Thm (Miller) Let $G$ be a finite group and $X$ a finite dimensional CW complex. Then

$$\text{Map} \left( BG, X \right) \cong X$$

evaluation

or

Inclusion via constant maps

- We can view $X$ as sSet

**Lemma:** For $G = \mathbb{Z}/2$, $\text{Map} \left( B\mathbb{Z}/2, (\mathbb{Z}/2)_\infty X \right) \cong (\mathbb{Z}/2)_\infty X$

To prove this recall: $R$ a ring, $X$ sSet, $RX \in \text{sSet}$

$$RX \in \text{Fun} \left( \Delta, \text{sSet} \right)$$

$$RX \cong \lim_{\text{nat}} \text{Map} \left( \Delta^n, RX \right)$$

$$R_{\infty} X = \lim_{\text{nat}} \text{Tot } RX$$

$$\subset \lim_{\text{nat}} \text{Tot } \text{Map} \left( \Delta^n, RX \right)$$

$$\text{Map} \left( \mathbb{Z}, R_{\infty} X \right) = \lim_{\text{nat}} \text{Map} \left( \mathbb{Z}, RX \right)$$

Associated spectral sequence $E_{s,t}^2 = \lim_{\text{nat}} \text{Tot } \text{Map}(\mathbb{Z}, RX)$

$$= \text{Ext}^t_{\mathbb{K}} \left( H^s(X), \Sigma^t H^\ast (\mathbb{Z}) \right)$$

$R = \mathbb{Z}/p$
\[ \sum^+ H^*(Z)^+ = \begin{cases} H^{*-+}(Z) & \ast \neq 0 \\ R & \ast = 0 \end{cases} \]

\[ K = \text{unstable alg over } A^* \]

\( b/c \quad R \rightarrow RX \text{ is a product of Eilenberg-MacLane spaces, whence } \text{Map}(Z', R \rightarrow RX) = \text{Hom}_K(H^*(R \rightarrow RX), H^*(Z')) \]

\[ H^*(RY) = g(X) \theta(\chi) H^*Y \]

when \( H^nY \) finite dimensional \( \forall n \)

Thus \( H^*(R \rightarrow RX) \rightarrow H^*(X) \) is a resolution. For proof we are about to give.

Caveat: Need \( H^nX \) finite dimensional \( \forall \). This is a result of using cohomology instead of homology. If we had developed same tools in categories of comodules/coalgebras, this hypothesis is not needed.

\[ \text{Proof: } R = \mathbb{Z}/2, G = \mathbb{Z}/2 \]

\[ R_{\infty}X \rightarrow \text{Map}(BG, R_{\infty}X)_{ev} \rightarrow R_{\infty}X \]

Gives morphisms of spectral sequences. On \( E^2 \) page:
algebra, i.e. $M \in K$, then $T M \in K$

and $\text{Hom}_K(T M, N') = \text{Hom}_K(M, N' \otimes H^* B G)$

$(T (M \otimes M_2) \cong T M \otimes TM_2)$

$(ii)$ $T$ is exact

$(iii)$ $\text{Ext}_K^s (M, N' \otimes H^* B G) = \text{Ext}_K^s (T M, N')$

$(i) \& (ii) \implies (iii)$: $\text{Ext}_K^s (T M, N')$ can be computed with the resolution

$T (\tilde{g}_0 \cdot M) \to TM$
Thus we have

\[ \text{Ext}^S_\mathbb{K} (M^*, M^* S^+) \rightarrow \text{Ext}^S_\mathbb{K} (M^*, \Sigma^+ M^* B G) \rightarrow \text{Ext}^S_\mathbb{K} (M^*, M^* H^+ \mathbb{K}) \]

\[ \sim \]

\[ \text{Ext}^S_\mathbb{K} (M^* X, M^* S^+ \otimes M^* B G) \]

\[ \sim \]

\[ \text{Ext}^S_\mathbb{K} (T M^* X, M^* S^+) \]

Last time: For \( X \) finite dimensional, there is a natural isomorphism

\[ TM^* X \rightarrow M^* X \]

This gives isomorphism \( \sim \). Thus \( i \) surjective

\( i \) injective b/c \( e_1 = \text{id} \).

Thus \( i: E^2_{S^+} (R^\infty X) \rightarrow E^2_{S^+} (\text{Map}(B G, R^\infty X)) \)
Apply Bousfield mapping lemma.

\[ e : E_{5,+}^2(\operatorname{Map}(BG, R^\infty X)) \xrightarrow{\sim} E_{5,+}^2(R^\infty X) \]

\[ R^\infty X \xrightarrow{\sim} \operatorname{Map}(BG, R^\infty X) \xrightarrow{e} R^\infty X \]

Caveat: really need \( \pi^S_\infty \pi^S_\infty = 0 \) too.

We'll omit.

Reference: Lannes appendix A, Cor A.1.2

\[ \operatorname{Map}_\ast(BG, R^\infty X) \xrightarrow{\sim} \ast \quad G = \mathbb{Z}/2, \quad R = \mathbb{Z}/2 \]

In fact, this result holds for \( G = \mathbb{Z}/p \), \( R = \mathbb{Z}/p \).

Recall that we showed:

Thm (Miller 1.5 - thanks to Bousfield)

\[ \operatorname{Map}_\ast(W, X) \xrightarrow{\sim} \operatorname{Map}_\ast(W/(\mathbb{Z}/p)\infty X) \]
For $X$ nilpotent & $W$ s.t. $H(W;\mathbb{Z}[rac{1}{p}])=0$.

**Cor:** For $X$ nilpotent and finite dim.'l

$$\text{Map}_*(B\mathbb{Z}/p, X) \xrightarrow{\sim} *$$

Thanks to M.J. Hopkins.

**Thm (Miller)** Let $G$ be a finite group

Then $X$ be a nilpotent space such that $\text{Map}_*(B\mathbb{Z}/p, X) \xrightarrow{\sim} *$

for all primes dividing $|G|$. Then

$$\text{map}_*(BG, X) \xrightarrow{\sim} *$$

**Pf** by induction on $|G|$. For $H < G$, let $A$

**Lemma** Let $H$ be a subgroup of $G$ s.t.

$$\text{map}_*(BK, X) \xrightarrow{\sim} *$$

for $K$ the intersection of any set of conjugates of $H$.

Then $\text{map}_*(G\setminus(\mathbb{Z}/p)^n, X) \xrightarrow{\sim} *$.
\[
\text{map}_{\ast} (\mathbb{G} \gamma X) \Rightarrow \ast
\]

Insert on previous page:

\[E(G/H)\]
be the category s.t.

\[
\text{ob} = G/H
\]

There is a unique morphism from any object to any other.

\[i.e. \ E(G/H) = E G / H\]

\[E(G/H) \in \text{ssSet}\]

\[E(G/H)[n] = \text{n composable morphisms in } E(G/H)\]

\[= \text{Fun}(C^n, E(G/H))\]
Lemma \implies \text{Thm}:

First suppose \(G\) is a \(p\)-group. Induct on order of \(G\).

Center of \(G\) is non-trivial. Choose elt of order \(p\) in center. Let \(H \cong \mathbb{Z}/p\) be subgroup generated by this element.

\[ g H g^{-1} = H \quad \forall g. \quad \implies K = H \]

Thus \(\text{Map}_* (B K, X) = \text{Map}_* (B \mathbb{Z}/p, X) \cong * \)

So we may apply Lemma.

Since \(H\) is normal, \( G \setminus E(G/H) = B(G/H) \)

and \(G\) acts through \(G/H\).
By induction, $\text{Map}_* (B G/H, X) \sim \ast$.

$\Rightarrow \quad \text{Map}_* (B G, X) \sim \ast$.

Lemma

Now drop the assumption that $G$ is a $p$-group.

Dror, Dwyer, Kan

A nilpotent $\Rightarrow$ have arithmetic square

$$
\begin{array}{ccc}
X & \longrightarrow & \Pi (\mathbb{Z}/p)_\infty X \\
\downarrow & & \downarrow \\
\mathbb{Q}_\infty X & \longrightarrow & \mathbb{Q}_\infty (\Pi (\mathbb{Z}/p)_\infty X)
\end{array}
$$

Since $H^* (B G, \mathbb{Q}) = 0$, $\text{Map}_* (B G, \mathbb{Q}_\infty ?) \sim \ast$

$\Rightarrow$ it suffices to show
\[ \map_* (B_\ell, (\mathbb{Z}/p)_\infty X) \xrightarrow{\sim} \ast \] for all \( p \).

Choose a \( p \), and let \( H \) be a \( p \)-Sylow.

Since \( K \) runs over \( p \)-groups, Lemma applies.

\[ \tilde{H}_\ast (G \setminus E(G/H) ; \mathbb{Z}/p) = 0 \] by averaging over cosets in \( G/H \). More specifically, an element of \( G \setminus E(G/H) \) is the \( G \)-orbit of

\[ [c_0, \ldots, c_n] \in (G \setminus H)^{n+1} \]

Define \( h \) by \( b/w \) \( id \) and \( 0 \) on

\[ A (\mathbb{Z}/p( G \setminus E(G/H))) \] by associated chain cplx.
\[ h(c_0, \ldots, c_n) = \frac{1}{[G:H]} \sum_{c \in G/H} [c, c_0, \ldots, c_n] \]

Thus \[ \text{map}_* (G \setminus E(G/H), X) \overset{\sim}{\to} * \]

Lemma
\[ \Rightarrow \] \[ \text{map}_* (BG, X) \overset{\sim}{\to} * \]

So Lemma \[ \Rightarrow \text{Thm} \]

Pf of lemma:
\[ BG \simeq \text{quotient by } G \text{ of} \]

Contradictible space w/ Free G-action

\[ \text{e.g. } EG/G \]

Let \[ W'_{BG} = \frac{E(G/H) \times EG}{G} \]
Let \( W = G \setminus E(G/H) \).

\( EG \to \ast \) induces \( W' \to W \).

Lemma is implied by claim:

\[
\text{Map}_\ast (W, X) \cong \text{Map}_\ast (W', X)
\]

To prove claim, it suffices to show

\[
\text{Map} \left( W^{\mathcal{CNJ}}, X \right) \cong \text{Map} \left( (G/H)^{n+1} \times EG, X \right)
\]

\( W^{\mathcal{CNJ}} \text{ viewed as } \text{Fun}(G^{op}, sSet) \)

Reference Prop 9.2 Miller

\[
(G/H)^{n+1} \cong \coprod_{G\text{-set}} G/K
\]

\( K \) is intersection of \( (n+1) \) conjugates of \( H \)

\[
\Rightarrow W^{\mathcal{CNJ}} \to W^{\mathcal{CNJ}}
\]

\[
\Rightarrow \text{BK} \to \ast
\]
Pf of Miller's Thm

It suffices to show

\[ \pi_n \text{ Maps}_* (BG, X) = 0. \]

Choose elt \( \pi_n \) represented by \( F : S^n \wedge BG \to X \)

For \( n > 1 \), \( S^n \wedge BG \) is simply connected,

\[ \xymatrix{ \bar{F} \ar[r] & \widetilde{X} \ar@{>->}[d] \quad \text{universal covering space} \\ S^n \wedge BG \ar[r]^f & X } \]

By previous thm, \( \bar{f} = 0 \) in \( \pi_n \text{ Maps}_* (BG, \tilde{X}) \)

\[ \Rightarrow f = 0. \]

In fact, same works for \( n = 1 \). Need to show \( BG \to X \) induces trivial map on \( \pi_1 \).

Reference: Miller Thm 10.1