ALGEBRAIC CURVES OVER $\mathbb{R}$

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1. Algebraic curves over $\mathbb{R}$

Graphing $y = x^2$ gives the familiar picture of the parabola, and it is reasonable to call this graph a curve. More generally, we could take any polynomial $f(x, y) \in \mathbb{R}[x, y]$ and consider all ordered pairs $(x, y) \in \mathbb{R}^2$ such $f(x, y) = 0$. The adjective 'algebraic' in the phrase 'algebraic curve' means that we are considering sets of points satisfying *polynomial* equations. For instance, we are not considering $y = e^x$.

It turns out to be very useful to not only consider points in $\mathbb{R}^2$ or $\mathbb{R}^n$ satisfying polynomial equations, but to study all points in $\mathbb{C}^n$ satisfying those same equations. One reason for this is that the complex points have nice topological properties which can be used to study the solutions to the equations, even over smaller fields like $\mathbb{R}$. This project is to investigate some of these topological properties.

Considering the complex valued points, however, does mean we need to change the 'picture' of a curve from a 'curvy line' to a 'surface.' For instance, note that the one-dimensional subset of $\mathbb{R}^2$ determined by $y = x^2$, becomes two-dimensional (over $\mathbb{R}$) when we take all points $(x, y) \in \mathbb{C}^2$ such that $y = x^2$.

Although we will be considering points whose coordinates are complex numbers, we will be considering real algebraic curves. The adjective 'real' in the phrase 'real algebraic curve' indicates that the coefficients of the polynomials determining the curve are real numbers.

1.1. Definition. An 'affine algebraic plane curve over $\mathbb{R}$' or a 'real affine algebraic plane curve' will mean a subset X of $\mathbb{C}^2$ of the form $\{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$ for some polynomial $f \in \mathbb{R}[x, y]$.

1.2. *Example.* The polynomial $f(x, y) = y^2 - x(x - 1)(x - \pi)$ determines a real affine algebraic plane curve. We will see below in Example 1.10, that this curve is a torus with one point removed.

It is also useful to add 'points at infinity' to the set of solutions to some polynomial equations. (The adjective 'affine' in the previous definition indicates that these 'points at infinity' have not been included.) In the previous example, there is one 'point at infinity,' and adding this point produces the entire torus. To include these points, we first define *projective space*.

1.3. Definition. For any positive integer n ($n \geq 1$), n-*dimensional complex projective space*, denoted $\mathbb{P}^n_{\mathbb{C}}$ is constructed as follows: for any $\lambda \in \mathbb{C}^*$ and $(x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} - (0, \ldots, 0)$,

$$\lambda(x_0, x_1, \ldots, x_n) = (\lambda x_0, \lambda x_1, \ldots, \lambda x_n)$$

determines another point of $\mathbb{C}^{n+1} - (0, \ldots, 0)$. We say that $\mathbb{C}^*$ *acts* on $\mathbb{C}^{n+1} - (0, \ldots, 0)$. Projective space is formed by quotienting $\mathbb{C}^{n+1} - (0, \ldots, 0)$ by this action

$$\mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} - (0, \ldots, 0))/\mathbb{C}^*$$

A point $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} - (0, \ldots, 0)$ determines a point $[x_0, \ldots, x_n] \in \mathbb{P}^n_{\mathbb{C}}$. The points of the form $[1, x_1, \ldots, x_n]$ determine a copy of $\mathbb{C}^n$ inside $\mathbb{P}^n_{\mathbb{C}}$. When looking at this copy of $\mathbb{C}^n$, the other points of $\mathbb{P}^n_{\mathbb{C}}$ can be viewed as 'points at infinity.'

Returning to Example 1.2, we can now add the 'missing point at infinity:' consider

$$X = \{[z, x, y] \in \mathbb{P}^2_{\mathbb{C}} : zy^2 = x(x - z)(x - \pi z)\}$$

Intersecting with the copy of $\mathbb{C}^2$ determined by $z = 1$ gives the curve of Example 1.2. Any point of $\mathbb{P}^2_{\mathbb{C}}$ either satisfies $z = 0$, or can be expressed in the form $[1, x, y]$, because for $[z, x, y]$ with $z \neq 0$, $[z, x, y] = [1, x/z, y/z]$. Thus the points of X not considered in Example 1.2 are $[0, x, y]$ satisfying $0 = x^3$. There is one such point, namely $[0, 0, 1]$.

1.4. Definition. A 'projective algebraic plane curve over $\mathbb{R}$' is a subset X of $\mathbb{C}^2$ of the form $\{[z, x, y] \in \mathbb{P}^2_{\mathbb{C}} : f(x, y) = 0\}$ for some polynomial $f \in \mathbb{R}[x, y]$. An 'algebraic plane curve over $\mathbb{R}$' will mean a projective algebraic plane curve with finitely many points removed.

We can consider algebraic curves which are not subsets of $\mathbb{C}^2$ or $\mathbb{P}^2_{\mathbb{C}}$, by taking subsets of $\mathbb{C}^n$ or $\mathbb{P}^n_{\mathbb{C}}$ which are determined by more polynomial equations. These polynomials need to be chosen so that the resulting subset has dimension 1 in an appropriate sense. As it is more tricky to say exactly when a set of polynomials $f_1, \ldots f_j \in \mathbb{C}[x_0, \ldots, x_n]$ or $\mathbb{C}[x_1, \ldots, x_n]$ determine a set of dimension 1, we leave this notion to your intuition, and give an example.[1]

1.5. *Example.* The set of points $\{[x, x^2, x^3, \ldots, x^n] : x \in \mathbb{C}\}$ subset $\mathbb{C}^n$ is called the *rational normal curve* , and is a real algebraic curve. See [Har95, Ex 1.16 pg 11].

For our purposes here, a *real algebraic curve* will either be a subset of $\mathbb{C}^n$ of the form

$$X = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : f_1(x_1, x_2, \ldots, x_n) = f_2(x_1, x_2, \ldots, x_n) = \ldots = f_j(x_1, x_2, \ldots, x_n) = 0\}$$

for polynomials $f_1, \ldots, f_j \in \mathbb{R}[x_1, \ldots x_n]$ such that X 'has dimension 1' or a subset of $\mathbb{P}^n_{\mathbb{C}}$ of the form

$$X = \{(x_0, x_1, \ldots, x_n) \in \mathbb{P}^n_{\mathbb{C}} : f_1(x_0, x_1, \ldots, x_n) = f_2(x_0, x_1, \ldots, x_n) = \ldots = f_j(x_0, x_2, \ldots, x_n) = 0\}$$

for homogeneous polynomials $f_1, \ldots, f_j \in \mathbb{R}[x_0, \ldots x_n]$ such that X 'has dimension 1,' or a set of the above form with finitely many points removed, where these points are required to satisfy the condition that if $(x_0, \ldots, x_n)$ has been removed, we must remove $(\overline{x_0}, \ldots, \overline{x_n})$, where $\overline{x_i}$ denotes the complex conjugate of x_i.

[1] A reasonable guess would be to take $n - 1$ polynomials in n variables. This usually works, but sometimes more equations are necessary.

A real algebraic curve is *compact* if it is determined by homogenous polynomials in $\mathbb{P}^n_{\mathbb{C}}$ without removing any points.

1.6. Genus. The genus of (an orientable[2], compact) surface is most easily understood with a picture. Please see figure 1.

1.7. *Example.* Here is a construction of a torus, which is a surface of genus 1: take a rectangle and glue the top to the bottom by identifying a point on the top side to the point on the bottom side directly below it. Similarly, glue the right side to the left side by identifying points lying on the same horizontal line. This is expressed by figure 2.

The resulting surface is a torus T.

Example 1.7 admits the following generalization:

1.8. *Example.* Let g be a positive integer. (g stands for 'genus.') Here is a construction of a surface S_g with genus g: take a polygon with $4g$ sides and glue these sides as shown by figure 3.

1.9. Every (orientable) compact surface is 'equivalent topologically' or, more precisely, every orientable compact surface is *homeomorphic* to a surface of genus g for a unique positive integer g. (See [Hat02] for the definition of the word *homeomorphic*. A proof that every compact orientable surface is homeomorphic to one of the surfaces constructed in Example 1.8 is given in [Mun75], but it is not at all necessary to read this for the purpose of studying the below questions.)

Compact algebraic curves determine compact (orientable) topological surfaces, because they are 'dimension 1' and a space which is one dimensional over $\mathbb{C}$ is two dimension over $\mathbb{R}$. We will be interested in the genera of the surfaces associated to compact algebraic curves. When a real algebraic curve is not-compact, we can add finitely many points, create a compact real algebraic curve, and then consider the genus of the compact algebraic curve.

1.10. *Example.* Consider the real algebraic curve $X \subset \mathbb{P}^2_{\mathbb{C}}$ defined by the homogeneous equation

$$zy^2 = x(x-z)(x-2z)(x-3z)$$

i.e.

$$X = \{[z, x, y] \in \mathbb{P}^2_{\mathbb{C}} : zy^2 = x(x-z)(x-2z)(x-3z)\}$$

Here is a lovely and very useful way of looking at this equation: consider the subset where $z = 1$. We get the affine real algebraic curve associated to the equation $y^2 = x(x-1)(x-2)(x-3)$. The map $(x, y) \mapsto x$ allows us to view this affine real algebraic curve as lying over the copy of $\mathbb{C}$ with coordinate x. In this copy of $\mathbb{C}$, make two 'cuts,' one along the interval where $x \in [0, 1]$ and one along the interval where $x \in [2, 3]$. Then consider

[2]all algebraic real curves are orientable, so there is no need to define this notion here, but see [Hat02] for a definition.

the subset of the affine real algebraic curve $y^2 = x(x-1)(x-2)(x-3)$ which maps to the complement of these cuts. In other words, make the corresponding cuts in the whole curve lying above the cut-up copy of $\mathbb{C}$. What happens is this: the curve separates into two pieces and on each of these pieces the map $(x, y) \mapsto x$ determines a homeomorphism of the piece with the cut-up copy of $\mathbb{C}$. In other words, we are left with two copies of the cut-up $\mathbb{C}$ both mapping by the identity to a cut-up copy of $\mathbb{C}$. See figure 4.

This allows us to see what the affine curve looks like. Namely, to reconstruct the affine curve, we need to glue together two copies of $\mathbb{C}$ each with two slits.

Consider for a moment the equation $y^2 = x$ and the map $(x, y) \mapsto x$. If one traces out a small circle around $x = 0$, one can lift the circle to an arc in the affine curve determined by the equation $y^2 = x$. The y-coordinate of this arc will trace out one half of a small circle around $y = 0$.

Near $(x, y) = (0, 0)$, the map $(x, y) \mapsto x$ from the curve $y^2 = x(x-1)(x-2)(x-3)$ to $\mathbb{C}$ has the same behavior as the map $(x, y) \mapsto x$ from $y^2 = -6x$ because $x - 1$ is just about equal to -1, $x-2$ is just about equal to -2 etc. This means that tracing out a small arc in $\mathbb{C}$ around $x = 0$ lifts to half of an arc around $(0, 0)$ in the curve. This means that we take one copy of the cut-up $\mathbb{C}$ and glue the bottom edge of the slit $[0, 1]$ to the top edge of the slit $[0, 1]$ on the other copy. Then we glue the top edge of the slit $[0, 1]$ on the first copy to the bottom edge of the slit $[0, 1]$ on the second copy. To visualize this, 'flip' the first copy so that the positive i axis is pointing along the negative i axis of the second copy. Then glue the top edge to the top edge and the bottom edge to the bottom edge, creating a picture that looks like figure 5.

Repeating with the other slit, gives the picture drawn in figure 6.

To form the projective curve from the affine curve, we add two points at infinity, as in figure 7.

We see that this curve has genus 1. It is worth the effort to internalize this example! The same argument shows that the genus of the curve corresponding to $y^2 = \prod_{i=1}^{4}(x - \lambda_i)$ is 1 for any choice of distinct λ_i.

1.11. *Problem.* Compute the genus of $y^2 = \prod_{i=1}^{2n}(x - \lambda_i)$. Compute the genus of $y^2 = \prod_{i=1}^{2n+1}(x - \lambda_i)$.

1.12. Homology.

The genus of a curve counts the number of holes. There is a more sophisticated way to do this, which is to define a group H_i which records information about i-dimensional holes for each i. There is a short explanation of the definition of H_i given in Hatcher's book "Algebraic Topology" in the sections "delta complexes" and "simplicial homology" [Hat02, pg 102 -108]. This is available on-line at

http://www.math.cornell.edu/ hatcher/AT/ATpage.html

For our purposes here, it is sufficient to learn enough about homology to be able to do the exercises in this section and then understand a $\mathbb{Z}/2$ action on the homology of a real algebraic curve, which we will discuss in the next section. So, please read pg 102-8 in Hatcher (or the equivalent) and think about the following:

1.13. *Problem.* Compute the homology of m circles glued together at one point as in figure 8.

This topological space is called the 'wedge of m circles' and is denoted by $\vee_m S^1$.

1.14. *Problem.* Let T be the torus of example 1.7 Compute $H_*(T, \mathbb{Z})$.

1.15. *Problem.* Take the $4g$-gon S_g of Example 1.8. Compute $H_*(S_g, \mathbb{Z})$.

A *deformation retraction* of a space X onto a subspace A is a (continuous) family of maps $f_t : X \to X$ such that $f_0 : X \to X$ is the identity map, $f_1(X) = A$, and for all $t \in [0, 1]$ and $a \in A$, $f_t(a) = a$. (See [Hat02, pg 2]. There are several great pictures on pg 2.)

1.16. *Problem.* Take the $4g$-gon S_g of Example 1.8 and remove m points. Denote the resulting surface by $S_{g,m}$. Find a positive integer N such that $S_{g,m}$ deformation retracts onto $\vee_N S^1$, where $\vee_N S^1$ is as in Problem 1.13. Then compute $H_*(S_{g,m}, \mathbb{Z})$

1.17. *Problem.* Let $X \subset \mathbb{P}^2$ be the curve $y^2 = \prod_{i=1}^{2n}(x - \lambda_i)$. Compute $H_1(X, \mathbb{Z})$.

1.18. Galois action.

A map $X \to X$ induces a homomorphism on homology groups. For example, consider $\vee_m S^1$ as in Problem 1.13. $\vee_m S^1$ can be built by attaching m edges to a single vertex. A permutation of the m edges then induces a map $\tau : \vee_m S^1 \to \vee_m S^1$. We also get a map $\tau_* : H_1(\vee_m S^1, \mathbb{Z}) \to H_1(\vee_m S^1, \mathbb{Z})$ as follows: since $\vee_m S^1$ is built by attaching m edges to 1 vertex, the homology of $\vee_m S^1$ is computed from the chain complex

$$\mathbb{Z}^m \xrightarrow{0} \mathbb{Z}$$

(Sorry to spoil the fun!) The permutation associated to τ induces a map $\mathbb{Z}^m \to \mathbb{Z}^m$ by the corresponding permutation matrix. This produces a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^m & \xrightarrow{0} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}^m & \longrightarrow & \mathbb{Z} \end{array}$$

which produces a map $\tau_* : H_*(\vee_m S^1, \mathbb{Z}) \to H_*(\vee_m S^1, \mathbb{Z})$.

This example generalizes: if X is built from cells (or X is a delta-complex as in Hatcher) and we have a map $\tau : X \to X$ given by permuting the cells, we have a map from the cellular chain complex used to compute $H_*(X, \mathbb{Z})$ to itself (i.e. we have a commutative diagram as above), and therefore we have a map $\tau_* : H_*(X, \mathbb{Z}) \to H_*(X, \mathbb{Z})$.

1.19. *Problem.* Consider the curve $X = \mathbb{C} - \{0, 1\}$. Let $\tau : X \to X$ be the map induced by complex conjugation. Show that X deformation retracts onto a delta complex on which τ acts by permuting the cells. Compute $\tau_* : H_*(X, \mathbb{Z}) \to H_*(X, \mathbb{Z})$.

In fact, for any space X and any map $X \to X$, there is an induced map on $H_*(X, \mathbb{Z})$. This is because homology can also be computed by what's called the singular chain complex, and it turns out to be easy to see that a map $X \to X$ produces a map of singular chain complexes. Even without learning the machinery associated to defining the singular chain complex, it is possible to compute the map $\tau_* : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ with a couple of tricks. Here they are: you can write down a basis for $H_1(X, \mathbb{Z})$ consisting of circles drawn on your curve. If you 'push' the circle around on your curve, you do not change the associated element of $H_1(X, \mathbb{Z})$. (To be precise about this, a map $f_0 : S^1 \to X$ determines a map $(f_0)_* : \mathbb{Z} \to H_1(X, \mathbb{Z})$ and given a continuous family of maps $f_t : S^1 \to X$ for $t \in [0, 1]$, $(f_0)_* = (f_t)_*$. See [Hat02, Th 2.10 pg 111].) A drawing is given in figure 9.

Furthermore, if you draw two circles on your curve starting at the same point, the sum of the corresponding elements of H_1 is the element of H_1 corresponding to drawing one of the circles first and then drawing the other. (More precisely, consider S^1 to be the unit circle in $\mathbb{C}$, so 1 is a point of S^1. Given $f_0, f_1 : S^1 \to X$ and such that $f_0(1) = f_1(1)$, one can define $f_2 : S^1 \to X$ by defining $f_2(z)$ to be $f_0(z^2)$ for $z = x + iy$ with $y > 0$, and defining $f_2(z)$ to be $f_1(z^2)$ for $z = x + iy$ with $y \leq 0$. We have the maps $(f_i)_* : \mathbb{Z} \to H_1(X, \mathbb{Z})$ as in the previous paragraph. Then $f_2(1) = f_0(1) + f_1(1)$.)

Here is an example of how to use these rules to compute τ_*

1.20. *Example.* Let T be the torus of example 1.7. T can also be viewed as the quotient of $\mathbb{R}^2$ obtained by identifying points $(x, y) \in \mathbb{R}^2$ with points $(x+a, y+b) \in \mathbb{R}^2$ for $(a, b) \in \mathbb{Z}^2$. Let α be circle in T which is the image of the interval $[0, 1] \times \{0\}$ in $\mathbb{R}^2$. Let β be circle in T which is the image of the interval $\{0\} \times [0, 1]$ in $\mathbb{R}^2$. Then $\{\alpha, \beta\}$ is a basis for $H_1(T, \mathbb{Z})$. (Check this!)

Let $\tau : T \to T$ be the map of the torus induced by the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ corresponding to the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a, b, c, d \in \mathbb{Z}$. Then the image of α under τ is the image in T of the line segment in $\mathbb{R}^2$ running between $(0, 0)$ and (a, c). By pushing this line segment, we see that this segment determines the same element of homology as the sum of the elements of homology corresponding to $[0, a] \times \{0\}$ and $\{0\} \times [0, c]$. In other words $\tau_*(\alpha) = a\alpha + c\beta$. See figure 10.

1.21. *Problem.* Find $\tau_*(\beta)$ in the previous example.

Let X be a real algebraic curve. Complex conjugation induces a map $\tau : X \to X$ defined by $\tau(x_0, \dots, x_n) = (\overline{x_0}, \dots, \overline{x_n})$, where $\overline{x_i}$ denotes the complex conjugate of x_i. This is what we mean by the *Galois action* on X.[3] We will be interested in what can be understood about a curve X from the map $\tau_* : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$.

1.22. *Example.* Let $X \subset \mathbb{P}^2_{\mathbb{C}}$ be the real algebraic curve determined by $y^2 z = \prod_{i=1}^4 (x - \lambda_i z)$ where the λ_i are distinct real numbers. Suppose that the λ_i are ordered such that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$. Let $f(x) \in \mathbb{R}[x]$ be the polynomial $f(x) = \prod_{i=1}^4 (x - \lambda_i)$. Because f is positive for $x < \lambda_1$, $\lambda_2 < x < \lambda_3$, and $\lambda_4 < x$, the real points of X form two copies of S^1. This is shown in figure 11 (there are two points missing at infinity).

Let $f_0 : S^1 \to X$ be a map tracing out the circle marked as α in figure 11. (So, a formula for this map could be given by $f_0(e^{t\pi i}) = (\lambda_3(1-t) + \lambda_2 t, \sqrt{f(\lambda_3(1-t) + \lambda_2 t)})$ for $t \in [0, 1]$, and $f_0(e^{t\pi i}) = (\lambda_3(1+t) + \lambda_2(-t), -\sqrt{f(\lambda_3(1+t) + \lambda_2(-t))})$ for $t \in [-1, 0]$. In fact, such formulas aren't usually very helpful.) We get a map $(f_0)_* : \mathbb{Z} \to H_1(X, \mathbb{Z})$. By abuse of notation, let α also denote $(f_0)_*(1)$.

Using Example 1.10 and Problem 1.14, we can choose a basis for $H_1(X, \mathbb{Z})$, consisting of the homology class α, and one more class $\beta \in H_1(X, \mathbb{Z})$ which we now define. Heuristically, β is the class determined by the circle 'around the cut' between λ_1 and λ_2, where this 'cut' is as described in Example 1.10. More precisely, let $f_1' : S^1 \to \mathbb{C}$ be the map $f_1'(e^{2\pi i t}) = (\lambda_1 + \lambda_2)/2 + ((\lambda_3 - \lambda_1)/2)e^{2\pi i t}$, so f_1' traces out a circle enclosing the segment $[\lambda_1, \lambda_2]$ and such that λ_3 and λ_4 are separated from λ_1 and λ_2 by the circle. The map $(x, y) \mapsto x$ from X to $\mathbb{C}$ (really from X minus the points at infinity to $\mathbb{C}$) is shown in figure 6. Call this map π. From this diagram, we see that there are two choices of map $f_1 : S^1 \to X$ such that $\pi \circ f_1 = f_1'$. (To be more precise about this notion, one can learn about *covering spaces*. The map π restricted to the complement of $\pi^{-1}(\lambda_i)$ is a covering space and therefore has what's called the 'path lifting' property.) Choose one such f_1. f_1 determines a map $(f_1)_* : \mathbb{Z} \to H_1(X, Z)$. Let $\beta = (f_1)_*(1)$. By Problem 1.14 and figures $4 - 7$, it follows that $\{\alpha, \beta\}$ is a basis for $H_1(X, Z)$.

It remains to compute τ_*. Since the image of f_0 consists entirely of real points, $\tau_*(\alpha) = \alpha$. β is equivalent to the class determined by a circle whose x coordinate goes from λ_2 to λ_1 and then back to λ_2. This circle should be chosen so that the y coordinate for a given x coordinate is different on the trip to λ_2 and on the trip back to λ_1. This circle has the property that the x coordinate is real, and the y coordinate is purely imaginary. Furthermore, taking the negative of the y coordinate corresponds to traversing the circle in the opposite direction because of the condition that the y coordinate changes on the trip to λ_2 and on the trip back to λ_1 (and because the square of the y coordinate is always $f(x)$). Since the negative of a purely imaginary number is equal to the complex conjugate $\tau_*(\beta)$ is the homology class associated to the circle traversed in the opposite direction. It follows that $\tau_*(\beta) = -\beta$.

[3] There is a general principal that to study solutions to equations over some field k, you can study the corresponding algebraic variety over the algebraic closure $\overline{k}$ of k along with the action of the Galois group. (The Galois group is the group automorphisms of $\overline{k}$ which fix k. A (quasi-projective) algebraic variety is a subset of $\mathbb{P}^n$ which is of the form $X_1 - X_2$, where each X_i is the set of simultaneous solutions to a set of homogeneous polynomials.)

1.23. *Problem.* Let $X \subset \mathbb{P}^2_{\mathbb{C}}$ be the curve determined by the equation $y^2z^{2n-2} = \prod_{i=1}^{2n}(x - \lambda_i z)$. Let $\tau : X \to X$ be the map induced by complex conjugation as above. Compute $H_1(X, \mathbb{Z})$ and the map $\tau_* : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$. (In other words 'compute the Galois action on $H_1(X, \mathbb{Z})$.')

1.24. *Problem.* Let $X \subset \mathbb{C}^2$ be the curve determined by the equation $y^2 = \prod_{i=1}^{2n}(x - \lambda_i)$. Compute the Galois action on $H_1(X, \mathbb{Z})$.

1.25. *Problem.* Let $X \subset \mathbb{C}^2$ be the curve in Problem 1.24. Let X_0 be $X - D$, where $D = \{(i, \sqrt{f(i)}), (-i, \overline{\sqrt{f(i)}})\}$, where $\sqrt{f(i)}$ denotes either choice of square root and where $f(x) = \prod_{i=1}^{2n}(x - \lambda_i)$. Compute the Galois action on $H_1(X_0, \mathbb{Z})$.

1.26. *Problem.* Let $E \subset \mathbb{P}^2_{\mathbb{C}}$ be the Fermat curve $x^3 + y^3 = z^3$ as in Problem 3.2. Compute the Galois action on $H_1(E, \mathbb{Z})$.

2. Questions

Let C be a compact real algebraic curve. By 1.9 and Problem 1.15, the homology $H_1(C, \mathbb{Z})$ determines the genus of C.

For non-compact curves, this is not the case:

2.1. *Problem.* Give two real curves X_1,X_2 such that $H_1(X_1, \mathbb{Z}) \cong H_1(X_2, \mathbb{Z})$, but $g_1 \neq g_2$, where g_i denotes the genus of C_i for $i = 1, 2$.

Now add the data of the Galois action $\tau : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$. Does this data determine the genus?

2.2. *Question.* Suppose you are given $H_1(X, \mathbb{Z})$, and $\tau : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$. Can you determine the genus of X?

2.3. *Question.* Suppose you are given real algebraic curves X, Y and an isomorphism $\theta : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ such that θ respects the Galois action, i.e. such that the following diagram commutes

$$\begin{array}{ccc} H_1(X, \mathbb{Z}) & \xrightarrow{\tau_*} & H_1(X, \mathbb{Z}) \\ \downarrow \theta & & \downarrow \theta \\ H_1(Y, \mathbb{Z}) & \xrightarrow{\tau_*} & H_1(Y, \mathbb{Z}) \end{array}$$

Is it true that X and Y are isomorphic? What if X and Y are both compact? What if θ is induced from a map $X \to Y$ and X and Y are compact?

2.4. *Question.* Suppose you are given a free abelian group $\mathbb{Z}^N$ and a homomorphism $\tau : \mathbb{Z}^N \to \mathbb{Z}^N$. Can you find necessary and sufficient conditions for the existence of a smooth real algebraic curve X such that $H_1(X, \mathbb{Z})$ with its Galois action is $\mathbb{Z}^N$ with the action of τ?

2.5. *Question.* Given $\mathbb{Z}^N$, τ as in Question 2.4, what are necessary and sufficient conditions for the existence of an X such that $X \subset \mathbb{P}^2$?

3. Tools that could be useful to study the questions

3.1. Euler characteristic and the Riemann-Hurwitz formula.

The abelian group $\mathbb{Z}^N$ is said to have *rank* N. We will only be dealing with spaces whose homology groups are of this form. We can then define the *Euler characteristic* $\chi(X)$ of a space X to be

$$\chi(X) = \sum_i (-1)^i \operatorname{rank} H_i(X, \mathbb{Z})$$

If X is a surface partitioned into polygons $\chi(X) = \text{vertices} - \text{edges} + \text{faces}$. (For a proof of this, see [Hat02, Th 2.44 p 146].)

The Euler characteristic behaves very well under maps between algebraic curves $X \to Y$, and this allows us to compute the genus, Euler characteristic, and homology of X from the the genus, Euler characteristic, or homology of Y. This is called the Riemann-Hurwitz formula. A reference is [For91]

3.2. *Problem.* Let $E \subset \mathbb{P}^2$ be the Fermat curve $x^3 + y^3 = z^3$. Compute $H_*(E, \mathbb{Z})$.

3.3. *Problem.* Let $f(x, y, z) \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of degree d and let $X \subset \mathbb{P}^2$ be $f(x, y, z) = 0$. Determine the genus of X.

3.4. The period mapping. A compact algebraic curve of genus g can be embedded in C^{2g}/Λ, where Λ is the *period lattice* $\Lambda \subset C^{2g}$. Λ with its action by complex conjugation is the same as H_1 with its Galois action. One reference is [For91, Section 21 pg 166].

4. Answers to some of the problems

Problem 1.15: S_g is formed from a $4g - gon$. Convince yourself that S_g has one 2-dimensional cell, $2g$ 1-dimensional cells, and one 0-dimensional cell. Futhermore, the boundary of each of these cells is trivial. Thus $H_*(S_g)$ is computed by the chain complex

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z}$$

Thus $H_1(S_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, $H_i(S_g, \mathbb{Z}) \cong \mathbb{Z}$ for $i = 1, 2$, and $H_i(S_g, \mathbb{Z}) \cong 0$ for $i > 2$.

Problem 1.17: By the solution to Problem 1.11, we have that X is a genus $n-1$ curve. By 1.9, X is homeomorphic to S_{n-1} of Example 1.8. Homology is invariant under homeomorphisms. Thus $H_i(X,\mathbb{Z}) \cong H_i(S_{n-1},\mathbb{Z})$. See the solution to Problem 1.15.

Problem 1.21: $\tau_*(\beta) = b\alpha + d\beta$.

5. Epilogue

One can define homology groups with a Galois action in a much more general context. Even for curves over finite fields, where topological intuition breaks down, one can define what's called *étale homology*. The étale homology of a curve over a finite field with its Galois action determines the number of solutions to the corresponding equations. The Galois action is futhermore controlled by the analogue of the Riemann hypothesis. There is a lovely description of this story given in the historical introduction in [FK88].

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