Friday: spectral sequences

Rmk: H^*: Top $\to \mathbb{Z}$ functors

L7 - 2/13/13

Algebraic description A^*

$$H^* = H^*(-;\mathbb{Z}/2)$$

$$A^* = \bigoplus_i \text{Nat'l trans compatible w/ suspension}(H^*, H^{*+i})$$

<u>Prop</u> (Adem relations) For $a < 2b$

$$Sq^a Sq^b = \sum_{c=0}^{[\frac{a}{2}]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

<u>Rmk:</u> $Sq^{a+b-c} Sq^c$ term s.t. $(a+b-c) \geq 2c$

(pf: $a+b > a + \frac{a}{2} > \frac{3}{2}a \geq 3c$)

<u>Presentation</u>: $A^* = \mathbb{F}_2\langle Sq^i \rangle$ (free alg) / Adem relations (proof in text)

$\Rightarrow A^*$ has a basis $\{ Sq^I \mid I \in \mathcal{I} \}$

where $\mathcal{I}$ is set of <u>admissible sequences</u>

$$\mathcal{I} = \{ (a_1, a_2, \ldots, a_r) \mid a_i \geq a_{i+1} \}$$

<u>Dual Steenrod algebra</u>

$$A_* = \mathrm{Hom}(A^*, \mathbb{F}_2)$$

coproduct $A^* \longrightarrow A^* \otimes A^*$ induces product $A_* \otimes A_* \to A_*$

$\Rightarrow A_*$ is an algebra

($\&$ prod A^* gives A_* a coproduct too)

Study A_* and A^* by action on

$H^*(\mathbb{R}P^\infty) = \mathbb{Z}/2$

Define $\xi_i \in A_{2^i - 1}$ by

$$\langle \xi_i, \theta \rangle x^{2^i} = \theta x \text{ in } H^{2^i}(\mathbb{R}P^\infty)$$

$$x \xrightarrow{Sq^1} x^2 \xrightarrow{Sq^2} x^4 \xrightarrow{Sq^4} \cdots \xrightarrow{Sq^{2^{i-1}}} x^{2^i}$$

$$I^i = (2^{i-1}, 2^{i-2}, \ldots, 2, 1)$$

$$\langle \xi_i, I^i \rangle = 1$$

<u>Thm</u> A_* as algebra is polynomial on ξ_i

Claim: $\langle \xi_K, Sq^I \rangle = \begin{cases} 1 & \text{if } I = I^K \text{ with zeros interspersed} \\ 0 & \text{otherwise} \end{cases}$

Pf: $Sq(x^{2^j}) = (Sq\, x)^{2^j} = (x+x^2)^{2^j}$

$= x^{2^j} + x^{2^{j+1}} = Sq^0 x^{2^j} + Sq^{2^j} x^{2^j}$

so only Sq^0 and Sq^{2^j} are non-zero on x^{2^j}

$x = x^0 = x^{2^j}$ $\square$

coprod $A^* \longrightarrow A^* \otimes A^*$ $(\star)$

$Sq^I \longmapsto \sum_{I=J+K} Sq^J \otimes Sq^K$

Sq^I's fit into basis $\{ Sq^I \mid I \in \mathcal{I} \}$

A_* has dual basis $\{ (Sq^I)^* \mid I \in \mathcal{I} \}$

$(Sq^J)^* (Sq^K)^* = (Sq^{J+K})^*$ by $(\star)$

$\Rightarrow (Sq^J)^*$ and $(Sq^K)^*$ commute. $\Rightarrow$ can also mult ξ_i's in any order

Consider $(a_1, \ldots, a_r) \in I$ = admissible sequences

$(a_1, \ldots, a_r, 0, \ldots)$

Let $R = \{(b_1, b_2, \ldots) \mid b_i \in \mathbb{Z}_{\geq 0}$ only finitely many of $b_i \neq 0\}$

Define $\xi^J = \prod \xi_i^{b_i}$ for $J \in R$, $J = (b_1, b_2, \ldots)$

To show: $\{\xi^J : J \in R\}$ is basis.

We will do this by: ordering I and R so that ξ^J acts by lower triangular matrices on Sq^I.

order elts of R lexicographically from right

e.g. $(2, 4, 1, 0, 0, \ldots) > (9, 8, 0, \ldots)$

$\gamma : I \longrightarrow R$ surjective

$(a_1, a_2, \ldots) \longmapsto (a_1 - 2a_2, a_2 - 2a_3, \ldots)$

It suffices to prove:

Lemma 2: $I, J \in \mathcal{I}$

$$\langle \xi^{\gamma(J)}, Sq^I \rangle = 1 \qquad I = J$$

$$\langle \xi^{\gamma(J)}, Sq^I \rangle = 0 \qquad I < J$$

reality check: $\deg(Sq^I) = \deg(\xi^{\gamma(I)})$:

$I = (a_1, \dots, a_r)$ $\quad \gamma(I) = (a_1 - 2a_2, a_2 - 2a_3, \dots)$

$$\deg(\xi^{\gamma(I)}) = \sum_{i=1}^{r} (a_i - 2a_{i+1}) \deg \xi_i$$

$$= \sum_{i=1}^{r} (a_i - 2a_{i+1})(2^i - 1)$$

$$= \sum_{i=1}^{r} 2^i a_i - \sum_{i=1}^{r} 2^{i+1} a^{i+1}$$

$$- \sum a_i + 2 \sum a_{i+1} = \sum a_i \quad \checkmark$$

pf lemma 2: $J = (a_1, \ldots, \underset{\neq 0}{a_k}, 0, 0, \ldots)$

$\geq$

$I = (b_1, \ldots, b_k, 0, 0, \ldots)$

Induct on J and a_k

$\gamma(J) = (a_1 - 2a_2, \ldots\ a_{k-1} - 2a_k, a_k)$

$\gamma(J) - (0, \ldots, 0, \underset{\substack{\uparrow \\ k\text{th slot}}}{1}, 0, \ldots) = (\qquad , a_{k-1} - 2(a_k - 1), a_k - 1, 0 \ldots)$

$= \gamma(J')$

$J' = (a_1 - 2^{k-1}, a_2 - 2^{k-2}, \ldots, a_k - 1, 0, \ldots)$
$= J - I_k$

$\Rightarrow \xi^{\gamma(J)} = \xi^{\gamma(J')} \xi_k$

$$\langle \xi^{\gamma(J)}, Sq^I \rangle = \langle \xi^{\gamma(J')} \xi_k, Sq^I \rangle$$

$$= \langle \xi^{\gamma(J')} \otimes \xi_k, \psi(Sq^I) \rangle$$

$$= \sum_{I_1 + I_2 = I} \langle \xi^{\gamma(J')}, Sq^{I_1} \rangle \langle \xi_k, Sq^{I_2} \rangle$$

By lemma 1, $\langle \xi_k, Sq^{I_2} \rangle = 0$ unless

$I_2 = I_k$ with zeros interspersed.

interspersed zeros make $I_2 > J$ which is not true

$\Rightarrow \langle \xi_k, Sq^{I_2} \rangle = 0$ unless $I_2 = I_k$

$$\Rightarrow \langle \xi^{\gamma(J)}, Sq^I \rangle = \begin{cases} 0 & a_k = 0 \\ \langle \xi^{\gamma(J - I_k)}, Sq^{I - I_k} \rangle & \text{otherwise} \end{cases}$$

Lemma follows by induction. $\square$

In fact, A_* has lovely description as Hopf alg (= Alg H s.t.

$\mathrm{Hom}_{Alg}(H, -)$ is a group valued functor.

In other words, H has prod, coprod, identity, inverse, satisfying properties coming from def'n group)

Here's the description:

$$\mathbb{R}P^\infty = \underbrace{\mathbb{R}(t)}_{} - \{0\} \Big/ \mathbb{R}^*$$

$$(\mathbb{R}(t) - \{0\}) \times (\mathbb{R}(t) - \{0\}) \xrightarrow[\text{functions}]{\text{mult rat'l}} (\mathbb{R}(t) - \{0\})$$

$$\Rightarrow \mathbb{RP}^\infty \times \mathbb{RP}^\infty \longrightarrow \mathbb{RP}^\infty$$

Apply H^*:

$$\mathbb{F}_2[[x,y]] \longleftarrow \mathbb{F}_2[[x]] \quad (\star)$$

$$x+y \xleftarrow{+} x$$

Gives $\mathbb{F}_2[[x]]$ Hopf alg structure.

Called <u>additive formal</u> group $\mathbb{G}_a$.

$\mathcal{A}^*$ action respects $(\star)$

$$\rightsquigarrow \quad \mathrm{Hom}(\mathcal{A}_*, \mathbb{F}_2) \longrightarrow \overset{\mathrm{Aut}}{\cancel{\mathrm{End}}}(\mathbb{F}_2[[x]], +),$$

Thm: $\forall$ commutative $\mathbb{F}_2$-algebras R, the action of A^* on $\mathbb{R}P^\infty$ gives a canonical bijection

$$\operatorname{Hom}_{alg}(A_*, R) = \{ x \mapsto x + a_1 x^2 + a_2 x^4 + a_3 x^8 + \dots \}$$

$a_i \in R$

$$\subset \text{automorphisms}(R[[x]], +) = \operatorname{aut} \mathbb{G}_{a,R}$$

$$\operatorname{Spf} A_* = \operatorname{aut} \mathbb{G}_{a,\mathbb{F}_2}$$