L6 2/11/13

Coprod $\mathcal{A}^*$

I owe: clean-up of dual Steenrod alg discussion.

Start: clean-up of consequences of Cartan formula

<u>Cartan formula</u>:

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

"exterior cup product"

$$H^*(X) \otimes H^*(Y) \xrightarrow{\pi_X^* \otimes \pi_Y^*} H^*(X \times Y) \otimes H^*(X \times Y) \xrightarrow{\cup} H^*(X \times Y)$$

Let $H^* = H^*(-; \mathbb{Z}/2)$

$$Sq^i(\pi_X^* \alpha \, \pi_Y^* \beta) = \sum_{j+k=i} \pi_X^* Sq^j \alpha \; \pi_Y^* Sq^k \beta$$

<u>Claim</u>: The Cartan formula defines a coproduct on $\mathcal{A}^*$

i.e. $\mathcal{A}^* \longrightarrow \mathcal{A}^* \otimes \mathcal{A}^*$

$$Sq^i \longmapsto \sum_{j+k=i} Sq^j \otimes Sq^k$$

determines an algebra map.

Assume (later): Sq^i's generate $\mathcal{A}^*$

Pf: Let T be the free $\mathbb{F}_2$-algebra on generators s^i.

$$s^i \longmapsto Sq^i$$

$$\begin{tikzcd}
s^i \arrow[d, mapsto] & T \arrow[r, "p"] \arrow[d, "\psi"] & \mathcal{A}^* \arrow[dl, dashed] \\
\sum\limits_{j+k=i} Sq^j \otimes Sq^k & \mathcal{A}^* \otimes \mathcal{A}^* &
\end{tikzcd}$$

necessary & sufficient: $\ker p \subset \ker \psi$

For X a space, $\alpha \in H^*(X)$ obtain $e_\alpha : \mathcal{A}^* \to H^*(X)$
$$s \longmapsto s\alpha$$

Take spaces X, Y and classes $\alpha \in H^*(X)$, $\beta \in H^*(Y)$

Thus have $X \times Y$ and $\gamma = \pi_X^* \alpha\, \pi_Y^* \beta \in H^*(X \times Y)$

<u>Lemma</u>: the diagram commutes

$$\begin{tikzcd}
T \arrow[r, "p"] \arrow[d, "\psi"'] & A^* \arrow[rrd, bend left, "e_\gamma"] & & \\
A^* \otimes A^* \arrow[r, "e_\alpha \otimes e_\beta"] & H^*(X) \otimes H^*(Y) \arrow[rr, "\text{external cup product}"'] & & H^*(X \times Y)
\end{tikzcd}$$

~~Pf~~: $e_\gamma\, p(\overset{\substack{T\\ \Downarrow}}{S^i}) = e_\gamma\, Sq^i = Sq^i(\gamma)$

$$\underset{\text{Cartan formula}}{=} \sum \pi_X^* Sq^j \alpha\ \pi_Y^* Sq^k \beta$$

$$= \underset{\text{cup}}{\text{ext}} \left(\sum Sq^j \alpha \otimes Sq^k \beta \right)$$

$$= \underset{\text{cup}}{\text{ext}}\ e_\alpha \otimes \alpha_\beta \left(\sum Sq^j \otimes Sq^k \right) \checkmark$$

But: $e_\alpha, e_\beta, e_\gamma$ are not algebra homomorphisms so it is not sufficient to check commutativity on alg gen

- maps in diagram are $\mathbb{F}_2$ vector space morphisms.

$\Rightarrow$ it suffices to check commutativity on $S^{i_1} S^{i_2} \cdots S^{i_r} =: S^I$

$$I = (i_1, i_2, \ldots, i_r)$$

$$e_\gamma P(S^I) = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_{r-1}} \Big(\sum_{j_r + k_r = i_r} Sq^{j_r} \alpha \, Sq^{k_r} \beta \Big)$$

$$= \sum_{J+K=I} Sq^J \alpha \, Sq^K \beta$$

$$= \underset{\text{cup}}{\text{ext}} (e_\alpha \otimes e_\beta) \; 4 \; Sq^I \qquad \square$$

Finish pf claim: Take $R \in \operatorname{Ker} p$

Since diagrams commutes,

$$\text{ext cup } e_\alpha \otimes e_\beta \, \psi R = 0$$

$$H^*(X) \otimes H^*(Y) \xrightarrow[\text{cup}]{\text{ext}} H^*(X \otimes Y)$$

is an isomorphism by Kunneth formula (prove later)

$\Rightarrow e_\alpha \otimes e_\beta \, \psi R = 0$

Since α, β arbitrary $\Rightarrow \psi R = 0$ $\square$

Recall: properties, Two alg approx Top

$$\alpha \in H^p \Rightarrow Sq^p \alpha = \alpha \cup \alpha$$

$$Sq^i \alpha = 0 \quad i > p$$

$$Sq^i(\alpha\beta) = \sum_{j+k=i} Sq^j \alpha \, Sq^k \beta$$