ref: Miller "The Sullivan conjecture on maps from classifying spaces"

L31 — 5/1/13
Sullivan's conjecture
Trivial action

<u>Thm:</u> (Miller) Let G be a finite group and X a finite dim'l CW cplx. Then

$$\mathrm{Map}(BG, X) \underset{ev}{\simeq} X \qquad \text{equivalently} \quad \mathrm{Map}_*(BG,X) \simeq *$$

or inclusion via constant maps

Last time: Assume additionally that X is nilpotent and $G = \mathbb{Z}/p$, then theorem holds.

we almost proved: Let X be nilpotent and G a finite group. If $\mathrm{map}_*(B\mathbb{Z}/p, X) \xrightarrow{\simeq} *$ for all $p \mid |G|$, then $\mathrm{map}_*(BG, X) \xrightarrow{\simeq} *$.

We showed: Lemma

<u>Lemma:</u> $H \underset{\text{subgroup}}{<} G$ finite group. X connected space.

Assume $\mathrm{Map}_*(BK, X) \xrightarrow{\simeq} *$ where K is any intersection of conjugates of H.

Then: $\mathrm{Map}_*(G\backslash E(G/H), X) \xrightarrow{\simeq} *$

$$\Downarrow$$

$$\mathrm{Map}_*(BG, X) \xrightarrow{\simeq} *$$

where: $\underline{E(G/H)}$ is category

ob: G/H

unique morphism $c_1 \to c_2$ $\forall c_1, c_2 \in G/H$

$E(G/H) \in \mathrm{SSet}$ $\quad E(G/H)[n] = \mathrm{Fun}([n], \underline{E(G/H)})$

$$= (G/H)^{n+1}$$

~~Pf~~: $BG \simeq$ quotient by G of contractible space w/ free G-action

e.g. $G \backslash EG$

$\simeq (E(G/H) \times EG)/G$

Let $\underset{\substack{\shortparallel \\ BG}}{W'} \simeq G \backslash (E(G/H) \times EG)$

Let $W = G \backslash E(G/H)$

The map $EG \to *$

induces $W' \to W$.

It suffices to show $\mathrm{Map}(W, X) \xrightarrow{\cong} \mathrm{Map}(W', X)$.

· Associated to W and W' are $\mathrm{Fun}(\Delta^{op}, sSet)$

$\underline{W}'$ $\quad [n] \longmapsto \left(E(G/H)[n] \times EG\right) \backslash G$

$$= \left((G/H)^{n+1} \times EG\right) \backslash G$$

$\underline{W}$ $\quad [n] \longmapsto G \backslash (G/H)^{n+1}$

· It is sufficient to show that

$(\star) \quad \mathrm{Map}(\underline{W}[n], X) \xrightarrow{\cong} \mathrm{Map}(\underline{W}'[n], X)$

reference: Prop 9.2

· To see $(\star)$, $(G/H)^{n+1} \underset{G\text{-set}}{\cong} \coprod G/K$

where K is intersection of $(n+1)$-conjugates of H.

Thus $\underline{W}'[n] \to \underline{W}[n]$ is

$$\coprod BK \longrightarrow \coprod *$$

Since $Map(BK, X) \xrightarrow{\simeq} Map(*, X)$, we have $Map(\underline{W}[n], X) \xrightarrow{\simeq} Map(\underline{W}'[n], X)$ $\square$

We've shown for X nilpotent.

<u>Pf of Miller's thm</u>: It suffices to show

$$\pi_n Maps_*(BG, X) = 0$$

choose elt of π_n represented by $f: S^n \wedge BG \to X$

For $n > 1$, $S^n \wedge BG$ is simply connected

$\Rightarrow$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\text{universal covering space}"] \\ S^n \wedge BG \arrow[r, "f"'] \arrow[ur, "\tilde{f}"] & X \end{tikzcd}$$

Since $\tilde{X}$ is nilpotent and finite dim'l, we have $\tilde{f} = 0$ in $\pi_n \mathrm{Map}_*(BG, \tilde{X})$

$\Rightarrow f = 0$.

Same works for $n=1$ once we show $\pi_1(BG \to X)$ is the trivial map.

<u>Thm 10.1</u>: Let X be finite dimensional CW complex. G torsion group. Then any

map $BG \to X$ is the trivial map on π_1.

Pf of this lemma motivates generalized cohomology thys

Pf: $BG \xrightarrow{f} X$

S'pose to contrary that $\exists\, \sigma \in G = \pi_1 BG$

s.t. $\pi_1 f(\sigma) \neq 1$ in $\pi_1 X$

Consider subgroup of $\pi_1 X$ generated by $\pi_1 f(\sigma)$

$$\langle \pi_1 f(\sigma) \rangle < \pi_1 X$$

This subgroup corresponds to a covering space Y

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & B\langle \pi_1 f(\sigma) \rangle \arrow[d] \\
X \arrow[r] & B\pi_1 X
\end{tikzcd}$$

$$\begin{tikzcd} & & Y \arrow[d] \\ B\langle\sigma\rangle \arrow[r] \arrow[urr, dashed, "g"] & BG \arrow[r, "f"] & X \end{tikzcd}$$

There exists g b/c $\operatorname{Im}(\pi_1(B\langle\sigma\rangle \to X)) \subset \operatorname{Im}(\pi_1 Y \to \pi_1 X)$

Thus $(*)$

$$\begin{tikzcd} B\langle\sigma\rangle \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & Y \arrow[r] & B\langle \pi_1 f \sigma\rangle \arrow[d, phantom, "\cong" rotate=90] \\ B(\mathbb{Z}/nm) & \text{finite dimensional} \arrow[u] & B(\mathbb{Z}/n) \end{tikzcd}$$

inducing quotient $\mathbb{Z}/nm \longrightarrow \mathbb{Z}/n$ on π_1

We show this is impossible:

try 1: Apply $H^*(-,R)$ to $B\mathbb{Z}/nm \to B\mathbb{Z}/n$

$$H^*(B\mathbb{Z}/n; R) \cong \begin{cases} \operatorname{coker}[n] & * \text{ odd} \\ \ker[n] & * \text{ even } > 0 \end{cases}$$

$[n]: R \to R$ mult by n

So for $m=1$ $B(\mathbb{Z}/nm) \longrightarrow B(\mathbb{Z}/n)$

induces non-zero map after applying $H^*(; \mathbb{Z}/n)$ for arbitrarily large $*$

Since $H^*(Y;R)=0$ for $* >> 0$, this is a contradiction.

Unfortunately, for $n=m=p$, this doesn't work
$H^2(B\mathbb{Z}/nm \to B\mathbb{Z}/n; R)$ takes periodicity elt in deg 2 to $m \cdot$ periodicity elt.

<u>Fix</u>: Generalized cohomology thys!

$$K_* = K\text{-thy}$$

Choose CW structure on $B\mathbb{Z}/n$ s.t.

$$H^i(B_n^{(2k)};\mathbb{Z}) = \begin{cases} \mathbb{Z}/n & i \text{ even} \quad 0 < i \leq 2k \\ \mathbb{Z} & i=0 \\ 0 & \text{otherwise} \end{cases}$$

Atiyah-Hirzebruch spectral sequence $\rightsquigarrow$

$$\Rightarrow K^0 B\mathbb{Z}/n^{(2k)} = \mathbb{Z}[x_n] / ((x_n+1)^n - 1, x_n^{k+1})$$

$$K^1 B\mathbb{Z}/n^{(2k)} = 0 \qquad x_n = \xi_n - 1 \quad \xi_n$$

line bundle on $B\mathbb{Z}/n$ corresponding to $\mathbb{Z}/n \subset \mathbb{C}^*$

$$K^0 B\mathbb{Z}/n = \mathbb{Z}[x_n] / ((x_n+1)^n - 1)$$

$$B\mathbb{Z}/nm \longrightarrow B\mathbb{Z}/n$$

$$\xi_{mn}^m \longleftarrow\!\shortmid \xi_n$$

Thus $K^0(B\mathbb{Z}/nm \to B\mathbb{Z}/n)$ is

$$(x_{nm}+1)^m - 1 \longleftarrow\!\shortmid x_n$$

This map can not factor through finite dimensional space. $\square$

This proves Miller's thm $\boxed{\smile}$

Non-trivial G-action