Nat'l trans $(H^m(-,\mathbb{Z}/2), H^n(-,\mathbb{Z}/2))$ determined by

To compute $H^m(K(\mathbb{Z}/2,n),\mathbb{Z}/2)$,

need spectral sequences. ← Not today

2/1/12

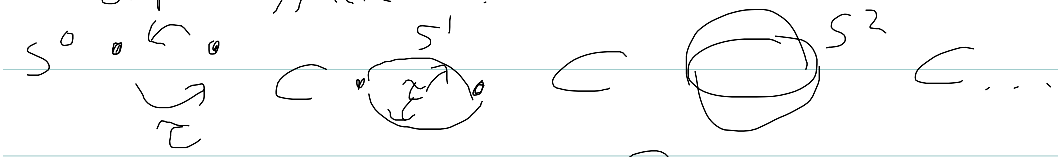
L3

Construction of Sq^i

For answer + computation, construct Sq^i ← today

- $E\mathbb{Z}/2$ denotes contractible space w/ $\mathbb{Z}/2$-action $\mathbb{Z}/2 = \langle \tau \rangle$

Explicitely, here's one:



$S^0 \subset S^1 \subset S^2 \subset \dots S^n$ ↖ $\tau(v) = -v$

τ acts by antipode

$\bigcup_n S^n = \operatorname{colim}_{\overrightarrow{n}} S^n =: S^\infty \circlearrowleft \mathbb{Z}/2$ ← inset CW ← insert corresponding chain cplx $\boxed{\text{w/ free cells } d_i}$

$E\mathbb{Z}/2 = S^\infty \quad B\mathbb{Z}/2 = (E\mathbb{Z}/2)/\mathbb{Z}/2 = \mathbb{R}P^\infty$ ← inset CW

- Space X, have $X \xrightarrow{\Delta} X \times X$

$\mathbb{Z}/2$ acts trivially on X; $\mathbb{Z}/2$ acts by switch on $X \times X$

Plan: "Free up" action using $E\mathbb{Z}/2$

Then get one cohomology op Sq_i for each cell in $\mathbb{R}P^\infty$

For X ~~space~~ simplicial complex, have $C_*(X)$ = simplicial chains

$$C_*(X \times Y) \underset{EZ}{\overset{AW}{\rightleftarrows}} C_*(X) \otimes C_*(X)$$

inverse chain equivalences. Here are formulas, but we don't need them

$$AW(a \times b) = \sum_{i=0}^{n} (\text{last } n-i \ a) \otimes (\text{first } i \ b)$$

n simplices

$$EZ(a \otimes b) = \sum_{\substack{(p,q) \text{ shuffles} \\ \sigma}} (-1)^{e(\sigma)} s_{\sigma(1)} \cdots s_{\sigma(p)} a \times s_{\sigma(p+1)} \cdots s_{\sigma(p)} b$$

sign

$$s_i : C_n \longrightarrow C_{n+1}$$

$$\sigma : \Delta^n \to X \mapsto \Delta^{n+1} \xrightarrow[\text{map}]{} \Delta^n \xrightarrow{\sigma} X$$

map corresponding to

$$(0,1,\ldots,n+1) \longrightarrow (0,1,\ldots,n)$$

i has two pre-images

Construct $W \otimes C_*(X) \xrightarrow{\varphi} C_*(X) \otimes C_*(X)$

$\mathbb{Z}/2$ $\qquad$ $\mathbb{Z}/2$

$\tau(w \otimes k) = \tau w \otimes k$

$\tau(k_1 \otimes k_2) = (-1)^{dg\, k_1\, dg\, k_2}\, k_2 \otimes k_1$

s.t. • φ is $\mathbb{Z}/2$-equivariant

• $\varphi(\sigma) \subset EZ(C_*(\sigma \times \sigma))$

These are acyclic

($\exists!$ up to htpy, as you can check with the same method)

• Note: $W \otimes C_*(W)$ has a $\mathbb{Z}[\mathbb{Z}/2]$ basis $\{d_i \otimes \sigma\}$

$$\varphi\big|_{d_0 \otimes C_*(X)} = AW \circ \Delta_*$$

$\varphi\big|_{\tau d_0 \otimes C_*(X)}$ determined by equivariance

We claim we can extend this φ

S'pose we have extended over all of

$$(W \otimes C_*(X))^q$$

Take $d_i \times \sigma \in (W \otimes C_*(X))^{q+1}$

$$\delta(d_i \times \sigma) = \sum a_j (d_{i_j} \times \sigma_j) \qquad a_j \in \mathbb{Z}[\mathbb{Z}/2]$$

$\varphi(d_{i_j} \times \sigma_j)$ has been defined.

want: $\varphi(d_i \times \sigma) \in EZ\, C_*(\sigma \times \sigma)$

s.t. $\delta\varphi(d_i \times \sigma) = \sum a_j\, \varphi(d_{i_j} \times \sigma_j)$

Since $\varphi\big|_{(W \otimes C_*(X))^q}$ is a chain

map, and since $\delta(\delta(d_i \times \sigma)) = 0$,

$\varphi(\delta(d_i \times \sigma))$ is a cocycle.

Since $EZ\, C_*(\sigma \times \sigma)$ is acyclic, can

choose b s.t. $\delta b = \varphi(\delta(d_i \times \sigma))$

Let $\varphi(d_i \times \sigma) = b$.

This constructs φ.

- diag approx aren't $\mathbb{Z}/2$ equiv. $AW \otimes \Delta_*$

<u>Rmk:</u> formula shows explicitely

- $\varphi|_{d_1 \otimes C_*(W)}$ is htpy b/w two differnt diagonal approx original, and is image under τ.
- This choice of htpy isn't $\mathbb{Z}/2$ equiv.
- $\varphi|_{d_2 \otimes -}$ is htpy b/w htpy and its image under τ

<u>Def:</u> For each $i \geq 0$ define "cup-i product" $C^p(X) \otimes C^q(X) \to C^{p+q-i}(X)$

$$u \otimes v \mapsto u \cup_i v$$

$$(u \cup_i v)(\kappa) = (u \otimes v)\varphi(d_i \otimes \kappa)$$

We will show: mod 2, these give homomorphisms

$$Sq_i : H^p(X, \mathbb{Z}/2) \to H^{2p-i}(X, \mathbb{Z}/2)$$

$$Sq_i(u) = u \cup_i u$$

$$\boxed{Sq^i|_{H^p} = Sq_{p-i}}$$

Rmk: $C_*(X)$ could be replaced by cplx K giving algebraic Steenrod operations, (ref. P. May)

- useful in stable Adams spectral sequence

$$\overset{\text{dg } p}{u} \cup_i \overset{\text{dg } q}{v}$$

To see when $u \cup_i v$ is a cocycle, compute boundary:

usual cup product δ formula:

$$\delta(u \cup v) = \delta u \cup v + (-1)^p u \cup v$$

i.e. $(u \cup v)(\delta c) = (\delta u \cup v)(c) +$
$(-1)^p (u \cup v)(c)$

why?: $\overset{\text{deg } p+q+1}{\delta c}$ is a sum of $p+q+2$ simplices of dim $p+q$

$(0-p)$ vertices of first $p+1$ is the same as δ of $(0-p)$ vertices of c

$(p-p+q)$ vertices of first $p+1$ is $(p-p+q)$ vertices of c

This gives $(\delta u \cup v)(c)$

$$\delta(U \cup_i V)(\overset{\deg\, p+q+1-i}{c}) = (U \otimes V)\,\varphi(d_i \otimes \delta c)$$

From usual cup product, we know

$$(u \otimes v)\,\delta\varphi(d_i \otimes c) =$$

$$(\delta u \otimes v)\,\varphi(d_i \otimes c) + (-1)^p (u \otimes \delta v)\,\varphi(d_i \otimes c)$$

$$\delta\varphi(d_i \otimes c) = \varphi\,\delta(d_i \otimes c) = \varphi(\delta d_i \otimes c) +$$

$$(-1)^i \varphi(d_i \otimes \delta c)$$

$$\delta d_i = d_{i-1} + (-1)^i \tau d_{i-1}$$

$$\Rightarrow \varphi(d_i \otimes \delta c) = (-1)^i \delta(d_i \otimes c) +$$

$$(-1)^{i+1} \varphi(d_{i-1} \otimes c) +$$

$$(-1)^{\cancel{2}i+1} \varphi(\tau d_{i-1} \otimes c)$$

$$\Rightarrow$$

$$\tau\varphi(d_{i-1} \otimes c)$$

$$\delta(u \cup_i v) = (-1)^i \delta u \cup_i v + (-1)^{i+p} u \cup_i \delta v$$
$$+ (-1)^{i+1} u \cup_{i-1} v +$$
$$(-1)^{pq+1} v \cup_{i-1} u$$

- Note, mod 2, when $u = v$, this gives
$$\delta(u \cup_i u) = \delta u \cup_i u + u \cup_i \delta u$$

- $\Rightarrow$ Have operation
$$Sq_i : Z^p \to Z^{2p-i}$$
$$u \mapsto u \cup_i u$$
$\mathbb{Z}/2$ coefficients

In fact, homomorphism on cohomology

$$Sq_i : H^p \longrightarrow H^{2p-i}$$

which is a natural transformation

Define $\boxed{Sq^i : H^p \longrightarrow H^{p+i}}$

by $u \longmapsto u \cup_{p-i} u$

Show: compatible w/ suspension

$$A^* := \text{Steenrod alg} := \bigoplus_i \text{Nat'l}(H^*, H^{*+i}) \text{ trns compatible w/ suspension.}$$

Have: $Sq^i \in A^*$

Fact: Sq^i generate & we can write down