Correction: φ s.t. $\varphi(\sigma) = \cancel{EZ}\ C_*(\sigma\times\sigma)$
$\qquad\qquad\qquad\qquad\qquad AW$

L5 2/6/13

A^*-modules, algebras, dual

$$C_*(X\times Y) \underset{EZ}{\overset{AW}{\rightleftarrows}} C_*(X)\otimes C_*(Y)$$

inverse equivalences

For proving Cartan:

$$H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) \underset{\text{graded ring}}{\cong} \mathbb{Z}/2[x] \quad (\text{or } \mathbb{Z}/2[[x]])$$

if a graded ring is $\prod$

- Follows from, for example, computing

$$C_*(\mathbb{R}P^\infty) \xrightarrow{\Delta} C_*(\mathbb{R}P^\infty\times\mathbb{R}P^\infty) \xrightarrow{AW} C_*(\mathbb{R}P^\infty)\times C_*(\mathbb{R}P^\infty)$$

or any φ s.t. $\varphi(\sigma)\in AW\ C_*(\sigma\times\sigma)$ for some model of $\mathbb{R}P^\infty$ as a simplicial set.

- $\mathbb{R}P^\infty = S^\infty/\mathbb{Z}/2$

$$\begin{array}{ccc} \mathbb{R}P^\infty & = & S^\infty/\mathbb{Z}/2 \\ \| & & \| \\ B\mathbb{Z}/2 & & (E\mathbb{Z}/2)/\mathbb{Z}/2 \end{array}$$

- This, in turn, follows from computing

$$C_*(E\mathbb{Z}/2) \xrightarrow{r} C_*(E\mathbb{Z}/2)\otimes C_*(E\mathbb{Z}/2)$$

s.t. 1) r $\mathbb{Z}/2$-equivariant with $\mathbb{Z}/2=\{\tau,e\}$ acting on $C_*(E\mathbb{Z}/2)\otimes C_*(E\mathbb{Z}/2)$ by $\tau(\sigma_1\otimes\sigma_2)=\tau\sigma_1\otimes\tau\sigma_2$

2) $r(\sigma) \in AW\, C_*(\sigma \times \sigma)$

Can construct r by same argument used to construct φ from Sq^i construction.

Messiness: it's been convenient to consider $E\mathbb{Z}/2$ as CW-cplx

$$E\mathbb{Z}/2 = S^\infty = \mathrm{colim}\Big(\overset{\circ}{d_0} \quad \overset{\circ}{\tau d_0} \subset$$

$$\subset \cdots \Big)$$

chains on simplicial cplx

and this isn't a simplicial complex.

$W = C_*(E\mathbb{Z}/2)$ $\quad W_p = \mathbb{Z} d_p \oplus \mathbb{Z} \tau d_p$ isn't

So same proof doesn't precisely work,

but the same idea of proof does work.

$\rightsquigarrow$ $r : W \longrightarrow W \otimes W$

Up to signs & τ's

$$r(d_i) = \sum_{\substack{p+q=i \\ p,q \geq 0}} d_p \otimes d_q$$

b/c $x^p \cup x^q = x^i$ in $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$

can check: $r(d_i) = \sum_{p+q=i} (-1)^{pq} d_p \otimes \tau^q d_q$

works.

Rmk: $\exists$ Similar formula for $B\mathbb{Z}/n$

See Brown "Cohomology of Groups"
Ch V §1

Recall construction Sq^i: X simplicial complex or set

$\varphi: W \otimes C_*(X) \to C_*(X) \otimes C_*(X)$ s.t.

1) φ $\mathbb{Z}/2$-equivariant $\quad \tau(w \otimes \sigma) = \tau w \otimes \sigma$, $\tau(\sigma_1 \otimes \sigma_2) = (-1)^{d_1 d_2} \sigma_2 \otimes \sigma_1$

2) $\varphi(w \otimes \sigma) \in AW_* \, C_*(\sigma \times \sigma)$

$\alpha \in C^p$, $(Sq^i \alpha)(\sigma) := (\alpha \cup_{p-i} \alpha)(\sigma) = (\alpha \otimes \alpha)(\varphi(d_{p-i} \otimes \sigma))$

$$X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$$

external cup product: $\pi_X^* \alpha \, \pi_Y^* \beta$ computed by identifying $C_*(X \times Y) = C_*(X) \times C_*(Y)$ $\quad \pi_X^* \alpha \, \pi_Y^* \beta (\sigma_x \otimes \sigma_y) = \alpha(\sigma_x) \beta(\sigma_y)$

<u>Prop</u> (Cartan formula):

$$Sq^i(\pi_X^* \alpha \, \pi_Y^* \beta) = \sum_{p+q=i} \pi_X^* Sq^p \alpha \, \pi_Y^* Sq^q \beta$$

<u>Pf</u>: compute $Sq^i(\pi_X^* \alpha \, \pi_Y^* \beta)$.

Need $\varphi_{X \times Y}$. Let $K = C_*(X)$, $L = C_*(Y)$.

Have $\varphi_X: W \otimes K \to K \otimes K$

$\varphi_Y: W \otimes L \to L \otimes L$

Want: $\varphi_{X \times Y}: W \otimes K \otimes L \to (K \otimes L) \otimes (K \otimes L)$

$\varphi_{X \times Y}$ can be defined

$$W \otimes K \otimes L \xrightarrow{r \otimes 1 \otimes 1} W \otimes W \otimes K \otimes L \xrightarrow{\text{reorder}} (W \otimes K) \otimes (W \otimes L)$$

$$\xrightarrow{\varphi_X \otimes \varphi_Y} K \otimes K \otimes L \otimes L \xrightarrow{\text{reordr}} (K \otimes L) \otimes (K \otimes L)$$

b/c it satisfies defining properties.

$$Sq^i(\pi_X^* \alpha \, \pi_Y^* B)(\sigma_X \otimes \sigma_Y) =$$

$$(\pi_X^* \alpha \, \pi_Y^* B) \otimes (\pi_X^* \alpha \, \pi_Y^* B) \, \varphi(d_{p+q-i} \otimes \sigma_X \otimes \sigma_Y)$$

$$r(d_{p+q-i}) = \sum_{l+m=p+q-i} d_l \otimes \tau^l d_m \quad \text{mod } 2$$

$$\Rightarrow \varphi(d_{p+q-i} \otimes \sigma_X \otimes \sigma_Y) = \underbrace{\sum_{l+m=p+q-i} \varphi_X(d_l \otimes \sigma_X) \otimes \varphi_Y(\tau^l d_m \otimes \sigma_Y)}_{\text{reordered}}$$

$$= \sum_{\substack{l+m= \\ p+q-i}} \varphi_X(d_l \otimes \sigma_X)\, \tau\, \varphi_Y(d_m \otimes \sigma_Y)$$

$$\Rightarrow \left(Sq^i\ \pi_X^* \alpha\, \pi_Y^* \beta\right)(\sigma_X \otimes \sigma_Y) \overset{\text{mod } 2}{=}$$

$$\sum_{\substack{l+m= \\ p+q-i}} \left((\alpha \otimes \alpha)\varphi_X(d_l \otimes \sigma_X)\right)\left((\beta \otimes \beta)\varphi_Y(d_m \otimes \sigma_Y)\right) =$$

$$\sum_{\substack{l+m= \\ p+q-i}} Sq^{p-l} \alpha\, \sigma_X\ Sq^{q-m} \beta\, \sigma_Y$$

Note: $p-l+q-m = i$

$$\Rightarrow Sq^i\, \pi_X^* \alpha\, \pi_Y^* \beta = \sum_{j+k=i} \pi_X^* Sq^j \alpha\ \pi_Y^* Sq^k \beta$$

$\square$

Take $Y = X$

$\Delta : X \longrightarrow X \times X$

$$\Delta^*\left(Sq^i\, \pi_X^*\alpha\, \pi_X^*\beta = \sum_{j+k=i} \pi_X^* Sq^j \alpha\, \pi_X^* Sq^k \beta \right)$$

$\rightsquigarrow$ $$\boxed{Sq^i \alpha\beta = \sum_{j+k=i} Sq^j\alpha\, Sq^k \beta}$$

$\Delta^* Sq^i = Sq^i \Delta^*$

<u>Cor</u>: $Sq = \sum_{i=0}^{\infty} Sq^i$ is a multiplicative homomorphism.

Properties: $Sq^0 = id$

$Sq^1 = \beta$

$Sq^p \alpha = \alpha \cup \alpha$ for $\alpha \in H^p$

$Sq^i \alpha = 0$ for $i > p$

(Cartan) $Sq^i(\alpha\beta) = \sum_{j+k} Sq^j \alpha \, Sq^k \beta$

Ex: Sq^i on $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[[X]]$

$Sq^0 X = X$

$Sq^1 X = X^2$

$Sq^i X = 0 \quad i > 1$

Cartan $\Rightarrow$

$Sq(X^n) = (Sq(X))^n = (X(1+X))^n$

$\Rightarrow Sq^i X^n = \binom{n}{i} X^{n+i}$

$Sq^i \in \mathcal{A}^* =$ Nat'l trans $(H^* \to H^{*+i})$ compatible w/ suspension

Composition of nat'l trans $\Rightarrow \mathcal{A}^*$ is algebra

cup product $H^* \otimes H^* \to H^*$ $\Rightarrow$ Have $\mathcal{A}^* \to$ nat'l trans $(H^* \otimes H^*)$

by $S \longmapsto \left\{ \alpha \otimes \beta \mapsto S(\alpha\beta) \right\}$

cup product compatible with Σ

$\Rightarrow \mathcal{A}^* \to$ nat'l transf $(H^* \otimes H^*)$

$\mathcal{A}^* \otimes \mathcal{A}^*$ compatible with Σ on each $\otimes$-factor

$$A^* \xrightarrow{\text{coproduct}} A^* \otimes A^*$$ gives A^* structure of co-algebra

Cartan Formula is the computation of this coproduct. (IOU : Sq^i generate A^*)

alg + coalg + ... = Hopf algebra

Dualizing, $A_* = \mathrm{Hom}(A^*, \mathbb{F}_2)$
coprod ⇄ prod $\mathbb{F}_2$-vector spaces

coproduct becomes product.

Cartan Formula $\Longleftrightarrow$ A_* is polynomial algebra

__Two algebraic approximations to Top__

$M = \{M^n\}_{n \in \mathbb{Z}}$ (or $\bigoplus M^n$ or $\prod M^n$) graded $\mathbb{F}_2$-vector space

M is an A^*-module when it is equipped with

$$A^* \underset{\mathbb{F}_2}{\otimes} M \xrightarrow{\alpha} M$$

$$\text{s.t.} \quad \begin{array}{ccc} A^* \otimes A^* \otimes M & \xrightarrow{1 \otimes \alpha} & A^* \otimes M \\ \downarrow {\scriptstyle prod_A \otimes 1} & & \downarrow \\ A^* \otimes M & \xrightarrow{\alpha} & M \end{array} \quad \text{commutes}$$

M is <u>unstable</u> if

$Sq^i x = 0$ for $x \in M^n$ $n < i$

- $\mathcal{U}$ = category of unstable A^*-modules

  morphisms = degree preserving module maps.

<u>ex</u>: $H^*(X; \mathbb{Z}/2) \in \mathcal{U}$. $Top \xrightarrow{H^*} \mathcal{U}$ is a functor.

- because A^* is coalgebra, A^*-modules has $\otimes$

$M, N \in \mathcal{U}$ $\quad A^* \otimes M \otimes N \xrightarrow{coprod} A^* \otimes A^* \otimes M \otimes N \to A^* \otimes M \otimes A^* \otimes N \xrightarrow{\alpha_M \otimes \alpha_N} M \otimes N$

(with 1)

Def: An unstable $\mathcal{A}^*$-algebra M is an unstable $\mathcal{A}^*$-module with

1) $M \otimes M \xrightarrow{\mu} M$ μ is $\mathcal{A}^*$-module map
2) $\mathbb{F}_2 \xrightarrow{\eta} M$

s.t. 1) + 2) make M an associative, graded-commutative algebra w/ 1

3) $x^2 = Sq^p x \;\; \forall \; x \in M^p$

$\mathcal{K}$ denotes the category of unstable $\mathcal{A}^*$-algebras

Ex: $H^*(X; \mathbb{Z}/2) \in \mathcal{K}$

$\mathcal{A}^*$ linear (1) $\Leftrightarrow$ cartan

(3) from properties

$\rightsquigarrow \mathrm{Top} \xrightarrow{H^*(-;\mathbb{Z}/2)} \mathcal{K}$ is a functor.

Rmk: There is a mod p Steenrod alg & analogous def'ns of $\mathcal{K}$ and $\mathcal{U}$

Let V be a finite p-group. X s.t. $\pi_1 X = 1$ and $\dim H^*(X;\mathbb{Z}/p) < \infty$ $\forall *$

Thm (Lannes) $[BV, Y] \to \mathrm{Hom}_{\mathcal{K}}(H^*(X;\mathbb{Z}/p), H^*(BV, \mathbb{Z}/p))$

is a bijection.

- proof later -

Next time: Full (algebraic) description $\mathcal{A}^*$ (still assuming Sq^i generate)

Adem relation