$$L25 - \pi^s \pi_t \, Map(Z, \underset{\sim}{RX}) \\ \shortparallel \\ Ext(\tilde{g}H^* X, \Sigma^t H^* Z^+)$$

Unstable Adams spectral sequence

$$Map(Z, R_\infty X) = Map(Z, Map(\underset{\sim}{\Delta}, \underset{\sim}{RX})) = Map(\underset{\sim}{\Delta}, Map(Z, \underset{\sim}{RX}))$$

$$= Tot \; Map(Z, \underset{\sim}{RX}) = \varprojlim Tot_m \, Map(Z, \underset{\sim}{RX})$$

Here: $Map(Z, \underset{\sim}{RX}) \in Fun(\Delta, sSet)$

$$Map(Z, \underset{\sim}{RX})[n] = Map(Z, \underbrace{R \circ \cdots \circ R}_{n+1} X)$$

Apply spectral sequence of tower of fib

obtain $\{E^r_{-s,t}, d^r\}$ with $E^2_{-s,t} = \pi^s \pi_t \, Map(Z, \underset{\sim}{RX})$

<u>Calculate</u>: $\pi_t \, Map(Z, \underset{\sim}{RX}[s])$

$$= \pi_0 Map(S^t \wedge Z_+, \underset{\sim}{RX}[s])$$

Assumption: Let $H_* = H_*(\; ; \mathbb{Z}/2)$

$X \in sSet$ has $H_n(X)$ finite dim'l $\forall n$

Recall: $\mathcal{A}, \mathcal{K}$ $\quad H^* : sSet \to \mathcal{K}$

Under this assumption, we saw $\pi_0 Map(S^t \wedge Z_+, \underset{\sim}{R}X[s]) = Hom_{\mathcal{K}}(H^* \underset{\sim}{R}S[s], H^*(S^t \wedge Z_+))$

b/c $\underset{\sim}{R}X[s]$ is product E.M. spaces, so maps htpy classes are tuples of elts of coh, these are images of fundamental classes, so they are recorded in $\mathcal{K}$.

$$H^*(S^t \wedge Z) = \begin{cases} H^{*-t}(Z) & \text{for } * \neq 0 \\ \mathbb{F}_2 & * = 0 \end{cases}$$

Notation $H^*(S^t \wedge Z) = \Sigma^t H^*(Z)^+$

Q: What is $H^*(\underset{\sim}{R}X[s])$?

$\mathcal{V}$ = cat of graded vector spaces

$\mathcal{O} : \mathcal{K} \to \mathcal{V}$ forgetful functor

$\mathcal{G} : \mathcal{V} \to \mathcal{K}$ left adjoint or free A^*-alg

Prop: Let $Y \in sSet$ be s.t.

$$\dim H^n Y < \infty \quad \forall n.$$

Then $\dim H^n RY < \infty \quad \forall n$

and $H^* RY \underset{K}{\cong} G \odot H^* Y$

Lemma: $H^* K(\mathbb{Z}/2, n) \cong G\, \Sigma^n \mathbb{F}_2$ ← graded v.s. with single generator in deg n

Pf: I admissible sequence excess $e(I) = q$

$I = (i_1, \dots, i_m)$

$q = i_1 - d((i_2, \dots, i_m))$

$\Rightarrow$ For $q > n$ and $x \in G^n$ $G \in K$

$$Sq^I x = Sq^{i_1} \underbrace{Sq^{(i_2, \dots, i_m)} x}_{\deg\, n + d(i_2, \dots, i_m) < q + d < i_1} = 0$$

For $q = n$, take r maximal so

$$I = (2^{r-1} i_r, \dots, 2 i_r, i_r)(i_{r+1}, \dots, i_{rm})$$

Then $e(I) = i_r - d(i_{r+1}, \dots, i_{rm})$

$$\Rightarrow \quad i_r = n + d(i_{r+1}, \dots, i_{r+m})$$

$$\Rightarrow \quad Sq^I x = \left(Sq^{(i_{r+1}, \dots i_{rm})} x\right)^{2^r}$$

and $e(i_{r+1}, \dots, i_{rm}) < e(I) = n$.

omit

$\Rightarrow$ $\mathcal{G}\mathcal{O}(\Sigma^n \mathbb{F}_2)$ is generated by

$Sq^I x \quad e(I) < n \quad x$ is basis $\Sigma^n \mathbb{F}_2$

We can define a map in $\mathcal{K}$

$$\mathcal{G} \odot \Sigma^n \mathbb{F}_2 \leftarrow H^*(K(\mathbb{Z}/2, n))$$

$$\| $$

$$\mathbb{F}_2[Sq^I \iota_n : e(I) < n]$$

$$x \longleftarrow\!\shortmid \iota_n$$
f

From universal property have

$$\mathcal{G} \odot \Sigma^2 \mathbb{F}_2 \to H^*(K(\mathbb{Z}/2, n))$$

Better: define f alg map $\square$

need $Sq^I f(y) = f(Sq^I y)$ $\forall\, y \in \mathbb{F}_2[Sq^I \iota_n : e(I) < n]$

Both sides satisfy Adm & Cartan $\Rightarrow$ sufficient to check $y = 2n$ & Sq^I admissible. $\square$

<u>Pf of Prop:</u>

Show iso $RY = \prod_m K(H_m Y, \mathbb{Z}/2)$

$$\Rightarrow H^* RY = \bigotimes_{m=0}^{\infty} H^* K(H_m Y, m)$$

Choose basis B_m for $H_m Y^* = H^m Y$

obtain iso

$$K(H_m Y, m) \cong \prod_{v \in B_m} K(\mathbb{Z}/2, m)$$

$$\Rightarrow H^* K(H_m Y, m) \cong \bigotimes_{v \in B_m} H^* K(\mathbb{Z}/2, m) \underset{\text{lemma}}{\cong} \bigotimes_{v \in B_m} \mathcal{G}(\Sigma^m \mathbb{F}_2)$$

☆ Have map. omit

$$\mathcal{G}\Theta H^* Y \longrightarrow H^* RY$$

adjoint to $\Theta H^* Y \longrightarrow \Theta H^* RY$

dual to $H_* RY \to H_* Y$

induced from $RRY \to RY$

Since $\mathcal{G}$ is a left-adjoint, $\mathcal{G}$ preserves $\underrightarrow{\mathrm{colim}}$ (HW 8). Here, $\bigotimes_{V \in B} \mathcal{G} = \mathcal{G}\left(\bigoplus_{V \in B}\right)$

categorical coproduct

$$\Rightarrow H^*(K(H_m Y, m)) \cong \mathcal{G}(\Sigma^m H^m Y)$$

$$\Rightarrow H^*(RY) \cong \mathcal{G}\left(\bigoplus_m \Sigma^m H^m Y\right)$$

$$= \mathcal{G}(\theta(H^* Y))$$

$\square$

For $V \in \mathcal{V}$, define $\tilde{\mathcal{G}} V \in \mathrm{Fun}(\Delta^{op}, \mathcal{K})$ by $\tilde{\mathcal{G}} V[n] = \underbrace{(\mathcal{G}\theta)(\mathcal{G}\theta) \circ \cdots \circ (\mathcal{G}\theta)}_{n+1 \text{ times}} V$

Have natural transformations

$$(g\theta) \xrightarrow{\psi} g\theta g\theta \qquad \text{From } id \to \theta g$$

$$(g\theta) \xrightarrow{\varepsilon} id$$

$$\tilde{g}V\left([n+1] \xrightarrow{s_i} [n]\right) : \underbrace{(g\theta)\circ\cdots\circ(g\theta)}_{n+1} V \xrightarrow{(g\theta)^i \psi (g\theta)^{n-i}} \underbrace{(g\theta)\cdots(g\theta)}_{n+2}$$

$$\tilde{g}V\left([n-1] \xrightarrow{d_i} [n]\right) : \underbrace{(g\theta) \quad (g\theta)}_{n+1} V \xrightarrow{(g\theta)^i \varepsilon (g\theta)^{n-i}} \underbrace{(g\theta\cdots g\theta)}_{n \text{ times}}$$

"Simplicial object associated to a comonad"

$$\underset{\sim}{R}X : \Delta \longrightarrow sSet$$

$$H^*(\underset{\sim}{R}X) : \Delta \xrightarrow{\underset{\sim}{R}X} sSet \xrightarrow{H^*} \mathcal{V}^{op}$$

Equivalently $H^*\underset{\sim}{R}X : \Delta^{op} \longrightarrow \mathcal{V}$

$$\boxed{H^*\underset{\sim}{R}X = \tilde{g}V}$$

Conclude: $\pi^s \pi_t \mathrm{Map}(Z, \underset{\sim}{R}X)$

$$= \mathrm{Hom}_{\mathcal{K}}(\widetilde{go}^s H^*X, \Sigma^t H^*Z^+)$$

$\widetilde{go}^s H^*X$ is the s^{th} term of the resolution of H^*X determined by $\widetilde{go}^\circ$

Write: $\mathrm{Hom}_{\mathcal{K}}(\widetilde{go}^s H^*X, \Sigma^t H^*Z^+)$

$$= \mathrm{Ext}^s(H^*X, \Sigma^t H^*Z^+)$$

- We will justify this notation

Unstable Adams spectral sequence:

$$\mathrm{Ext}^s_K(H^*X, \Sigma^t H^* Z^t) \Rightarrow \pi_{t-s} \mathrm{Map}(Z, R_\infty X)$$

$$\pi_{t-s} \mathrm{Map}(Z, R_\infty X) = \pi_{t-s} \mathrm{Map}(Z, X)$$

X nilpotent