additive polynomials

L8 2/15/13

Algebraic description of A^* & A_*

$f \in R[x]$

$f(x+y) = f(x) + f(y)$

ex: R char p, x^{p^i} is additive.

Non-example: x^n for n not a power of p.

Thus any additive ~~polynomial~~ is $\sum a_i x^{p^i}$
power series.

$\Rightarrow \mathbb{F}_p[a_0, a_1, \dots]$ represents functor

$\mathbb{F}_p\text{-algebras} \longrightarrow \text{groups} \quad =: \operatorname{End} \mathbb{G}_a$

$R \longmapsto \{\text{additive power series of } R[x]\}$

This notation comes from:

$\mathbb{R}[[x]]$ and map $\mathbb{R}[[x]] \longrightarrow \mathbb{R}[[x, y]]$
$x \longmapsto x+y$

is a topological Hopf algebra (equivalently formal group scheme)
Called $\mathbb{G}_a$

Picture: $(\mathbb{R}, +)$ $\quad \mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}$
$(x, y) \longmapsto x+y = z$

Corresponding map on functions $\mathbb{R}[x, y] \longleftarrow \mathbb{R}[z]$
$x+y \longleftarrow z$

Define $\mathrm{End}^s(\mathbb{G}_a) : \mathbb{F}_p\text{-algebras} \to \text{Groups}$

$R \longmapsto \{$ additive power series in $R[[X]]$ with linear term X $\}$ ← means: act by identity on tangent space

use ξ_i

$\mathrm{End}^s(\mathbb{G}_a)$ represented by $R[a_0, a_1, \ldots] / \langle a_0 - 1 \rangle$

Set $p=2$, work in category of $\mathbb{F}_2$-algebras

Can check: $\mathrm{End}^s(\mathbb{G}_a) = \mathrm{Aut}^s(\mathbb{G}_a)$

$\mathbb{G}_a$ occurs topologically

$X = \mathbb{RP}^\infty$ is a group

One way to see this: $\mathbb{R}(t)$ = field of rational functions $\mathbb{R}$-coefficients

$$\mathbb{R}(t) \times \mathbb{R}(t) \xrightarrow{\text{multiplication}} \mathbb{R}(t)$$

defines $(\mathbb{R}(t) - \{0\}) \times (\mathbb{R}(t) - \{0\}) \to (\mathbb{R}(t) - \{0\})$

$$\mathbb{R}P^\infty = (\mathbb{R}(t) - \{0\}) / \mathbb{R}^*$$

giving $\mathbb{R}P^\infty \times \mathbb{R}P^\infty \longrightarrow \mathbb{R}P^\infty$

Get entire group structure $X = \mathbb{R}P^\infty$

$H^* = H^*(\ ; \mathbb{Z}/2)$

Applying H^* gives group structure on

$$\mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(H^*(X), -)$$

i.e, $H^*(X)$ is Hopf-algebra

<u>exercise</u>: $G_a \simeq H^*(X)$

Hopf algebra

$$A^* = \bigoplus_i \text{Nat'l transformations compatible w/ suspension} (H^*, H^{*+i})$$

It is sometimes useful to consider cohomology as negatively graded. We'll do this now.

$$A^* \otimes H^*(X) \xrightarrow{\text{tautological action}} H^*(X)$$ preserves degree

<u>Exercise</u>: $\bar{H}^*(X) = \mathrm{Hom}(H_*(X), \mathbb{F}_2)$

whence $A^* \otimes H_*(X) \longrightarrow H_*(X)$

Dualize (i.e. apply $\mathrm{Hom}_{\substack{\text{graded} \\ \mathbb{F}_2\text{-mod}}}$)

obtain $H^*(X) \xrightarrow{\alpha} H^*(X) \otimes A_*$

$$\mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(A_*, R) \longrightarrow \mathrm{End}_{R\text{-alg}}(H^*(X) \otimes R)$$

$$\varphi \longmapsto \bar{\varphi}$$

$$\begin{tikzcd} H^*(X) \arrow[r, "\alpha"] \arrow[dr] & H^*(X) \otimes A_* \arrow[r, "1 \otimes \varphi"] & H^*(X) \otimes R \\ & H^*(X) \otimes R \arrow[ur, dashed] & \end{tikzcd}$$

using Kunneth $H^*(X \times X) = H^*(X) \otimes H^*(X)$

+ diagonal, this is alg map

S'pose X is a group.

Then coaction α respects $\underbrace{X \times X \to X}$ group law

whence $H^*(X) \longrightarrow H^*(X) \otimes H^*(X)$

We obtain:

$$\mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(A_*, R) \to \mathrm{End}_{\substack{\text{Hopf}\\\text{alg}}}\left(H^*(X) \otimes R\right)$$

a natural transformation of functors to category of groups

- Now take $X = \mathbb{RP}^\infty$. Obtain

$$\mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(A_*, R) \longrightarrow \mathrm{End}\, \mathbb{G}_a(R)$$

In fact,

$$\begin{array}{ccc} & & \operatorname{End}(\mathbb{G}_a)(R) \\ & \nearrow & \cup \\ \operatorname{Hom}_{\mathbb{F}_2\text{-alg}}(A_*, R) & \longrightarrow & \operatorname{End}^s(\mathbb{G}_a) \end{array}$$

is an isomorphism.

<u>Thm</u> (Serre–Milnor)

$$\operatorname{Hom}_{\mathbb{F}_2\text{-alg}}(A_*, R) \simeq \{ x \mapsto x + a_1 x^2 + a_2 x^{2} + a_3 x^4 \}$$

$$\simeq \{ x \mapsto \sum a_i x^{2^i} \quad a_0 = 0 \quad a_i \in R \}$$

as group valued functors.

These are ξ_i from before:

$\Longrightarrow$

$A_* = \mathbb{F}_2[\xi_1, \xi_2 \ldots]$ polynomial algebra

with coproduct determined by power series composition.

(This describes that ugly product on A^*!)

<u>Explicitly</u>: $f(x) = \sum a_i x^{2^i}$

$g(x) = \sum b_i x^{2^i}$

$$gf(x) = \sum b_i \left(f(x)\right)^{2^i}$$

$$= \sum b_i \left(\sum a_j x^{2^j} \right)^{2^i}$$

$$= \sum b_i \left(\sum a_j^{2^i} x^{2^{i+j}} \right)$$

$$= \sum_n \sum_{i+j=n} a_j^{2^i} \otimes b_i$$

$\Rightarrow$ coproduct on A_*

$$\psi : A_* \longrightarrow A_* \otimes A_*$$

$$\psi(\xi_k) = \sum_n \sum_{i+j=n} \xi_j^{2^i} \otimes \xi_i$$

Milnor diagonal.

Pf of Serre-Milnor:

1) Sq^i generate $\Rightarrow \langle Sq^1, \dots \rangle \twoheadrightarrow A^*$
(later)

2) Adem relation (omitted)

$a < 2b$ $\lfloor \frac{a}{2} \rfloor$

$$Sq^a Sq^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

$$I = \{ (a_1, a_2, \dots) \in \mathbb{Z}^n_{\geq 0} \mid \text{only finitely many terms non-zero}$$

$$a_i \geq 2a_{i+1} \}$$

"admissible sequences"

For $I \in \mathcal{I}$

$$Sq^I := Sq^{a_1} Sq^{a_2} \cdots$$

$\{Sq^I : I \in \mathcal{I}\}$ spans $\mathcal{A}^*$

3) $X = K(\mathbb{Z}/2, n)$

$\iota \in H^n(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$

Fundamental class.

$$\{Sq^I \iota : I \in \underset{\deg \leq n}{\mathcal{I}}\} \subset H^*(K(\mathbb{Z}/2, n), \mathbb{Z}/2)$$

is linearly independent.

(We'll see this when we calculate $H^*(K(\mathbb{Z}/2, n), \mathbb{Z}/2)$ next week)

$\Longrightarrow \{Sq^I : I \in \mathcal{I}\}$ basis for $\mathcal{A}^*$

4) Using $\{Sq^I : I \in \mathcal{I}\}$ basis, we showed

Wednesday that $A_* \cong \mathbb{F}_2[\xi_1, \dots]$

- Since A^* acts on $H^*(\mathbb{RP}^\infty)$ by $x \mapsto \sum x + \xi_1 x^2 \dots$

map from Serre-Milnor

$$\mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(A_*, R) = \{ x \mapsto x + \xi_1 x^2 + \xi_2 x^3 + \dots \}$$

$\xi_i \in R$

is $\xi_i \mapsto \xi_i$

$\Rightarrow$ iso.

Correction from Wednesday; Thanks to Akhil & Denis and Haynes

S'pose we know $\{ Sq^I : I \in \mathcal{I} \}$ basis for A^*

Let $\{ (Sq^I)^* : I \in \mathcal{I} \}$ be dual basis.

It is not true that $(Sq^I)^* (Sq^J)^* = (Sq^{I+J})^*$

ex: $(Sq^2)^\star (Sq^1)^\star \neq (Sq^3)^\star$ b/c

$\langle Sq^2 Sq^1, (Sq^3)^\star \rangle = 0$ by def'n

$\langle Sq^2 Sq^1, (Sq^2)^\star (Sq^1)^\star \rangle =$

$\langle \mathrm{coprod}_{A^\times}(Sq^2 Sq^1), (Sq^2)^\star \otimes (Sq^1)^\star \rangle =$

$\langle (Sq^2 \otimes Sq^0 + Sq^1 \otimes Sq^1 + Sq^0 \otimes Sq^2)(Sq^0 \otimes Sq^1 + Sq^1 \otimes Sq^0), \dots \rangle$

$= 1.$