No class 4/22–4/26
I'll put up last HW by next weekend

L27 – 4/15/13
Derived Functors II
Brown-Gitler

# Homotopical algebra

- $\mathcal{C}$ cat with $w \subset \mathrm{Mor}\,\mathcal{C}$ "weak equivalences"

  iso's $\in w$ and $\forall$
$$\begin{tikzcd} \cdot \arrow[r, "f"] & \cdot \arrow[d, "g"] & \\ & \cdot \arrow[r, "h"] & \cdot \end{tikzcd}$$
"2-of-6"

$$gh,\ gf \in w \Rightarrow f, g, h, hgf \in w$$

- $\mathrm{Ho}\,\mathcal{C} = \mathcal{C}[w^{-1}]$ formaly invert weak equivalences

- $F: \mathcal{C} \to \mathcal{D}$ functor b/w categories w/ weak equivalences

- $F(f)$ for $f \in w_{\mathcal{C}}$ may not be in $w_{\mathcal{D}}$

Ex!: $\mathcal{C} = Ch_+(ab)$ $\mathcal{D} = Ch_-(ab)$ with $w = \{H_* \text{ isos}\}$

abelian groups

$$\begin{array}{ccc} \downarrow & & \downarrow \\ 0 & & \mathbb{Z} \\ \downarrow & \xrightarrow{f} & \downarrow p \\ \mathbb{Z}/p & & \mathbb{Z} \end{array}$$

$F(A_\cdot) = \mathrm{Hom}(A_\cdot, \mathbb{Z}/p) \in Ch_+$

$\mathrm{Hom}(A_\cdot, \mathbb{Z}/p)_m = \mathrm{Hom}(A_{\cdot+m}, \mathbb{Z}/p)$

(Apply $F$

actually let's do $\mathrm{Hom}(\mathbb{Z}/p, -)$

$$\begin{array}{ccc} \mathbb{Z}/p & \xrightarrow{F(f)} & \mathbb{Z}/p \\ \uparrow & & \uparrow 0 \\ 0 & & \mathbb{Z}/p \\ \uparrow & & \uparrow \\ 0 & & 0 \\ \uparrow & & \uparrow \end{array}$$

$F(f) \notin w_{\mathcal{D}}$

$\underline{\text{Ex 2}}$: $I$ cat w/ one object $*$, mor $= G$

$\underline{\text{note}}$: $X$ sSet with $G$-action $\quad X = \mathcal{X}(*)$

$$\begin{array}{ccc} X \text{ sSet with } G\text{-action} & & X = \mathcal{X}(*) \\ \| & & \uparrow \\ \text{Functor } \mathcal{X}: I \to sSet & & \mathcal{X} \end{array}$$

$$\lim_{\overleftarrow{I}} \mathcal{X} = X^G$$

$\mathcal{C} = \mathrm{Fun}(I, sSet) \qquad W = \{X \xrightarrow{f} Y : f \text{ w.e.}\}$

$\mathcal{D} = sSet$

$F: \mathcal{C} \to \mathcal{D}$

$X \mapsto X^G$

$$F(EG \to *) \notin W_{\mathcal{D}}$$

(with $EG \to * \in W_{\mathcal{C}}$)

- Approximate $\mathcal{F}$ by a functor preserving weak equivalences

<u>Ex 1</u>: $R\mathcal{F}(A_\cdot) = \mathrm{Hom}(\mathbb{Z}/p, I_\cdot)$

$A_\cdot \to I_\cdot$ quasi-iso $I_n$ injective $\forall\, n$

(i.e. $\mathrm{Hom}(-, I_n)$ is exact)

So $\pi_i \Gamma R\mathcal{F}\begin{pmatrix} \downarrow \\ 0 \\ \downarrow \\ M \end{pmatrix} = \mathrm{Ext}^i(\mathbb{Z}/p, M)$

$H_i R\mathcal{F}\begin{pmatrix} \downarrow \\ 0 \\ \downarrow \\ M \end{pmatrix} =$

<u>Ex 2</u>: $R\mathcal{F}(X) = X^{hG}$

<u>Ex 4</u>: $R\,\underleftarrow{\lim} = \mathrm{ho}\underleftarrow{\lim}$ $\underleftarrow{\lim} : \mathrm{Fun}(I, sSet) \to sSet$

<u>Ex 5</u>: $\underleftarrow{\lim}^1$ as functor on towers of abelian

groups is $\pi^1$ or $\lim^1$ of $R\varprojlim$

( replace $\mathbb{N} \to ab$ by

$\mathbb{N} \to Ch_+(ab)$

with transition functions required to be surjective

$$\{a_n\} \to \left\{ \begin{array}{c} \prod_{k\le n-1} a_n \\ \uparrow {\scriptstyle 1-f} \\ \prod_{k\le} a_n \end{array} \right\}$$ )

<u>Definition of derived functor:</u>

Let $F: \mathcal{C} \to \mathcal{D}$ be a functor b/w categories w/ notions of weak equivalence

A right derived functor $RF$

is $\mathcal{C} \xrightarrow{R\mathcal{F}} \mathcal{D}$ and $\mathcal{F} \Rightarrow R\mathcal{F}$

nat'l transformation

s.t. $R\mathcal{F}$ preserves w.e.

and is initial w.r.t. this property.

Reference: E. Riehl "categorical homotopy theory" 2.1 – 2.2

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathcal{F}"] \arrow[d, "\gamma"'] & \mathcal{D} \arrow[d, "\delta"] \\
\mathrm{Ho}\,\mathcal{C} \arrow[r, "\mathrm{Lan}_\gamma \delta\circ\mathcal{F}"'] & \mathrm{Ho}\,\mathcal{D}
\end{tikzcd}
$$

defined as left adjoint

$$\mathrm{Mor}_{\mathrm{Fun}(\mathrm{Ho}\,\mathcal{C},\, \mathrm{Ho}\,\mathcal{D})}(\mathrm{Lan}_\gamma \delta\circ\mathcal{F}, \mathcal{G})$$

$$\|$$

$$\mathrm{Mor}_{\mathrm{Fun}(\mathcal{C},\, \mathrm{Ho}\,\mathcal{D})}(\delta\circ\mathcal{F}, \mathcal{G}\gamma)$$

$\Rightarrow \exists$ tautological map $\delta F \to \mathrm{Lan}_\gamma \delta \mathcal{F} \circ \gamma$

<u>Def'</u>

$\mathbb{R}\mathcal{F}$ is derived functor if $\mathbb{R}\mathcal{F}(W_{\mathcal{C}}) \subset W_{(\mathcal{D})}$

$\Uparrow$

$\mathcal{F}$

and $\mathcal{F} \Rightarrow \mathbb{R}\mathcal{F}$ satisfies condition

$$\mathrm{Lan}_\gamma \delta \mathcal{F} \circ \gamma = \delta \circ \mathbb{R}\mathcal{F}$$

and $\delta F \Rightarrow \delta \circ \mathbb{R}\mathcal{F} = \mathrm{Lan}_\gamma \delta \mathcal{F} \circ \gamma$

is the tautological map above

<u>Common condition/construction of derived functors</u>:

$\mathcal{C}$ has "good objects" s.t. restriction of

$F$ to these preserves w.e.

<u>Def</u>: A right deformation is

$$I : \mathcal{C} \to \mathcal{C}$$

w/ $1 \overset{\sim}{\Rightarrow} I$

- Let $\mathcal{C}_I$ be full subcategory containing $I(\mathcal{C})$

<u>Thm</u> If $F|_{\mathcal{C}_I}$ preserves w.e.
(cf. 2.2.8 Riehl)

Then $F \circ I$ is a right derived functor of $F$.

- This gives the derived functors in all the above examples.

**Recall**: $\mathcal{K}$ = unstable algs over $A^*$

$\mathcal{U}$ = unstable modules over $A^*$

**Unstable Adams Spectral Sequence**

$$\boxed{E^2_{-s,t} = \mathrm{Ext}^s_{\mathcal{K}}(H^*X, \Sigma^t H^* Z^+) \Rightarrow \pi_* \mathrm{Map}(Z, R_\infty X)}$$

Bousfield connectivity lemma relates contractibility $\mathrm{Map}_*(Z, R_\infty X)$ and condition $\mathrm{Ext}^s_{\mathcal{K}}(H^*X, \Sigma^t H^* Z^+) = 0$

**Next**: $H^*BV$ is injective in $\mathcal{U}$ for $V$ finite p-group

Friday: $\mathcal{K}$ vs $\mathcal{U}$

Mon 4/29: $\mathrm{Map}_*(BV, X) \simeq *$

Wed 5/1: Sullivan conjecture

Notation switch:

$$\begin{tikzcd} \mathcal{U} \arrow[r, bend left, "\mathcal{O}"] \arrow[r, phantom, "\top"] & \mathcal{V} \arrow[l, bend left, "\mathcal{G}"] \end{tikzcd}$$

forgetful

free

graded vector spaces

<u>Lemma</u>: $\mathcal{G}(\Sigma^n \mathbb{F}_2) = \Sigma^n \left( A / \left\langle Sq^I : \begin{matrix} I \text{ admissible} \\ e(I) > n \end{matrix} \right\rangle \right)$

<u>Pf</u>: From $\mathcal{G} \circ \mathcal{H}^*(Y) = H^*(RY)$ pf, we saw

$Sq^I \iota_n = 0$ (didn't use alg structure)

$Sq^{a_1} Sq^{(a_2, \dots, a_r)}$ $\quad a_1 - d(a_2, \dots, a_r) = e(I) > n$

Remains to check: $\left\langle Sq^I : \begin{matrix} I \text{ admissible} \\ e(I) > n \end{matrix} \right\rangle$ is a sub-module. Exercise.

- For $M \in \mathcal{U}$, $\mathrm{Hom}_{\mathcal{U}}(g\Sigma^n \mathbb{F}_2, M)$

$$\mathrm{Hom}(\Sigma^n \mathbb{F}_2, M) = M^n$$

$M^n$ ← degree $n$ piece of $M$

Prop: The functor $\mathcal{U}^{op} \longrightarrow \mathrm{Vect}$ ← non-graded

$M \longmapsto \mathrm{Hom}_{\mathrm{Vect}}(M^n, \mathbb{F}_2)$

is $\mathrm{Hom}_{\mathcal{U}}(-, J(n))$

Rmk: • $J(n)^m = \mathrm{Hom}_{\mathcal{U}}(g\Sigma^m \mathbb{F}_2, J(n))$

$= (g\Sigma^m \mathbb{F}_2)^{n*}$

dual vector space

• $J(n)^n = \mathbb{F}_2$

• $J(0) = \mathbb{F}_2$ in deg $0$

Rmk: $J(n)$ is nth Brown-Gitler module. They control when a manifold embedds in $\mathbb{R}^{n+\dim}$

Pf: $J(n)$ is determined by:

$$J(n)^m = \left(g \Sigma^m \mathbb{F}_2\right)^{n*}$$

$\alpha \in A^*$ $\quad \alpha : J(n)^m \to J(n)^{|\alpha|+m}$

is map obtained by applying

$$\operatorname{Hom}_{\mathrm{Vect}}\left((-)^n, \mathbb{F}_2\right)$$

to $g \Sigma^{|\alpha|+m} \mathbb{F}_2 \to g \Sigma^m \mathbb{F}_2$

$$2^{|\alpha|+m} \mapsto \alpha\, 2^m$$

check: defines $\mathcal{A}$-module structure

Nat'l transformation $D(n)$

$$\mathrm{Hom}_{\mathcal{U}}(\ \ , J(n)) \xrightarrow[\substack{\text{deg}\\ n\\ \text{piece}}]{\substack{\text{read}\\ \text{off}}} \mathrm{Hom}_{vect}(M^n, \mathbb{F}_2)$$

$D(n)(M)$ is iso:

$$g_0 g_0(M) \xrightarrow{d_1 - d_0} g_0(M) \longrightarrow M \longrightarrow 0, \quad (\star)$$

Both $\mathrm{Hom}_{\mathcal{U}}(\ \ , J(n))$ and

$\mathrm{Hom}_{vect}((\ \ )^n, \mathbb{F}_2)$

take $(\star)$ to a left exact sequence.

$D(n)(g_0 M')$ iso by construction.

(direct sums are taken to $\prod$) $\square$