L4 - Properties of the Steenrod algebra

2/4/13

Section: 8-9pm Monday Science center 310
problem sets: Tuesday
(This info on syllabus.)
New room: SC 110

- defined Sq^i last time. This time, we'll show some properties. These properties allow us to define categories of "unstable" modules & algebras over $\mathcal{A}^*$. These categories are alg approximations to Top, which are useful.
- So here are some properties of Sq^i. These properties are also interesting in their own right.

Recall: $Sq^i : H^*(;\mathbb{Z}/2) =: H^* \longrightarrow H^{*+i}$ nat'l transformation group homomorphism compatible w/ suspension

Defined with $\varphi : C_*(E\mathbb{Z}/2) \otimes C_*(X) \to C_*(X) \otimes C_*(X)$

s.t. 1) $\mathbb{Z}/2$-equivariant

2) $\varphi(d_i \otimes \sigma) \in EZ\ C_*(\sigma \otimes \sigma)$

$u \in C^p$

$Sq^i(u)\sigma = (u \cup_{p-i} u)\sigma := (u \otimes u)\varphi(d_{p-i} \otimes \sigma)$

Do $Sq^p u = u \cup u$ $u \in H^p$ and $Sq^i u = 0$ for $i > p$. Then do $Sq^0 = 1$

- A SES $0 \to N \to M \to Q \to 0$ of abelian groups induces LES $H^*(X; -)$

Boundary $H^p(X;Q) \to H^{p+1}(X,N)$ is described:

$$\begin{array}{ccc} H^p(X;Q) & \to & H^{p+1}(X,N) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ u & \longmapsto & [\delta x] \end{array}$$

Choose $x \in C^p(X; M)$ $x \mapsto u$ $\delta x \in C^{p+1}(X;M) \mapsto 0$ in $C^{p+1}(X,Q)$

$\Rightarrow \delta x \in C^{p+1}(X, N)$

$(*) \quad 0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$

<u>Def</u>: Let $\beta : H^p(X; \mathbb{Z}/2) \to H^{p+1}(X; \mathbb{Z}/2)$

be boundary map associated with $(*)$

Here: $[\delta x]$ is $[\frac{1}{2} \delta x]$.

<u>Prop</u>: $\beta \, Sq^i = \begin{cases} Sq^{i+1} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$

<u>Pf</u>: compute $\beta \, Sq^i$:

$$\beta \, Sq^i u = \left[\tfrac{1}{2} \delta \left(\bar{u} \cup_{p-i} \bar{u} \right) \right] \qquad \bar{u} \in C^p(X; \mathbb{Z}/4)$$

Last time: computed $\delta(u \cup_j v)$ w/ $\underline{d(d_j \otimes \sigma)} = \underline{d(d_j) \otimes \sigma}$ $\boxed{+(-1)^j d_j \otimes d\sigma}$

$$\delta(u \cup_j v) = (-1)^j \delta u \cup_j v + (-1)^{j+p} u \cup_j \delta v \qquad ☆$$

$$+ (-1)^{j+1} u \cup_{j-1} v + (-1)^{pq+1} v \cup_{j-1} u \qquad ☆$$

$$\Rightarrow \delta\left(\bar{u} \cup_{p-i} \bar{u}\right) = (-1)^{p-i}\, \delta\bar{u} \cup_{p-i} \bar{u}$$

$$+ (-1)^{-i}\, \bar{u} \cup_{p-i} \delta\bar{u}$$

$$+ (-1)^{p-i+1}\, \bar{u} \cup_{p-i-1} \bar{u}$$

$$+ \overbrace{(-1)^{p+1}}^{p^2 \overset{(2)}{\equiv}}\, \bar{u} \cup_{p-i-1} \bar{u}$$

$\delta\bar{u} = 2a$

$$\Rightarrow \beta Sq^i u = \Big[(-1)^{p-i}\, a \cup_{p-i} \bar{u}$$

$$+ (-1)^{-i}\, \bar{u} \cup_{p-i} a$$

$$+ \varepsilon\, \bar{u} \cup_{p-i-1} \bar{u} \Big]$$

$\varepsilon = 1$ i even

$\phantom{\varepsilon = {}}0$ i odd

$$\left[\bar{u} \cup_{p-(i+1)} \bar{u}\right] = Sq^{i+1} u$$

First two terms are a coboundary mod 2

$\delta(\bar{u} \cup_{p-i+1} a)$

$\square$

Prop: $Sq^0 = \mathrm{id}$

Pf: Nat'l trans $(H^p, H^p) =$

$$H^p(K(\mathbb{Z}/2, p), \mathbb{Z}/2).$$

Universal coef thm $\Longrightarrow$

$$0 \to \mathrm{Ext}^1(\overset{0 \,/\!/\, \text{Hurewicz}}{H_{p-1}(K(\mathbb{Z}/2,p))}, \mathbb{Z}/2) \to H^p(K(\mathbb{Z}/2,p), \mathbb{Z}/2)$$

$$\to \mathrm{Hom}(\underset{\underset{\mathbb{Z}/2}{\parallel \ \text{Hurewicz}}}{H_p(K(\mathbb{Z}/2,p), \mathbb{Z})}, \mathbb{Z}/2)$$

$$\Longrightarrow H^p(K(\mathbb{Z}/2,p), \mathbb{Z}/2) = \mathbb{Z}/2$$

compatibility with $\Sigma \Longrightarrow Sq^0 = 1$ or 0

$Sq^0 u = u \cup u$ for $u \in H^0 \Longrightarrow Sq^0 = 1$ $\square$

Cor: $Sq^1 = \beta$

Prop (Cartan formula)

$Sq^i(uv) = \sum Sq^j u \, Sq^{i-j} v$ in $C^n(X \times Y)$

$u \in C^p(X)$, $v \in C^q(Y)$, $n = p + q - i$

Cor: Pulling back by $\Delta : X \to X \times X$

in $C^n(X)$, $Sq^i(uv) = \sum Sq^j \, Sq^{i-j}$

Cor: $Sq := \sum Sq^i$ is a homomorphism

Pf: Compute $Sq^i \; \pi_1^* u \; \pi_2^* v$.

$W = C_*(E\mathbb{Z})$ $\mathbb{Z}/2 = \langle \tau \rangle$

$W_j = \mathbb{Z} d_j \oplus \mathbb{Z} \tau d_i$

Construct $r : W \longrightarrow W \otimes W$ $\mathbb{Z}/2$-equiv chain map

$\tau(u \otimes v) = \tau u \otimes \tau v$

suitable for computing cup products on $\mathbb{R}P^\infty$

Assert: $r(d_i) = \sum_{j=0}^{i} (-1)^{j(i-j)} d_j \otimes \tau^j d_{i-j}$

works.

Construct φ for $X \times Y$ from φ_X, φ_Y:

after identifying $C_*(X \times Y) = C_*(X) \otimes C_*(Y)$

Let $K = C_*(X)$, $L = C_*(Y)$

φ given by

$$\begin{tikzcd}
W \otimes K \otimes L \arrow[r, "r \otimes 1"] & W \otimes W \otimes K \otimes L \arrow[r] & W \otimes K \otimes W \otimes L \arrow[d, "\varphi_X \otimes \varphi_Y"] \\
& & K \otimes K \otimes L \otimes L \arrow[d] \\
& & K \otimes L \otimes K \otimes L
\end{tikzcd}$$

Can forget signs!

$$Sq^i(uv)(a \otimes b) =$$

$$= (u \otimes v \otimes u \otimes v)\, \varphi(d_n \otimes a \otimes b)$$

$$= (u \otimes u \otimes v \otimes v)\, (\varphi_X \otimes \varphi_Y)\big(\underbrace{r(d_n) \otimes a \otimes b}_{\text{re-ordered}}\big)$$

$$= (u \otimes u \otimes v \otimes v) \quad \Sigma\Big(\varphi_X(d_j \otimes a) \otimes \tau^j \varphi_Y(d_{n-j} \otimes b)\Big)$$

$$= \Sigma \; (u \cup_j u)(a) \otimes (v \cup_{n-j} v)(b)$$

$$= \Sigma \; Sq^{p-j} u \, (a) \otimes Sq^{q-n-j} v \quad (a \otimes b)$$

$\square$

<u>Adem relation:</u> For $i < 2j$

$$Sq^i Sq^j = \Sigma_c \binom{j-c-1}{i-2c} Sq^{i+j-c} \; Sq^c$$