L9 2/20/12

Hopf algebras

$$A^{\cancel{*}\,-*} = \bigoplus_i NT^{\Sigma}\left(H^{\cancel{*}\,-*}, H^{\cancel{*+i}\,-*-i}\right)$$

Has product by definition

Constructed $Sq^i \in A^{-i}$

Next week (Friday?) we'll show: Sq^i generate A^* as algebra

<u>Prop</u>: $\exists$ algebra map $\psi : A^* \longrightarrow A^* \otimes A^*$ (called coproduct)

s.t. $\forall\ \alpha \in H^*(X)$, $\beta \in H^*(Y)$, $s \in A^*$

$$(\underset{\cup}{\text{ext}})(\psi s)(\alpha \otimes \beta) = s(\alpha \underset{\cup}{\text{ext}}\, \beta) \quad \text{in } H^*(X \times Y)$$

<u>Pf</u> (2/11/13 - used that Sq^i generate)

$\Rightarrow A^*$ is a <u>Hopf algebra</u>

More specifically: graded, connected, Hopf algebra

concentrated in non-neg (or non-pos)

This means: A^* is a graded vector space 1 w/ A^0 = ground field $= R$

graded alg w/ 1

equipped with 1) $A^* \otimes A^* \xrightarrow{\text{v.s. hom}} A^*$ giving an associative product w/ $1 \in A^0$

coalg = 2) - requirement ψ alg hom

2) $A^* \xrightarrow{\psi} A^* \otimes A^*$ hom of alg w/ 1

where alg structure on $A^* \otimes A^*$

is $(a_1 \otimes a_2)\cdot(a_3 \otimes a_4) = (-1)^{\dim a_2 \dim a_3}\left(a_1 a_3 \otimes a_2 a_4\right)$

and s.t. $\Psi(a) = a \otimes 1 + 1 \otimes a +$

$\sum b_i \otimes c_i$

$\dim a < 0$ with $\dim b_i, c_i < 0$

This last condition comes from dual of: unit: $\mathbb{F}_2 \to A^0 \to A^*$

$$\begin{tikzcd} & A^* \otimes A^* \arrow[dr, "\text{Prod}"] & \\ A^* \arrow[ur, "id \otimes unit"] \arrow[dr, "unit \otimes id"'] & & A^* \\ & A^* \otimes A^* \arrow[ur, "\text{prod}"'] & \end{tikzcd}$$

- A^* connected Hopf alg $\Rightarrow$ $\mathrm{Hom}(A^*, K)$ is.
- notions commutativity, associativity
- S'pose A connected coalg
  B connected alg

Let $\mathrm{Mor}^{0}_{K\text{-mod}}(A, B) = \left\{ A \xrightarrow{f} B \text{ graded Mod hom s.t. } f_0 : A_0 \to B_0 = id \right\}$

There is a composition on $\mathrm{Mor}^{\circ}_{R\text{-mod}}(A,B)$

Take $f,g \in \mathrm{Mor}^{\circ}_{R\text{-mod}}(A,B)$,

define $f \star g \in \mathrm{Mor}^{\circ}_{R\text{-mod}}(A,B)$

as the composition

$$A \xrightarrow{\text{coprod}} A \otimes A \xrightarrow{f \otimes g} B \otimes B \xrightarrow{\text{Prod}} B$$

<u>Prop</u>: $\mathrm{Mor}^{\circ}_{R\text{-mod}}(A,B)$ is a group under $\star$ with identity $A \to R \to B$

<u>Pf</u>: What we need to check is that $f \in \mathrm{Mor}^{\circ}_{R\text{-mod}}(A,B)$ has an inverse

f^{-1}. S'pose have defined f^{-1} in $\deg < n$ s.t. $f^{-1} * f = \mathrm{id}$ up to deg $n-1$.

Take $x \in A^n$.

$$\text{coprod } x = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$$

$n > \deg > 0$

$$\text{prod}(f^{-1} \otimes f)\,\text{coprod} = \underbrace{f(x)}_{f^{-1}(1) f(x)} + f^{-1}(x) + \sum f^{-1}x' \, f x''$$

Define $f^{-1}(x) = -f(x) - \sum f^{-1}x' \, f x''$ $\square$

$\Rightarrow$ a connected Hopf algebra has a canonical conjugation $c: A \to A$

inverse to identity

Facts: 1) $c(a_1 a_2) = (-1)^{\dim a_1 \dim a_2} c(a_2) c(a_1)$

2) conjugation in Hopf alg & its dual are dual homomorphisms

3) $C^2 = \mathrm{id}_A$ if A is either commutative or cocommutative.

4) A connected commutative alg.

A Hopf alg $\Longleftrightarrow$

$\mathrm{Hom}_{Alg}(A, -)$ is a group-valued functor

Thank Akhil & maybe someone else

e.g. A^* coprod is com $\Rightarrow A_*$ comm $\Rightarrow \mathrm{Hom}_{alg}(A_*, -)$ group

references: Milnor-Moore "Structure Hopf alg"

Milnor "Steenrod Alg & its dual"

Ex: $\psi 1 = 1 \otimes 1$ (ψ is assumed alg hom)

$id = \text{prod}(c \otimes 1)\psi \Rightarrow 1 = c(1)$

Ex: $\psi Sq^1 = Sq^1 \otimes 1 + 1 \otimes Sq^1$

$\Rightarrow 0 = Sq^1 \cdot c(1) + 1 \cdot c(Sq^1)$

$0 = Sq^1 + c(Sq^1)$

$c(Sq^1) = -Sq^1$

Def: Elts s.t. $\psi x = 1 \otimes x + x \otimes 1$ are called primitives. $\Rightarrow c(x) = -x$

$\{\text{primitives}\}_A \overset{=}{\Longleftrightarrow} \text{Mor}_{\text{hopf alg}}(\mathbb{G}_a, A)$

$\mathbb{G}_a = R[[x]]$ ← graded ring

$R[[x]] \to R[[x]] \otimes R[[y]]$ ← graded

$R[[x]] \otimes R[[y]] = R[[x,y]]$

$x \mapsto x + y = x \otimes 1 + 1 \otimes x$

Tautologically,

$$A^* \otimes H^*(X) \to H^*(X) \quad \text{action}$$

$$H_*(X) = \mathrm{Hom}(H^*, \mathbb{F}_2) \Rightarrow$$

$$H_*(X) \otimes A^* \to H_*(X) \quad \text{action} \Rightarrow$$

$$H^*(X) \to H^*(X) \otimes A_* \quad \text{coaction.}$$

Defines $\mathrm{Hom}_{alg}(A_*, R) \xrightarrow{\theta} \mathrm{End}_{R\text{-mod}}(H^*(X) \otimes R)$

functorial in R, group hom

Since A^* consists of natural transformations, θ compatible with $X \to X \times X$ $\Rightarrow$ $\mathrm{im}\,\theta \subset \mathrm{End}_{R\text{-alg}}(H^*(X) \otimes R)$

(Lemma 3 Milnor Steenrod alg)

For X a group, apply to $X \times X \to X$, obtain

$$\mathrm{Hom}_{alg}(A_*, R) \to \mathrm{End}_{\text{Hopf-alg}}(H^*(X) \otimes R)$$

Last time/problem set: $\mathbb{G}_a \cong H^*(\mathbb{R}P^\infty)$ Hopf alg

Thus we have

$$\mathrm{Hom}(A_*, R) \xrightarrow{\Theta'} \mathrm{End}(\mathbb{G}_a \otimes R)$$

functorially in R.

<u>Last time</u>: $\mathrm{End}(\mathbb{G}_a \otimes R) = \left\{ \begin{array}{c} \text{additive power} \\ \text{series coefs in } R \end{array} \right\}$

$$= \{ r_0 X + r_1 X^2 + r_2 X^4 + \dots$$

$$= \mathrm{Hom}_{alg}(\mathbb{F}_2[r_0, r_1, \dots], R)$$

coproduct from composition of power series.

Thus $\mathrm{Hom}(A_*, A_*) \longrightarrow \mathrm{Hom}_{alg}(\mathbb{F}_2[r_0, r_1, \dots], R)$

$$\mathrm{id} \longmapsto \Theta$$

$\Theta : \mathbb{F}_2[r_0, r_1, \ldots] \to A_*$ morphism Hopf algebras b/c Θ' group hom

Thm (Serre–Milnor) Θ induces an isomorphism

$$\mathbb{F}_2[r_0, r_1, \ldots]/r_0 \to A_*$$

$$r_i \longmapsto \xi_i$$

Pf: Coaction $H^*(\mathbb{R}P^\infty) \xrightarrow{\alpha} H^*(\mathbb{R}P^\infty) \otimes A_*$

$$H^*(\mathbb{R}P^\infty) = \mathbb{Z}/2[[x]]$$

Tautologically $\alpha x = \sum_{i=1}^{\infty} x^i \otimes \overline{\xi_i}$

where $\sum_i \overline{\xi_i}(S) x^i = Sx$.

$$H^*(\mathbb{R}P^\infty) \otimes A_* \xrightarrow{\overline{\alpha}} H^*(\mathbb{R}P^\infty) \otimes A_*$$

is Hopf alg hom $\Longrightarrow$

$\sum \overline{\xi}_i x^i$ is an additive power series

$\Longrightarrow$ $\overline{\xi}_i = 0$ for $i \neq 2^i$.

Thus $\alpha x = \sum_{i=0}^{\infty} \xi_i x^{2^i}$

$\xi_0 = 1$ ξ_i are defined as before

Thus $r_i \longmapsto \xi_i$.

It remains to show iso.

$$R = \{(a_1, a_2, \ldots) : a_i \in \mathbb{Z}_{\geq 0}\}$$
$$\cup$$
$$I = \{ \quad '' \quad : a_i \geq 2a_{i+1} \}$$

ordered lexicagraphically from right

$$\gamma : I \to R$$
$$(a_1, \ldots) \longmapsto (a_1 - 2a_2, a_2 - 2a_3, \ldots)$$

bijective

<u>Prop</u> (2/13/13) $I, J \in I$

$$\langle \xi^{\gamma(J)}, Sq^I \rangle = \begin{cases} 0 & I < J \\ 1 & I = J \end{cases}$$

$\Rightarrow$ $\{ Sq^I : I \in I \}$ are lin indep.

(Omar!)

By Adem relation & $\overset{\text{alg}}{\text{spanning}}$ of Sq^i,

$\{ Sq^I : I \in \mathcal{I} \}$ basis.

By prop, we have $\{ \xi^{\gamma(I)} : I \in \mathcal{I} \}$

$$\| $$

$\{ \xi^J : J \in R \}$

basis.

Friday: Spectral Sequences.