L31 – 5/1/13

Generalized Sullivan Conjecture

Generalized Sullivan conjecture

$G \curvearrowright X$ space

group

$$X^G \longrightarrow \mathrm{Map}(EG, X) \quad \text{inclusion via constant maps}$$

$$\mathrm{Map}(Z, X) \to \mathrm{Map}(Z, RX) \rightsquigarrow R\,\mathrm{Map}(Z,X) \to \mathrm{Map}(Z, RX)$$

$$\rightsquigarrow R_\infty \mathrm{Map}(Z,X) \to \mathrm{Map}(Z, R_\infty X)$$

$$R_\infty X^G \to R_\infty \mathrm{Map}(EG, X) \to \mathrm{Map}(EG, R_\infty X)$$

This is an equivariant map.

$\Rightarrow$ Have natural map

$$R_\infty(X^G) \to \mathrm{Map}(EG, R_\infty X)^G =: (R_\infty X)^{hG}$$

Dwyer–Miller–Neisendorfer

Thm (Carlsson, Lannes, Miller)

For G a finite p-group and X a finite dim'l G-cplx

$$(\mathbb{Z}/p)_\infty X^G \to ((\mathbb{Z}/p)_\infty X)^{hG}$$

is a weak equivalence.

Say after proof sketch

- Unstable Adams spectral sequence methods need additional hypothesis: X finite. Carlsson's method proves f.d.

Heuristic: fiber square

$$\begin{tikzcd} EG \times X \arrow[r] \arrow[d] & EG \times_G X = (EG \times X)/G \arrow[d] \\ EG \arrow[r] & BG \end{tikzcd}$$

quotient by G gives

$\Rightarrow \mathrm{Map}(EG, EG \times X)^G = \mathrm{Map}(BG, EG \times_G X)$

require map $EG \to EG \times X$

$$\begin{tikzcd} EG \arrow[r] \arrow[dr, "id"'] & EG \times X \arrow[d] \\ & EG \end{tikzcd}$$

Same as requiring map $BG \to EG \times_G X$ to be a section

$$\Rightarrow \mathrm{Map}_{EG}(EG, EG \times X)^G = \mathrm{Map}_{BG}(BG, EG \times_G X)$$

$$\underset{\mathrm{Map}(EG, X)^G = X^{hG}}{\|}$$

$$X^{hG} = \mathrm{Map}_{BG}(BG, EG \times_G X)$$

See: Lannes "Sur les espaces..." Ch 4

Schwartz "Unstable modules over..." Ch 9

Ex
p-completion is needed in statement

$G = \mathbb{Z}/2$ acts on S^2 by flipping northern & southern hemisphere

Carlsson Thm B(b) "Equivariant stable htpy & Sullivan's conjecture"

Constructs: $X^{hG} \to (\mathbb{Z}/2)_\infty X^G$ with connected fiber

$$\Rightarrow \quad \pi_1 X^{hG} \longrightarrow \pi_1 (\mathbb{Z}/2)_\infty S^1 \longrightarrow 0$$

$$\pi_1 (\mathbb{Z}/2)_\infty S^1 = \mathbb{Z}_2$$

$$\Rightarrow \quad X^G \overset{\not\simeq}{\longrightarrow} X^{hG}$$