

No class 4/22-4/26  
 I'll put up last HW by next weekend

L27 - 4/15/13

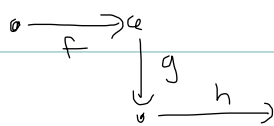
Derived Functors II

Brown-Gitler

Homotopical algebra

- $\mathcal{C}$  cat with  $\omega \subseteq \text{Mor } \mathcal{C}$  "weak equivalences"

iso's  $\in \omega$  and  $\forall$



"2-of-6"

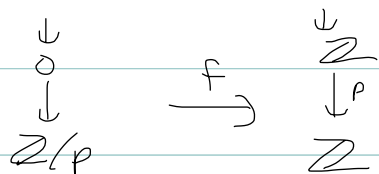
$$gh, gf \in \omega \Rightarrow f, g, h, hgf \in \omega$$

- $\text{Ho } \mathcal{C} = \mathcal{C}[\omega^{-1}]$  formally invert weak equivalences

- $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$  functor b/w categories w/ weak equivalences

- $\mathcal{F}(f)$  for  $f \in \omega_{\mathcal{C}}$  may not be in  $\omega_{\mathcal{D}}$

Ex 1:  $\mathcal{C} = \text{Ch}(ab)$   $\mathcal{D} = \text{Ch}(ab)$  with  $\omega = \{ \text{Mor-iso's} \}$

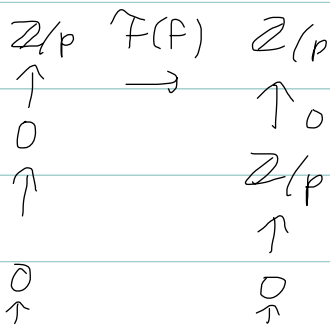


$$\mathcal{F}(A_\bullet) = \text{Hom}(A_\bullet, \mathbb{Z}/p) \in \text{Ch}_+$$

$$\text{Hom}(A_\bullet, \mathbb{Z}/p)_m = \text{Hom}(A_{\bullet+m}, \mathbb{Z}/p)$$

Apply  $\mathcal{F}$

actually let's do  $\text{Hom}(\mathbb{Z}/p, -)$



$\mathcal{F}(f) \notin \omega_{\mathcal{D}}$

Ex 2:  $I$  cat w/ one object,  $\text{mor} = G$

note:  $X$  sSet with  $G$ -action  $X = X(*)$   
 $\parallel$   $\uparrow$   
 Functor  $\mathcal{X}: I \rightarrow \text{sSet}$   $\mathcal{X}$

$$\lim_{\leftarrow I} \mathcal{X} = X^G$$

$$\mathcal{C} = \text{Fun}(I, \text{sSet}) \quad \omega = \{X \xrightarrow{f} Y : f \text{ w.e.}\}$$

$$\mathcal{D} = \text{sSet}$$

$$\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$$

$$X \mapsto X^G$$

$$\omega_{\mathcal{C}} \downarrow$$

$$\mathcal{F}(\mathcal{C}G \rightarrow *) \notin \omega_{\mathcal{D}}$$

• Approximate  $\mathcal{F}$  by a functor preserving weak equivalences

Ex 1:  $R\mathcal{F}(A_\bullet) = \text{Hom}(\mathbb{Z}/p, I)$

$A_n \rightarrow I_n$  quasi-iso  $I_n$  injective  $\forall n$

(i.e.  $\text{Hom}(-, I_n)$  is exact)

So  $\prod_i R\mathcal{F}(\mathbb{Z}/p) = \text{Ext}^i(\mathbb{Z}/p, M)$

Ex 2:  $R\mathcal{F}(X) = X \wedge G$

Ex 4:  $R\lim_{\leftarrow} = h_0 \lim_{\leftarrow} \quad \lim_{\leftarrow} : \text{Fun}(I, \text{SSet})$

Ex 5:  $\lim_{\leftarrow}$  as functor on towers of abelian  $\rightarrow \text{SSet}$

groups is  $\Pi'$  or  $M'$  of  $R \lim$

(replace  $N \rightarrow ab$  by

$N \rightarrow Ch_+(ab)$

with transition functions required

to be surjective

$$\{a_n\} \rightarrow \left\{ \begin{array}{c} \prod_{k \leq n} a_k \\ \uparrow 1-f \\ \prod_{k \in \mathbb{N}} a_k \end{array} \right\}$$

Definition of derived functor:

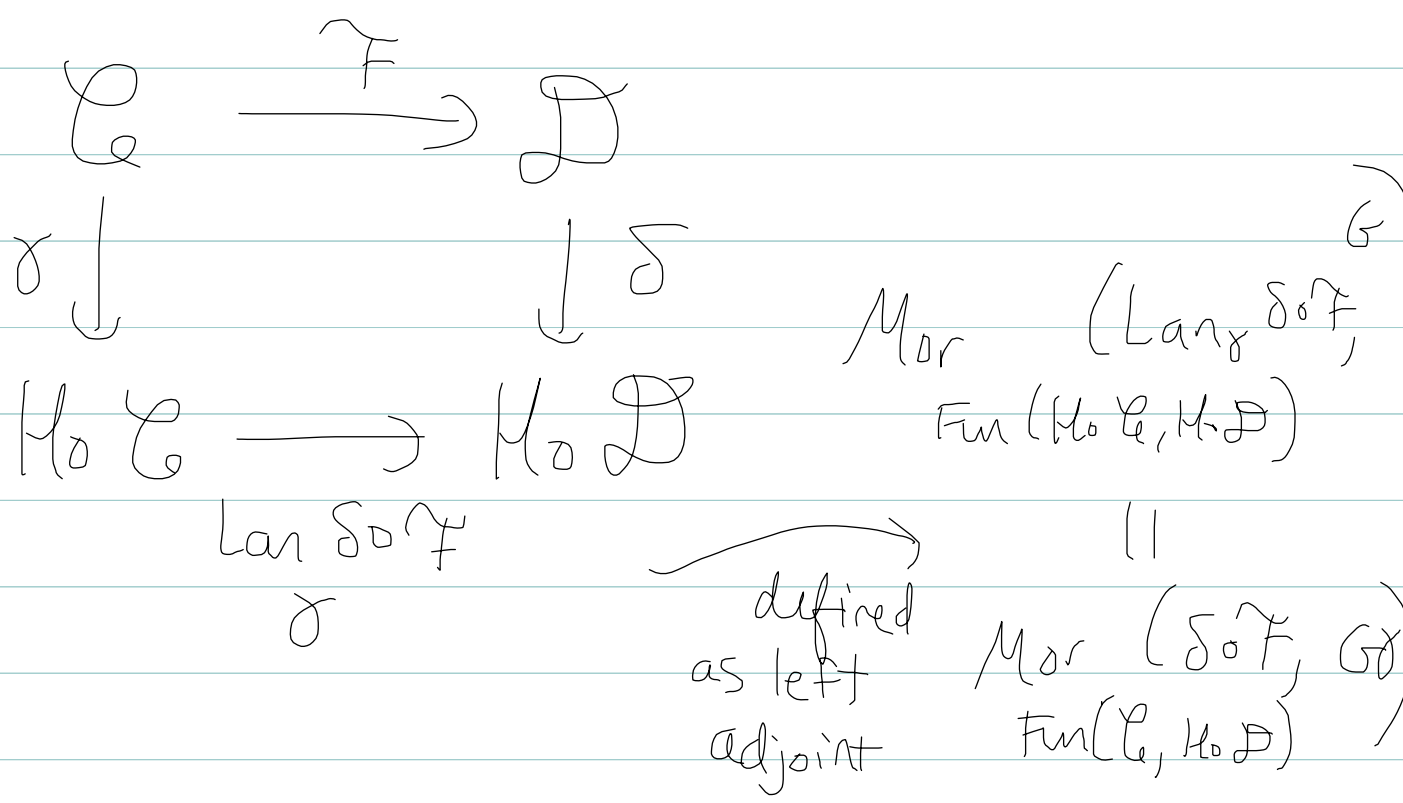
Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor  
b/w categories w/ notions of  
weak equivalence

A right derived functor  $R\tilde{F}$

is  $\mathcal{C} \xrightarrow{R\mathcal{F}} \mathcal{D}$  and  $\mathcal{F} \Rightarrow R\mathcal{F}$   
 $\uparrow$   
 nat'l transformation

s.t.  $R\mathcal{F}$  preserves w.e.,  
 and is initial w.r.t. this property.

Reference: E. Riehl "Categorical homotopy theory"  
 2.1-2.2



$\Rightarrow \exists$  tautological map  $\delta F \rightarrow \text{Lan}_\gamma \delta F \circ \gamma$

Def'

$\mathbb{R}\mathcal{F}$  is derived functor if  $\mathbb{R}\mathcal{F}(w_\alpha) \xrightarrow{w_\alpha} \mathcal{F}$   
 $\mathbb{R}\mathcal{F} \xrightarrow{\eta} \mathcal{F}$

and  $\mathcal{F} \Rightarrow \mathbb{R}\mathcal{F}$  satisfies condition

$$\text{Lan}_\gamma \delta F \circ \gamma = \delta \circ \mathbb{R}\mathcal{F}$$

and  $\delta F \Rightarrow \delta \circ \mathbb{R}\mathcal{F} = \text{Lan}_\gamma \delta F \circ \gamma$

is the tautological map above

Common condition/construction of derived functors

$\mathcal{C}$  has "good objects" s.t. restriction of

$\mathcal{F}$  to these preserves w.e.

Def: A right deformation is

$$\mathcal{I} : \mathcal{C} \rightarrow \mathcal{C}$$

$$\rightsquigarrow \mathcal{I} \xrightarrow{\sim} \mathcal{I}$$

• Let  $\mathcal{C}_{\mathcal{I}}$  be full subcategory containing  $\mathcal{I}(\mathcal{C})$

Thm (4.2.2.8 Riehl) If  $\mathcal{F}|_{\mathcal{C}_{\mathcal{I}}}$  preserves w.e.

Then  $\mathcal{F} \circ \mathcal{I}$  is a right derived functor of  $\mathcal{F}$ .

• This gives the derived functors in all the above examples.

$K =$  unstable algs  
over  $A^*$

Recall :

$\mathcal{U} =$  unstable modules over  $A^*$

Unstable Adams Spectral Sequence

$$E_{-s,t}^2 = \text{Ext}_K^s(H^*X, \Sigma^t H^*Z^+) \Rightarrow \Pi_* \text{Map}(Z, R_{\infty}X)$$

Bousfield connectivity lemma relates  
contractibility  $\text{Map}_*(Z, R_{\infty}X)$

and condition  $\text{Ext}_K^s(H^*X, \Sigma^t H^*Z^+) = 0$

Next:  $H^*BV$  is injective in  $\mathcal{U}$  for  $V$  finite  $p$ -group

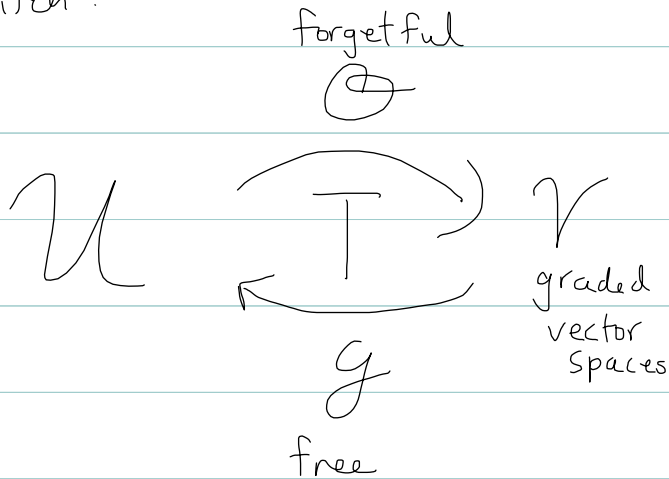


Friday:  $\mathcal{K}$  vs  $\mathcal{U}$

Mon 4/29:  $\text{Map}_*(BV, X) \cong *$

Wed 5/1: Sullivan Conjecture

Notation switch:



Lemma:  $\mathcal{G}(\Sigma^n \mathbb{F}_2) = \sum^n (A \langle \text{Sq}^I : \begin{array}{l} I \text{ admissible} \\ e(I) > n \end{array} \rangle)$

Pf: From  $\mathcal{G} \circ \mathcal{H}^*(Y) = \mathcal{H}^*(RY)$  pf, we saw

$$\text{Sq}^I z_n = 0 \quad (\text{didn't use alg structure})$$

$\text{Sq}^{a_i} \text{Sq}^{(a_2, \dots, a_r)} \quad a_i - d(a_2, \dots, a_r) = e(I) > n$

Remains to check:  $\langle \text{Sq}^I : \begin{array}{l} I \text{ admissible} \\ e(I) > n \end{array} \rangle$  is a sub-module. Exercise.

• For  $M \in \mathcal{U}$ ,  $\text{Hom}_{\mathcal{U}}(g \Sigma^n \mathbb{F}_2, M)$

$$\text{Hom}_{\mathcal{U}}(\Sigma^n \mathbb{F}_2, M) \cong M^n$$

degree  $n$   
piece of  $M$

Prop: The functor  $\mathcal{U}^{\text{op}} \rightarrow \text{Vect}^{\leftarrow \text{non-graded}}$   
 $M \mapsto \text{Hom}_{\text{Vect}}(M^n, \mathbb{F}_2)$

is  $\text{Hom}_{\mathcal{U}}(-, J(n))$

Rmk: •  $J(n)^m = \text{Hom}_{\mathcal{U}}(g \Sigma^m \mathbb{F}_2, J(n))$   
 $= (g \Sigma^m \mathbb{F}_2)^n$   $\star$

dual  
vector  
space

•  $J(n)^n = \mathbb{F}_2$

•  $J(0) = \mathbb{F}_2$  in deg 0

Rmk:  $J(n)$  is  $n$ th Brown-Gitler module. They control when a manifold embeds in  $\mathbb{R}^{n+\dim}$ .

pf:  $J(n)$  is determined by:

$$J(n)^m = (g \Sigma^m \mathbb{F}_2)^{n \star}$$

$$\alpha \in A^* \quad \alpha: J(n)^m \rightarrow J(n)^{|a|+m}$$

is map obtained by applying

$$\text{Hom} \left( \text{Vect} \left( (-)^n, \mathbb{F}_2 \right) \right)$$

$$\text{to } g \Sigma^{|a|+m} \mathbb{F}_2 \rightarrow g \Sigma^m \mathbb{F}_2$$

$$2\mu^{|a|+m} \mapsto \alpha 2\mu^m$$

check: defines  $A$ -module structure  
 Nat'l transformation  $D(n)$

$$\text{Hom}_u(\quad, J(n)) \xrightarrow[\substack{\text{deg} \\ n \\ \text{piece}}]{\text{read off}} \text{Hom}_{\text{vect}}(M^n, \mathbb{F}_2)$$

$D(n)(M)$  is iso:

$$g \circ g \circ (M) \xrightarrow{d_i - d_0} g \circ (M) \rightarrow M \rightarrow 0 \quad (\star)$$

Both  $\text{Hom}_u(\quad, J(n))$  and

$$\text{Hom}_{\text{vect}}(\quad)^n, \mathbb{F}_2$$

take  $(\star)$  to a left exact  
 sequence.

$D(n)(g \circ M')$  iso by construction.  
 (direct sums are taken to  $\mathbb{F}_2$ )  $\square$