplices are composable sequences of morphisms $C_0 \to C_1 \to \ldots \to C_n$.

- Let $\Lambda^n_i$ denote the $i$th horn, the simplicial subset of $\Delta^n$ obtained by removing the interior and the face opposite the $i$th vertex.

**Fact.** A simplicial set $S$ is isomorphic to the nerve of a category $\mathcal{C}$ for all $0 \leq i < n$ and $\Lambda^n_i \to S$ there is a unique $\Delta^n \to S$. 
Def: An \(\infty\)-category (also called \((\infty,1)\)-cat and quasicategory) is a simplicial set \(\mathcal{C}\) satisfying the condition: for all \(0 < i < n\) and \(\Lambda^n_i \rightarrow \mathcal{C}\) there is \(\Delta^n \rightarrow \mathcal{C}\) s.t.

\[
\begin{array}{ccc}
\Delta^n & \overset{g}{\longrightarrow} & \mathcal{C} \\
\Lambda^n_i & \overset{f}{\longrightarrow} & \mathcal{C} \\
\end{array}
\]

\(\text{Ex: } i = 1, n = 2\)

\[
\begin{array}{ccc}
X & \overset{g}{\longrightarrow} & Z \\
\downarrow^{f} & \nearrow^{f} & \searrow^{g} \\
st & \overset{g \circ f}{\longrightarrow} Z \\
\end{array}
\]

\(\mathcal{C}_0 = \text{"objects"}\)

\(\mathcal{C}_1 = \text{"morphisms"}\) multiple choices of compositions

using previous example unique up to (a lot of) homotopy
Can do homotopy theory in $\mathcal{C}$, (a) fiber sequences (b) limits.

- For objects $A, B \in \mathcal{C}$, can associate a simplicial set $\text{Map}(A, B)$, $\text{Tot}\text{Map}(A, B) \cong \text{hhtpy classes of maps } A \to B$.

Lurie L2 Ex 11 L3 Ex 6

Example (sketch) $R$ ring

There is an $\infty$-category $\text{Dperf}(R)$.

0-simplices: bounded chain complexes of finitely generated projective $R$-modules $\ldots \to 0 \to 0 \to P_N \to P_{N-1} \to \ldots \to P_0 \to 0 \to \ldots$.

1-simplices: pairs $(P, Q, \phi)$ of 0-simplices together with a map of chain complexes $f: P \to Q$.

2-simplices: diagrams $R$ with a chain homotopy from $h$ to $g \circ f$. 

Example (sketch) $R$ ring

There is an $\infty$-category $\text{Dperf}(R)$.
\[ \text{ho } \mathcal{D}_{\text{perf}}(R) \text{ is the derived category of } \]
\[ R \text{ from homological algebra} \]

\[ \text{see Lurie L3 Def 9} \]

\[ \mathcal{D}_{\text{perf}}(R) \text{ is a } \mathbf{stable} \ \mathbf{\infty}\text{-category} \]

\[ \text{0 object } = \ldots \to 0 \to 0 \to \ldots \]

\[ \text{maps have fibers and cofibers} \]

\[ \text{fiber sequences } = \text{cofiber sequences } = \text{mapping cones} \]

\[ \text{objects can be desuspended by shift} \]

\[ (\Sigma X = \text{cofib } (X \to 0)) \]

\[ \text{can do stable homotopy theory in a stable } \]
\[ \mathbf{\infty}\text{-category} \]

\[ X, Y \text{ objects of a stable } \mathbf{\infty}\text{-cat} \]

\[ \text{The simplicial sets } \mathbf{\mathbf{E}} \text{ Map}_E (Y, \Sigma^n X)^\mathbb{Z} \]
\[ \text{define a spectrum, i.e. object in classical stable htpy} \]

\[ \text{let } \text{More } (X, Y) \text{ denote this spectrum.} \]

\[ \text{True Story : You can take the derivative} \]
\[ \text{of a functor, and the second, third, ... etc. derivatives} \]

\[ \text{“GoodWillie Calculus”} \]
\[ \text{ex} \]
\[ M, N \text{ mfld} \quad \text{Emb}(M, N) = \text{embeddings} \quad M \hookrightarrow N \]
\[ \text{Imm}(M, N) = \text{immersions} \quad M \rightarrow N \]
\[ \text{First derivative} \quad \text{Emb}(-, N) = \text{Imm}(-, N) \]

The first derivative is \underline{linear} in the sense that it takes fiber sequences to fiber sequences.

The functor taking an object of, say \( \text{Def}_R(\mathbb{R}) \) to all the quadratic forms or bilinear forms valued in some line bundle is itself a \underline{quadratic} functor, meaning that it can be recovered from its first two derivatives.

For the definition of a quadratic functor see Lurie 24 Def 6. We will use the following examples:
\[ \text{Ex.1: } \mathcal{S}p = \text{stable } \infty\text{-category} \]
\[ \text{corresponding to classical } \]
\[ \text{stable htpy theory} \]

"\( \mathcal{S}p \)" for "spectra" (not related to \( \text{spec } R \))

\( \mathcal{S} \in \mathcal{S}p \) sphere spectrum \( \mathcal{S} = \Sigma^\infty \mathbb{p}^+ \)

\( \mathcal{B} : \mathcal{S}p^{\mathcal{B}op} \times \mathcal{S}p^{\mathcal{B}op} \rightarrow \mathcal{S}p \)

\( \mathcal{B}(X, Y) = \text{Mor}_{\mathcal{S}p}(X \wedge Y, \mathcal{S}) \)

\[ \text{Rmk } \mathcal{B}(X, Y) = \text{Mor}_{\mathcal{S}p}(Y, \mathcal{D}X) \]

\( \mathcal{Q}^s : \mathcal{S}p^{\mathcal{B}op}_f \rightarrow \mathcal{S}p \quad \mathcal{Q}^s(X) = \mathcal{B}(X, X)_{h\mathbb{C}_2} \)

\( \mathcal{Q}^a : \mathcal{S}p^{\mathcal{B}op}_f \rightarrow \mathcal{S}p \quad \mathcal{Q}^a(X) = \mathcal{B}(X, X)_{h\mathbb{C}_2} \)

\( \mathcal{Q}^s \) and \( \mathcal{Q}^a \) are \underline{quadratic functors}
Ex 2: $R$ ring, $M$ projective $R$-module

$\n \forall n \in \mathbb{Z}$

$\varGamma: M \to M, \quad \varGamma^2 = 1$

$B: D^{\text{perf}}(R)^{\text{op}} \times D^{\text{perf}}(R)^{\text{op}} \to \text{Sp}$

$B(P, Q) = \text{Mor}_{D^{\text{perf}}(R)}(P \otimes Q, M[-n])$

$C_2$

Chain Complex with the single $R$-module $M$ in degree $-n$

$Q_s: D^{\text{perf}}(R)^{\text{op}} \to \text{Sp}$

$Q_s(P) = B(P, P)$

"Spectrum of symmetric $M\text{-}n$ valued forms on $P."

$Q_u: D^{\text{perf}}(R)^{\text{op}} \to \text{Sp}$

$Q_u(P) = B(P, P)_{\text{h}C_2}$

"Spectrum of $M\text{-}n$ valued quadratic forms on $P."

Ex: $M$ compact oriented manifold dim $n$
Singular cochain complex $C^\ast(M; \mathbb{Z})$ determines object $D^\text{perf}(\mathbb{Z})$.

Intersection pairing

$$C^\ast(M; \mathbb{Z}) \otimes C^\ast(M; \mathbb{Z}) \to C^\ast(M; \mathbb{Z}) \to \mathbb{Z}[\langle n \rangle]$$

determines a point $b_m \in S^\infty_Q, (C^\ast(M; \mathbb{Z}))$.

\underline{\text{$L$-theory and Hermitian $K$-theory}}

\underline{\text{input:}}

- $(\mathcal{E}, Q)$
  - $\mathcal{E}$ stable $\infty$-category
  - $Q$ quadratic functor (nondegenerate)

\underline{\text{output:}}

- Spectrum thus $\text{htpy}$ groups
  - "$L$-groups"
  - "Hermitian $K$-theory"

\underline{\text{references}}

- Lurie 1, 2, 3, 4, 13
- Land 1, 1