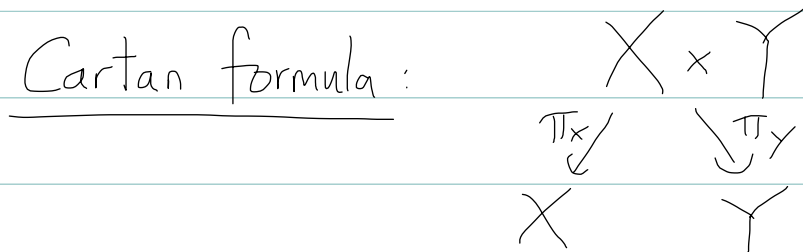


LG 2/11/13

Coproduct A^*

I owe: clean-up of dual Steenrod alg discussion.

Start: clean-up of consequences of Cartan formula



"exterior cup product"

$$H^*(X) \otimes H^*(Y) \xrightarrow{\pi_X^* \otimes \pi_Y^*} H^*(X \times Y) \otimes H^*(X \times Y) \xrightarrow{\cup} H^*(X \times Y)$$

Let $H^* = H^*(-; \mathbb{Z}/2)$

$$Sq^i(\pi_X^* \alpha \pi_Y^* \beta) = \sum_{j+k=i} \pi_X^* Sq^j \alpha \pi_Y^* Sq^k \beta$$

Claim: The Cartan formula defines a coproduct on A^*

i.e. $A^* \longrightarrow A^* \otimes A^*$

$$Sq^i \mapsto \sum_{j+k=i} Sq^j \otimes Sq^k$$

determines an algebra map.

Assume (later): Sqⁱ's generate A^{*}

Pf: Let T be the free \mathbb{F}_2 -algebra

on generators s^i .

$$\begin{array}{ccc}
 s^i & T & \xrightarrow{p} A^* \\
 \downarrow & \downarrow \psi & \swarrow \text{---} \\
 \sum_{j+k=i} sq^j \otimes sq^k & A^* \otimes A^* &
 \end{array}$$

necessary & sufficient: $\ker p \subset \ker \psi$

For X a space, $\alpha \in H^*(X)$ obtain $e_\alpha: A^* \rightarrow H^*(X)$
 $s \mapsto s\alpha$

Take spaces X, Y and classes

$\alpha \in H^*(X), \beta \in H^*(Y)$

Thus have $X \times Y$ and $\gamma = \pi_X^* \alpha \wedge \pi_Y^* \beta \in H^*(X \times Y)$

Lemma: the diagram commutes

$$\begin{array}{ccc}
 T & \xrightarrow{p} & A^* \\
 \psi \downarrow & & \searrow e_\gamma \\
 A^* \otimes A^* & \xrightarrow{e_\alpha \otimes e_\beta} & H^*(X) \otimes H^*(Y) \xrightarrow{\text{external cup product}} H^*(X \times Y)
 \end{array}$$

$$\begin{array}{c}
 T \\
 \downarrow \\
 e_\gamma p(s^i) = e_\gamma Sq^i = Sq^i(\gamma)
 \end{array}$$

$$= \sum \pi_X^* Sq^j \alpha \wedge \pi_Y^* Sq^k \beta$$

 Cartan formula

$$= \text{ext}_{\text{cup}} \left(\sum Sq^j \alpha \otimes Sq^k \beta \right)$$

$$= \text{ext}_{\text{cup}} e_\alpha \otimes e_\beta \left(\sum Sq^j \alpha \otimes Sq^k \beta \right) \quad \checkmark$$

But: $e_\alpha, e_\beta, e_\gamma$ are not algebra homomorphisms

So it is not sufficient to check commutativity on alg gen

maps in diagram are \mathbb{F}_2 vector space morphisms.

\Rightarrow it suffices to check commutativity on $S^{i_1} S^{i_2} \dots S^{i_r} =: S^I$

$$I = (i_1, i_2, \dots, i_r)$$

$$e_{\sigma} p(S^I) = S_q^{i_1} S_q^{i_2} \dots S_q^{i_{r-1}} \left(\sum_{j_1 + k_r = i_r} S_q^{j_1} \alpha S_q^{k_r} \beta \right)$$

$$= \sum_{J+K=I} S_q^J \alpha S_q^K \beta$$

$$= \text{ext}_{\text{cup}}(e_A \otimes e_B) \psi S_q^I$$

□

finish pf claim: Take $R \in \ker p$

Since diagram commutes,

$$\text{ext cup } e_\alpha \otimes e_\beta \psi R = 0$$

$$H^*(X) \otimes H^*(Y) \xrightarrow{\text{ext cup}} H^*(X \otimes Y)$$

is an isomorphism by Kunneth formula (prove later)

$$\Rightarrow e_\alpha \otimes e_\beta \psi R = 0$$

Since α, β arbitrary $\Rightarrow \psi R = 0 \quad \square$

Recall: properties, Two alg approx Top

$$\alpha \in M^p \Rightarrow Sq^p \alpha = d \vee \alpha$$

$$Sq^i \alpha = 0 \quad i > p$$

$$Sq^i (\alpha \beta) = \sum_{j+k=i} Sq^j \alpha Sq^k \beta$$