L26 - 4/11/13
Derived functors

Unstable Adams Spectral sequence $\quad \pi_*(Z, X) \quad Z, X \in sSet$

$R = \mathbb{Z}/2 \quad (\text{or } \mathbb{Z}/p) \qquad H^* = H^*(-;R)$

$Map(Z, R_\infty X) = \mathrm{Tot}\, Map(Z, \underset{\sim}{R}X) = \varprojlim \mathrm{Tot}_m\, Map(Z, \underset{\sim}{R}X)$

$Map(Z, \underset{\sim}{R}X)[s] = Map(Z, \underbrace{R \circ \cdots \circ R}_{s+1 \text{ times}} X) \in sSet$

Have spectral sequence:

$\{E^r_{-s,t},\ d^r : E^r_{-s,t} \to E^r_{-s-r,t+r-1}\} \quad E^2_{-s,t} = \pi^s \pi_t\, Map(Z, \underset{\sim}{R}X)$

Compute $E^2_{-s,t}$ in terms of A^* = Steenrod alg:

$\mathcal{K}$ = category of unstable algebras over A^* $\qquad \mathcal{V}$ = category of graded vector spaces

$\mathcal{K} \xrightarrow{\mathcal{O}} \mathcal{V}$ forgetful functor

$\mathcal{V} \xrightarrow{\mathcal{G}} \mathcal{K}$ free functor

$Mor_{\mathcal{K}}(\mathcal{G}V, H) = Mor_{\mathcal{V}}(V, \mathcal{O}H)$

Prop: $Y \in sSet$ s.t. $\dim H^n(Y) < \infty\ \forall n \Rightarrow \dim H^n(RY) < \infty\ \forall n$

and $H^*(RY) \cong \mathcal{G}\mathcal{O} H^*(Y)$ in $\mathcal{K}$

Here's a natural way to describe this isomorphism:

$$g \otimes H^* Y \longrightarrow H^* R Y \quad \text{in } K$$

is equivalent to a map

$$\otimes H^* Y \longrightarrow \otimes H^* R Y \quad \text{in } V$$

is equivalent to

$$\otimes H^* R Y^{\star} \longrightarrow \otimes H^* Y^{\star}$$

dual vector space

$$\parallel$$

$$H_* R Y \longrightarrow H_* Y$$

This is induced from $R R Y \to R Y$

Where $R \circ R \xrightarrow{\psi} R$

$\mathrm{id} \xrightarrow{\varepsilon} R$

are as in the definition of $\underset{\sim}{R}X$.

Applying H^* "dualizes" the construction $\underset{\sim}{R}X$:

$$\Delta \xrightarrow{\underset{\sim}{R}X} sSet \xrightarrow{H^*} K^{op}$$

is the same as data

$$\Delta^{op} \xrightarrow{\underset{\sim}{R}X} sSet^{op} \xrightarrow{H^*} K^{op}$$

$$[s] \longmapsto \underbrace{R \circ \ldots \circ R}_{s+1 \text{ times}} X \longrightarrow \underbrace{(g\theta)(g\theta)\circ \ldots \circ (g\theta)}_{s+1 \text{ times}} H^* X$$

$$\underset{\sim}{R}X\left([n-1] \xrightarrow{d_i} [n]\right) : R^n X \xrightarrow[R^i \varepsilon_{R^{n-i}}]{} R^{n+1} X$$

$$\underset{\sim}{R}X\left([n+1] \xrightarrow{s_i} [n]\right) : R^{n+2} X \xrightarrow[R^i \psi_{R^{n-i}}]{} R^n X$$

Dually, for $H \in \mathcal{K}$ define

$$\widetilde{g\Theta H} : \Delta^{op} \longrightarrow \mathcal{K}$$

$$\widetilde{g\Theta H}\,[s] = \underbrace{g\Theta \circ g\Theta \circ \cdots \circ g\Theta}_{s+1}\; H$$

Need to define: $\widetilde{g\Theta}\left([n-1] \xrightarrow{d_i} [n]\right)$

$\widetilde{g\Theta}\left([n+1] \xrightarrow{s_i} [n]\right)$

Analogues / duals ε, ψ:

$g\Theta \xrightarrow{\eta} id$ defined $Map_{\mathcal{K}}(g\Theta H, H) = Map(\Theta H, \Theta H)$

$\eta(H) \longleftarrow\!\shortmid id$

$g\Theta \xrightarrow{\nu} g\Theta \circ g\Theta$ defined $Map_{\mathcal{V}}(V, \Theta g V) = Map_{(gV, gV)}$

$g\Theta(V) \xrightarrow{g\,\varepsilon(V)\Theta} g\Theta \circ g\Theta V$ $\quad \varepsilon(V) \longleftarrow\!\shortmid id$

$$\tilde{g\theta}\left([n-1] \xrightarrow{d_i} [n]\right) = g\theta^{(n+1)} \xrightarrow{g\theta^i \eta\, g\theta^{n-i}} g\theta^n$$

$$\tilde{g\theta}\left([n+1] \xrightarrow{s_i} [n]\right) = g\theta^{(n+1)} \xrightarrow{g\theta^i \nu\, g\theta^{n-i}} g\theta^{n+2}$$

up to checking that d_i's, s_i's work

Prop $\Rightarrow$ $\boxed{M^* \underset{\sim}{R} X = \tilde{g\theta}\, M^* X}$

only depends on $M^* X$

$\tilde{g\theta} M^* X \to M^* X$ is a resolution or cofib replacement

Observation: Given adjoint functors

$$\mathcal{C} \xrightarrow{\theta} \mathcal{E}$$

$$\mathcal{E} \xrightarrow{g} \mathcal{C}$$

$$\mathrm{Mor}_{\mathcal{C}}(gV, M) = \mathrm{Mor}_{\mathcal{E}}(V, \theta M)$$

and $\quad C \subset \mathcal{C}$

Define $\tilde{g\theta}\, C : \Delta^{op} \to \mathcal{C}$

by above formulas.

- Suppose given $F : \mathcal{C} \to Ab$
(resp $\mathcal{C}^{op} \to Ab$)

(really suppose $\exists\ \tau : F \to F \circ g\theta$ s.t.

$F(\eta_C) \circ \tau(C) : F(C) \to F(C)$ is identity)

Derived Functor of F (resp $R^i F(C)$)

Calc Def $L^i F(C)$ ~~Λ~~ w.r.t (g, θ) is the

i^{th} homology of

$NF(\tilde{g\theta}\, C_\bullet)$, $d\ N_i \to N_{i-1}$

(Recall $NA_\bullet$ is chain complex $NA_n = A_n \cap \bigcap_{i=0}^{n-1} \ker d_i$

equiv $NA_n = A_n / \oplus \operatorname{im} s_i$ $\quad d = \Sigma (-1)^i d_i$ $\quad d = d_n$)

resp the i^{th} homology of

$$\{ N \mathcal{F} \tilde{g} \theta c, d_i \}$$

where $N_n \mathcal{F} \tilde{g}\theta c = \mathcal{F} \tilde{g} \theta^i c \cap \bigcap_{i=0}^{n-1} \ker s_i$

$d = \sum (-1)^i d_i$

i.e. $\pi^i \mathcal{F} \tilde{g} \theta c$

Rmk: one could define a more general simplicial res w.r.t. (g, θ) and the derived functors of $\mathcal{F}$ would be independent of the resolution

Ex: $\mathcal{C}$ = graded algebras $\underset{g}{\overset{\theta}{\rightleftarrows}}$ $\mathcal{E}$ = graded vector spaces

$A_0 = R$

"cotangent space"

$$\mathcal{F}(A) = \frac{A_{\geq 1}}{A_{\geq 1} \cdot A_{\geq 1}} = I / I^2$$

$L^i \mathcal{F}(A)$ = André-Quillen cohomology

$\underline{Ex}$

$$\mathcal{C} = \mathcal{K} \underset{g}{\overset{\theta}{\rightleftarrows}} \mathcal{E} = \mathcal{V}$$

$$\mathcal{F} = \mathrm{Hom}_{\mathcal{K}}(\ \ , M) \qquad M \in \mathcal{K}$$

$$R^i \mathcal{F}(A) = \mathrm{Ext}^i_{\mathcal{K}}(A, M)$$

$$\Rightarrow \pi^s \pi_t \mathrm{Map}(Z, \underset{\sim}{R}X) = \mathrm{Ext}^s_{\mathcal{K}}(M^*X, \Sigma^t M^* Z^+)$$

$$\begin{aligned}
&\cong \\
&\pi^s \pi_0 \mathrm{Map}(S^t \wedge Z_+, \underset{\sim}{R}X) \\
&\cong \\
&\pi^s \mathrm{Hom}_{\mathcal{K}}(M^* \underset{\sim}{R}X, \Sigma^t M^* Z^+) \\
&\cong \\
&\pi^s \mathrm{Hom}_{\mathcal{K}}(g\tilde{\theta} M^*X, \Sigma^t M^* Z^+) = \text{r.h.s}
\end{aligned}$$

defined $\quad \begin{cases} M^{*-t} Z & * \neq 0 \\ R & * = 0 \end{cases}$

$$\boxed{E^2_{-s,t} = \mathrm{Ext}^s_{\mathcal{K}}(M^*X, \Sigma^t M^* Z^+)}$$

Homological / homotopical algebra

(Grothendieck, Quillen)

- many categories $\mathcal{C}$ have natural notion of weak equivalence in analogy to htpy thy of topological spaces

- leads to notion of homotopy category

$$\mathrm{Ho}\,\mathcal{C} = \mathcal{C}[\underbrace{W^{-1}}_{\text{invert all weak equivalences}}]$$

ex $\mathcal{A}$ abelian cat

$\rightsquigarrow$

$Ch_+(\mathcal{A})$ with quasi-iso := maps inducing iso on H_* as weak equivalence

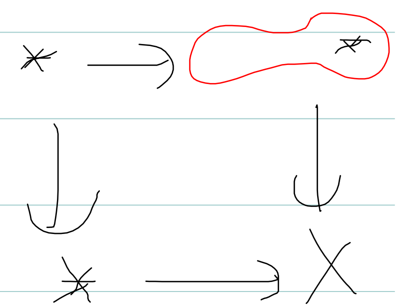
$\mathrm{Ho}(Ch_+(\mathcal{A})) = D(\mathcal{A})$ derived category

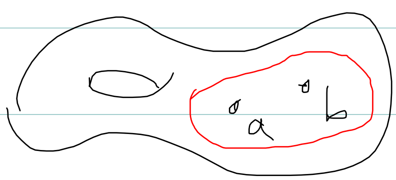
- many functors of interest do not preserve natural notion of weak equivalence

<u>ex 1</u>: Fiber product

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & * \arrow[d, "a"] \\ * \arrow[r, "b"] & X \end{tikzcd}$$

vs





More generally, $\varprojlim$

<u>ex</u>: $Ch_+(\mathbb{Z})$

$$\longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z}$$

quasi-iso $\qquad \otimes \mathbb{Z}/p$

$$\longrightarrow 0 \longrightarrow \mathbb{Z}/p$$

degree 0

- Approximate such a functor by a functor preserving weak-equivalences

<u>ex 1</u>: $\underleftarrow{\text{holim}}$ $\qquad$ <u>ex 2</u>: Tor

Def Derived functor: Let $F: \mathcal{C} \to \mathcal{D}$ be a functor b/w categories w/ notions of w.e.

A right derived functor RF is

$$\mathcal{C} \xrightarrow{RF} \mathcal{D}$$

preserving weak equivalences together with $F \Rightarrow RF$, initial with this property.

More formally: see E. Riehl "Categorical homotopy theory" 2.1

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \arrow[d, "\delta"] \\ \mathrm{Ho}\,\mathcal{C} \arrow[r, "\mathrm{Lan}_\gamma \delta F"'] & \mathrm{Ho}\,\mathcal{D} \end{tikzcd}$$

$$\mathrm{Lan}_\gamma \delta \mathcal{F} \circ \gamma = \delta \circ R\mathcal{F}$$

and $\mathcal{F} \Rightarrow R\mathcal{F}$ satisfies

$$\delta \mathcal{F} \Rightarrow \delta R\mathcal{F} = \mathrm{Lan}_\gamma \delta \mathcal{F} \circ \gamma$$

is tautological $\delta \mathcal{F} \Rightarrow \mathrm{Lan}_\gamma \delta \mathcal{F} \circ \gamma$

- $\mathcal{C}$ may have a collection of good objects on which $\mathcal{F}$ respects w.e. Find $\mathcal{C} \to$ good objects

$\rightsquigarrow$ derived functors via deformations

- To compute $c \in \mathcal{C}$

$c \to \underline{I}$ resolution
fibrant replacement

$$R\mathcal{F}(c) = \mathcal{F}(I)$$