$R = \mathbb{Z}/2$

See last pg for announcements

L29 4/19/13

T_G for $\mathcal{K}$

$\mathcal{U}$ = cat unstable mod over $\mathcal{A}^*$ Steenrod alg

Def: $M \in \mathcal{U}$ is locally finite if $\forall\, m \in M$, $\mathcal{A}^* m$ is finite dim'l.

Ex: $H^*(X)$ is locally finite for X finite dim'l.

Non-ex: $H^*(\mathbb{R}P^\infty) = \mathbb{Z}/2[[x]]$ $\quad Sq^{2^q} Sq^{2^{q-1}} \cdots Sq^1 x = x^{2^{q+1}}$

$$\begin{array}{ccc} m & \longmapsto & m \otimes 1 \\ M & \longrightarrow & M \otimes H^* BG \end{array} \quad \text{induces} \quad (M ; H^*BG)_{\mathcal{U}} \longrightarrow M$$

Assume $H^n BG$ finite dimensional $\forall\, n$

$(M ; H^*BG)_{\mathcal{U}} = T_G M$

Prop: $M \in \mathcal{U}$ locally finite, G finite 2-group $\Rightarrow$ $T_G M \xrightarrow{\sim} M$ is iso

pf $G = \mathbb{Z}/2$: Equivalent to show $\mathrm{Hom}_{\mathcal{U}}(M, J(m))$ $\downarrow$ iso $\mathrm{Hom}_{\mathcal{U}}(T_G M, J(m))$.

$\Longleftrightarrow \quad \mathrm{Hom}_{\mathcal{U}}(M, J(m)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}}(M, H^*(BG) \otimes J(m))$

$\Longleftrightarrow \quad \mathrm{Hom}_{\mathcal{U}}(M, \tilde{H}^*(BG) \otimes J(m)) = 0$ because $H^*(BG) \cong \mathbb{F}_2 \oplus \tilde{H}^*(BG)$

Filter $J(m)$ by degree

$$\left(\tilde{H}^*(BG) \otimes J(m)^{\leq n}\right) \Big/ \left(\tilde{H}^*(BG) \otimes J(m)^{\leq n-1}\right)$$

$$\cong \bigoplus \Sigma^n \tilde{H}^*(BG)$$

For any $0 \neq x \in \bigoplus \Sigma^n \tilde{H}^*(BG)$, $A^* x$ is infinite.

$\Rightarrow$ For any $0 \neq x \in \tilde{H}^*(BG) \otimes J(m)$, $A^* x$ is infinite.

Thus $0 \neq m \in M$ can not map to x.

$$\Rightarrow \mathrm{Hom}_{\mathcal{U}}\left(M, \tilde{H}^*(BG) \otimes J(m)\right) = 0 \qquad \square$$

$\mathcal{K}$ = cat of unstable alg over A^* $\qquad \mathcal{K} \xrightarrow{\mathcal{O}(\mathcal{K},\mathcal{U})} \mathcal{U}$

<u>Prop 1</u>: For $H \in \mathcal{K}$, $T_G(\mathcal{O}(\mathcal{K},\mathcal{U})H)$ has canonical structure of algebra in $\mathcal{K}$ s.t.

$$\mathrm{Hom}_{\mathcal{K}}(T_G\, \mathcal{O}(\mathcal{K},\mathcal{U})\, H, K) = \mathrm{Hom}_{\mathcal{K}}(H, K \otimes BG)$$

This justifies the notation $T_G M = T_G \mathcal{O}(G,u) M$

<u>Prop 2</u>: $\mathrm{Ext}^s_{K}(H^*X, H^*BG \otimes H^*Y)$

$$\cong$$

$$\mathrm{Ext}^s_{K}(T_G H^*X, H^*Y)$$

Ext can be computed w/ a resolution:

$$\mathrm{Ext}^s_K(M,K) := \pi^s \operatorname{Hom}(\widetilde{(g\sigma)}M, K)$$

$$\begin{tikzcd} K \arrow[r, bend left, "\sigma"] & V \arrow[l, bend left, "g"] \end{tikzcd}$$

More generally, $\mathrm{Ext}^s_K(M,K) = \pi^s(\mathrm{Res}\,M, K)$

for $\mathrm{Res}\,M : \Delta^{op} \to K$

s.t. 1) $\exists\ \mathrm{Res}\,M[0] \xrightarrow{\varepsilon} M$ s.t. $d_1 \varepsilon = d_0 \varepsilon : \mathrm{Res}\,M^{(1)} \to M$

2) $\mathrm{Res}\, M[i] = g\mathcal{O}(L_i)$ for some $L_i \in K$

3) $N\,\mathrm{Res}\, M \in Ch_+(\mathbb{F}_2\text{-vector spaces})$

$$
\begin{array}{c}
N\,\mathrm{Res}\, M \\
\parallel \\
\to \mathrm{Res}\, M[i] \cap \bigcap_{j=0}^{i-1} \ker d_j \xrightarrow{(-1)^i d_i} \mathrm{Res}\, M[i-1] \cap \bigcap_{j=0}^{i-2} \ker d_j \\
\downarrow \\
\vdots \\
\downarrow \\
M[0]
\end{array}
$$

is s.t. $N\,\mathrm{Res}\, M \to M \to 0$ $\quad H_* = 0$

Pf of Prop 2 assuming Prop 1:

$$\mathrm{Ext}^s(H^*X, H^*BG \otimes H^*Y) \cong$$

$$\pi^s \operatorname{Hom}(\widetilde{g\sigma} H^*(X), H^*BG \otimes H^*Y) \cong$$

$$\pi^s \operatorname{Hom}(T_G \widetilde{g\sigma} H^*(X), H^*Y)$$

Prop 2 follows from claim that $T_G \widetilde{g\sigma} H^*X$ is a res of Ext:

$T_G \widetilde{g\sigma} H^*(X)[0] \longrightarrow T_G H^*(X)$ b/c $\widetilde{g\sigma} H^*(X) \downarrow H^*(X)$

Thus 1) satisfied

3) satisfied b/c T_G is exact

For 2) $\operatorname{Hom}_K(T_G\, g\sigma \circ \cdots \circ g\sigma\, H^*X, M)$

$$= \operatorname{Hom}_K(g\sigma \circ \cdots \circ g\sigma\, H^*X, H^*BG \otimes M)$$

$$= \mathrm{Hom}_{\mathcal{V}}\left(\theta \circ g \circ \theta \circ \ldots \circ g \circ H^*X,\ \underbrace{\underset{\left(H^*BG\right)^{**}}{\overset{}{H^*BG}} \otimes M}_{\mathrm{Hom}\left(H^*BG^*,\, M\right)}\right)$$

$$= \mathrm{Hom}_{\mathcal{V}}\left(\underbrace{\theta \circ g \circ \theta \circ \ldots \circ g \circ H^*X \otimes H^*BG^*}, \ M\right)$$

$$\Rightarrow T_G\, g\theta \circ \ldots \circ g\theta\, H^*X = g\left(\underbrace{\qquad\qquad}\right)$$

$\square$

Proof of Prop 1 Needs $T(M \otimes M) \cong T(M) \otimes T(M)$

ref: Lannes "Sur les espaces fonctionnels"

2.2 and 2.3

$$M \longrightarrow K \otimes (M; K)_{\mathcal{U}} \quad \text{adjoint to } \mathrm{id}_{(M;K)_{\mathcal{U}}}$$

Apply $\otimes^2$

$$M_1 \otimes M_2 \longrightarrow K \otimes K \otimes (M_1; K)_{\mathcal{U}} \otimes (M_2; K)_{\mathcal{U}}$$

By adjointness obtain

$$(M_1 \otimes M_2 : K \otimes K)_{\mathcal{U}} \longrightarrow (M_1; K)_{\mathcal{U}} \otimes (M_2; K)_{\mathcal{U}}$$

$K = H^*BG$, $\quad H^*BG \otimes H^*BG \to H^*BG$

$$(M \otimes M ; H^*G)_{\mathcal{U}} \longrightarrow (M \otimes M ; H^*G \otimes H^*G)_{\mathcal{U}}$$

defined

$$\mathrm{Hom}_{\mathcal{U}}(M \otimes M, \underbrace{H^*G \otimes X}_{\substack{\uparrow \\ X}}) \longleftarrow \mathrm{Hom}_{\mathcal{U}}(M \otimes M, \underbrace{H^*BG \otimes H^*BG \otimes X}_{\substack{\uparrow \\ X}})$$

Thus $\quad T_G(M_1 \otimes M_2) \xrightarrow{\mu_{M_1,M_2}} T_G(M_1) \otimes T_G(M_2)$

Special property of BG G 2-grp:

<u>Thm 2.2.1</u>: μ_{M_1,M_2} is isomorphism

<u>Pf</u>: $\displaystyle\bigoplus_{\alpha \in A} F(\Sigma^{n_\alpha}\mathbb{F}_2) \longrightarrow \bigoplus_{\beta \in B} F(\Sigma^{n_\beta}\mathbb{F}_2) \to M_1 \to 0$

$$\begin{tikzcd} \mathcal{U} \arrow[r, bend right, "\mathcal{O}(\mathcal{U},\mathcal{V}) \text{ forgetful}"'] \arrow[r, phantom, "\perp"] & \mathcal{V} \arrow[l, bend right, "F \text{ free}"'] \end{tikzcd}$$

Apply T_G, obtain

$$\bigoplus T_G F \longrightarrow \bigoplus T_G F \longrightarrow M_1 \longrightarrow 0$$

and $\bigoplus T_G(F \otimes M_2) \longrightarrow \bigoplus T_G(F \otimes M_2) \longrightarrow M_1 \otimes M_2 \longrightarrow 0$

($\otimes$ is right exact)

$\Longrightarrow$ We may assume $M_1 = F(\Sigma^n \mathbb{F}_2)$

Similarly " $M_2 = F(\Sigma^m \mathbb{F}_2)$

$\mu_{F(\Sigma^n \mathbb{F}_2), F(\Sigma^m \mathbb{F}_2)}$ is shown to be an iso by induction.

Tools for induction:

- $\Sigma : \mathcal{U} \longrightarrow \mathcal{U}$

<u>Lemma</u>: $T\Sigma = \Sigma T$ $\qquad T = T_{\mathbb{Z}/2}$

Pf omitted – see 2.2.3 – uses the fact that $\mathrm{Hom}_{\mathcal{U}}(\Sigma M, H^*\mathbb{R}P^\infty) = 0 \quad \forall M \in \mathcal{M}$

this property has name: $H^*\mathbb{R}P^\infty$ is <u>reduced</u>

- $\Omega : \mathcal{U} \longrightarrow \mathcal{U}$ defined $\Omega M = (M; \Sigma \mathbb{F}_2)_{\mathcal{U}}$

$$\Sigma\Omega M = \Sigma\,\Omega^\infty\Bigg(M \otimes \underbrace{\begin{matrix}\vdots\\ 0\\ \mathbb{Z}/2 - \deg 0\\ \mathbb{Z}/2 - \deg -1\\ 0\\ \vdots\end{matrix}}_{\deg n \text{ part } = \underbrace{M_n \oplus M_{n-1}}_{\text{quotient } Sq^i x \quad i > |x|}}\Bigg)$$

For $x \in \Omega M^n$ s.t. $x \in M^{n-1}$

$$Sq^n_{\Omega M} x = Sq^n_M x = 0$$

b/c $n > n-1$

Sq on $M \otimes DS^1$ by Cartan but S^1, whence DS^1, no non-trivial squares

In Fact, $\Sigma \Omega M \cong M \big/ \langle Sq^{|x|} x : x \in M \rangle$

Define ΦM by

$$\Phi M^n = \begin{cases} M^{n/2} & n \equiv 0 \ (2) \\ 0 & n \equiv 1 \ (2) \end{cases}$$

$$Sq^i(\Phi x) = \begin{cases} \Phi Sq^{i/2} x & i \equiv 0 \ (2) \\ 0 & \text{otherwise} \end{cases}$$

Fact:

$$\Phi M \longrightarrow M \longrightarrow \Sigma\Omega M \longrightarrow 0$$

is exact.

Pf shows: $\Omega \mathcal{U}_{F(\Sigma^n \mathbb{F}_2)\, F(\Sigma^m \mathbb{F}_2)}$ iso

and degree 0 part

and that this suffices (2.3.5.1)

omitted. $\square$

<u>Pf of Prop</u>: $T = T_{\mathbb{Z}/2}$

For $H \in \mathcal{K}$ have

mult $H \otimes H \xrightarrow{\text{mult}} H$

$$TH \otimes TH \cong T(H \otimes H) \xrightarrow[T(\text{mult})]{} T(H)$$

This defines structure of unstable algebra (ref: 2.3.1)

For any $K_1, K_2 \in \mathcal{K}$

$\mathrm{Hom}_{\mathcal{K}}(K_1, K_2)$ can be expressed as equalizer

$$\mathrm{Hom}_{\mathcal{K}}(K_1, K_2) \longrightarrow \mathrm{Hom}_{\mathcal{U}}(K_1, K_2) \underset{\mathrm{mult}_{K_2}{}^* \circ \, \mathrm{id}_{K_1}^* \otimes \mathrm{id}_{K_1}^*}{\overset{\mathrm{mult}_{K_1}^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{U}}(K_1 \otimes K_1, K_2)$$

$K_2 = H^* \mathbb{R}P^\infty \otimes K \qquad K_1 = H$

expresses $\mathrm{Hom}_{\mathcal{K}}(H, H^* \mathbb{R}P^\infty \otimes K)$ as equalizer

$$\mathrm{Hom}_{\mathcal{U}}(H, H^* \mathbb{R}P^\infty \otimes K) \rightrightarrows \mathrm{Hom}_{\mathcal{U}}(H \otimes H, H^* \mathbb{R}P^\infty \otimes K)$$

$$\begin{array}{ccc} \| & & \| \ \text{Thm} \\ \mathrm{Hom}_{\mathcal{U}}(TH, K) & & \mathrm{Hom}_{\mathcal{U}}(TH \otimes TH, K) \end{array}$$

$\rightrightarrows$ are the corresponding maps for TH

$$\Rightarrow \mathrm{Hom}_{\mathcal{K}}(H, H^*\mathbb{R}P^\infty \underset{\mathcal{K}}{\otimes}) \simeq \mathrm{Hom}_{\mathcal{K}}(TH, K)$$

$\square$

Announcement:

HW #9 posted

No class Friday 4/26

Mon 4/29 $\mathrm{Map}_*(BG, X) \simeq *$

Wed 5/1 Sullivan conjecture