reference: H. Miller "The Sullivan conjecture on maps from classifying spaces"

L30 - 4/29

Miller's theorem

__Thm__ (Miller) Let G be a finite group and X a finite dimensional CW complex. Then

$$\mathrm{Map}(BG, X) \underset{\substack{\text{evaluation} \\ \text{or} \\ \text{inclusion via} \\ \text{constant maps}}}{\simeq} X$$

- We can view X as sSet

__Lemma__: For $G = \mathbb{Z}/2$, $\mathrm{Map}(B\mathbb{Z}/2, (\mathbb{Z}/2)_\infty X) \underset{\substack{\text{evaluation} \\ \text{or} \\ \text{inclusion}}}{\simeq} (\mathbb{Z}/2)_\infty X$ $\quad \substack{ev \\ i}$

To prove this recall: R a ring, X sSet, $RX \in \mathrm{sSet}$

$$\underset{\sim}{RX} \in \mathrm{Fun}(\Delta, \mathrm{sSet}) \qquad \underset{\sim}{RX}[n] = \overbrace{R \cdots R}^{n+1 \text{ times}} X$$

$$R_\infty X = \mathrm{Tot}\, \underset{\sim}{RX}.$$
$$\subset \prod_n \mathrm{Map}(\Delta^n, \overbrace{RR \cdots R}^{n+1} X)$$

$$\mathrm{Map}(Z, R_\infty X) = \mathrm{Tot}\, \mathrm{Map}(Z, \underset{\sim}{RX}.) = \varprojlim \mathrm{Tot}^m \mathrm{Map}(Z, \underset{\sim}{RX}.)$$ tower of fibrations

based at chosen constant map

Associated spectral sequence $E^2_{-s,t} = \pi^s \pi_t \mathrm{Map}(Z, \underset{\sim}{RX}.)$

$$= \mathrm{Ext}^s_K(H^*(X), \Sigma^t H^*(Z)) \qquad R = \mathbb{Z}/p$$

$$\Sigma^t H^*(Z)^+ = \begin{cases} H^{*-t}(Z) & * \neq 0 \\ R & * = 0 \end{cases}$$

$\mathcal{K}$ = unstable alg over $\mathcal{A}^*$

(b/c $R \cdots R X$ is a product of Eilenberg-MacLane spaces, whence $\mathrm{Map}(Z', R\cdots RX] = \mathrm{Hom}_{\mathcal{K}}(H^*(R\cdots RX), H^*(Z'))$

$$H^*(RY) = g\binom{Y}{K}\,\theta\binom{K}{V}\,H^*Y$$

when H^nY finite dimensional $\forall n$

Thus $H^*(R\cdots RX) \to H^*(X)$ is a resolution)

Caveat: Need H^nX finite dimensional for proof we are about to give. This is a result of using cohomology instead of homology. If we had developed same tools in categories of comodules/coalgebras, this hypothesis is not needed.

Pf: $R = \mathbb{Z}/2$ $G = \mathbb{Z}/2$

$$R_\infty X \to \mathrm{Map}(BG, R_\infty X) \xrightarrow[ev]{} R_\infty X$$

Gives morphisms of spectral sequences. On E^2 page:

$$\begin{tikzcd} \mathrm{Ext}^s_K(H^*X, \Sigma^t \overset{\substack{H^*S^t \\ \|}}{H^*(pt)^+}) \arrow[r, "i"] \arrow[dr, equal] & \mathrm{Ext}^s_K(H^*X, \Sigma^t H^*BG^+) \arrow[d, "ev"] \\ & \mathrm{Ext}^s_K(H^*X, H^*S^t) \end{tikzcd}$$

$$\Sigma^t H^*BG^+ \longrightarrow H^*(S^t) \otimes H^*BG$$

Induces

$$\mathrm{Ext}^s_K(H^*X, \Sigma^t H^*BG^+) \to \mathrm{Ext}^s_K(H^*X, H^*(S^t) \otimes H^*BG)$$

<u>Lanne's T-functor</u>: $T : \mathcal{U} \longrightarrow \mathcal{U}$

unstable modules over A^*

$$\mathrm{Hom}_{\mathcal{U}}(TM, N) = \mathrm{Hom}_{\mathcal{U}}(M, N \otimes H^*BG)$$

Needed properties of T:

(i) If M has structure of unstable

algebra, i.e. $M \in \mathcal{K}$, then $TM \in \mathcal{K}$

and $\mathrm{Hom}_{\mathcal{K}}(TM, N') = \mathrm{Hom}_{\mathcal{K}}(M, N' \otimes H^*BG)$

$(T(M_1 \otimes M_2) \cong TM_1 \otimes TM_2)$

(ii) T is exact

(iii) $\mathrm{Ext}^s_{\mathcal{K}}(M, N' \otimes H^*BG) =$

$\mathrm{Ext}^s_{\mathcal{K}}(TM, N')$

(i) & (ii) $\Rightarrow$ (iii): $\mathrm{Ext}^s_{\mathcal{K}}(TM, N')$ can

+ T free = free

be computed with the resolution

$$T(\tilde{g}_0 . M) \to TM$$

Thus we have

$$\begin{tikzcd}
\mathrm{Ext}^s_{\mathcal{K}}(H^*X, H^*S^t) \arrow[r, "i"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "\cong"'] & \mathrm{Ext}^s_{\mathcal{K}}(H^*X, \Sigma^t H^*BG^+) \arrow[r, "e"] \arrow[d] & \mathrm{Ext}^s_{\mathcal{K}}(H^*X, H^*S^t) \\
& \mathrm{Ext}^s_{\mathcal{K}}(H^*X, H^*S^t \otimes H^*BG) \arrow[d, equal] & \\
& \mathrm{Ext}^s_{\mathcal{K}}(TH^*X, H^*S^t) &
\end{tikzcd}$$

Last time: For X finite dimensional, there is a natural isomorphism

$$TH^*X \xrightarrow[\cong]{} H^*X$$

(adjoint to $H^*X \longrightarrow H^*X \otimes H^*BG$
$x \longmapsto x \otimes 1$)

This gives isomorphism $\xleftarrow{\cong}$. Thus i surjective.
i injective b/c $ei = \mathrm{id}$.

Thus $i: E^2_{s,t}(R_\infty X) \xrightarrow{\cong} E^2_{s,t}(\mathrm{Map}(BG, R_\infty X))$

$$\Rightarrow \quad e : E^2_{s,t}(\mathrm{Map}(BG, R_\infty X)) \xrightarrow{\cong} E^2_{s,t}(R_\infty X)$$

Apply Bousfield mapping lemma.

$$\Rightarrow \quad R_\infty X \xrightarrow[i]{\cong} \mathrm{Map}(BG, R_\infty X) \xrightarrow[e]{\cong} R_\infty X$$

Caveat: really need $\pi^s \pi_s = 0$ too

We'll omit

reference: Lannes appendix A

Cor A.1.2

pointed maps

$$\Rightarrow \mathrm{Map}_*(BG, R_\infty X) \xrightarrow{\simeq} * \quad G = \mathbb{Z}/2 \quad R = \mathbb{Z}/2$$ $\square$

In fact, this result holds for $G = \mathbb{Z}/p$ $R = \mathbb{Z}/p$.

Recall that we showed:

<u>Thm</u> (Miller 1.5 - thanks to Bousfield)

$$\mathrm{Map}_*(W, X) \xrightarrow{\cong} \mathrm{Map}_*(W, (\mathbb{Z}/p)_\infty X)$$

For X nilpotent & W s.t. $\tilde{H}(W; \mathbb{Z}[\frac{1}{p}]) = 0$.

Cor: For X nilpotent and finite dim'l

$$\mathrm{Map}_*(B\mathbb{Z}/p, X) \xrightarrow{\simeq} *$$

Thanks to M.J. Hopkins

Thm (Miller) Let G be a finite group. Let X be a nilpotent space such that $\mathrm{Map}_*(B\mathbb{Z}/p, X) \xrightarrow{\simeq} *$ for all primes dividing $|G|$. Then

$$\mathrm{map}_*(BG, X) \xrightarrow{\simeq} *$$

Pf by induction on $|G|$. For $H < G$ subgroup, let (A) insert from below

Lemma Let H be a subgroup of G s.t. $\mathrm{map}_*(BK, X) \xrightarrow{\simeq} *$ for K the intersection of any set of conjugates of H. Then $\mathrm{map}_*(G \backslash E(G/H), X) \xrightarrow{\simeq} * \Rightarrow$

$$\mathrm{map}_*(BG, X) \xrightarrow{\simeq} *$$

Insert on previous page:

$\underline{E(G/H)}$ be the category s.t.

$$ob = G/H$$

There is a unique morphism from any object to any other.

i.e. $\underline{E(G/H)} = \underline{EG}/H$

$$E(G/H) \in sSet$$

$E(G/H)[n] = n$ composable morphisms in $\underline{E(G/H)}$

$$= \mathrm{Fun}([n], \underline{E(G/H)})$$

Lemma $\Rightarrow$ thm:

First suppose G is a p-group. Induct on order of G.

Center of G is non-trivial
Choose elt of order p in center.
Let $H \cong \mathbb{Z}/p$ be subgroup generated by this element.

$$gHg^{-1} = H \ \forall g. \Rightarrow K = H$$

Thus $\mathrm{Map}_*(BK, X) = \mathrm{Map}_*(B\mathbb{Z}/p, X) \xrightarrow{\simeq} *$

So we may apply Lemma.

Since H is normal, and G acts through G/H

$$G \backslash E(G/H) = B(G/H)$$

By induction, $\mathrm{Map}_*(B(G/H), X) \xrightarrow{\simeq} *$.

$\underset{\text{Lemma}}{\Rightarrow}$ $\mathrm{Map}_*(BG, X) \xrightarrow{\simeq} *$.

Now drop the assumption that G is a p-group.

Dror, Dwyer, Kan

X nilpotent $\Rightarrow$ have arithmetic square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \prod (\mathbb{Z}/p)_\infty X \arrow[d] \\
\mathbb{Q}_\infty X \arrow[r] & \mathbb{Q}_\infty\left(\prod (\mathbb{Z}/p)_\infty X\right)
\end{tikzcd}$$

Since $H^*(BG, \mathbb{Q}) = 0$, $\mathrm{Map}_*(BG, \mathbb{Q}_\infty ?) \xrightarrow{\simeq} *$

$\Rightarrow$ it suffices to show

$$\mathrm{Map}_*(BG, (\mathbb{Z}/p)_\infty X) \xrightarrow{\simeq} * \quad \text{for all } p.$$

Choose a p, and let M be a p-Sylow

Since K runs over p-groups, Lemma applies

- $\tilde{H}_*(G \backslash E(G/M); \mathbb{Z}/p) = 0$ by

averaging over cosets in G/M. More specifically, an element of $G \backslash E(G/M)[n]$

is the G-orbit of

$$[c_0, \ldots, c_n] \in (G \backslash M)^{n+1}$$

Define htpy b/w id and 0 on

$$A(\mathbb{Z}/p(G \backslash E(G/M)))$$ by

associated chain cplx

$$h[c_0, \ldots, c_n] = \frac{1}{[G:H]} \sum_{c \in G/H} [c, c_0, \ldots, c_n]$$

Thus $\mathrm{map}_*(G \backslash E(G/H), X) \xrightarrow{\sim} *$

$\overset{\text{Lemma}}{\Longrightarrow}$ $\mathrm{map}_*(BG, X) \xrightarrow{\sim} *$. $\square$

So Lemma $\Rightarrow$ Thm

<u>Pf of lemma</u>:

$BG \simeq$ quotient by G of contractible space w/ free G-action $\cong \dfrac{E(G/H) \times EG}{G}$

e.g. EG/G

Let $W' = \dfrac{E(G/H) \times EG}{G}$

$\quad \simeq BG$

$$W = G \backslash \underline{E(G/M)}$$

$EG \to *$ induces $W' \to W$

Lemma is implied by Claim:

$$\mathrm{Map}_*(W, X) \underset{\simeq}{\longrightarrow} \mathrm{Map}_*(W', X)$$

To prove claim, it suffices to show

$$\mathrm{Map}(W[n], X) \xrightarrow{\simeq} \mathrm{Map}(\underbrace{(G/M)^{n+1} \times EG}_{= W'[n]}, X)$$

viewed as $\mathrm{Fun}(\Delta^{op}, sSet)$

reference Prop 9.2 Miller

$$(G/M)^{n+1} \underset{G\text{-set}}{\cong} \coprod G/K$$

K is intersection of $(n+1)$ conjugates of M

$$\Rightarrow \quad \begin{array}{ccc} W'[n] & \longrightarrow & W[n] \\ \| & & \| \\ \coprod BK & & \coprod * \end{array}$$

$\square$

Pf of Miller's thm

It suffices to show

$$\pi_n \mathrm{Maps}_*(BG, X) = 0.$$

Choose elt π_n represented by $f : S^n \wedge BG \to X$

For $n > 1$, $S^n \wedge BG$ is simply connected,

$$\Rightarrow \begin{tikzcd} & \tilde{X} \arrow[d, "\text{universal covering space}"] \\ S^n \wedge BG \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

By previous thm, $\tilde{f} = 0$ in $\pi_n \mathrm{Map}_*(BG, \tilde{X})$

$\Rightarrow f = 0.$

In fact, same works for $n=1$. Need to show $BG \to X$ induces trivial map on π_1.

reference: Miller thm 10.1