- PS 8 & suggestions for final paper posted on website

**L24 – 4/8/13**
Arithmetic square
Miller 1.5

$X \in sSet$, $R$ ring

$X \to L_R X$ localization, characterized as terminal $H_*(-;R)$-equiv

i.e.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d, dashed, "\text{unique up to htpy}"] \\ & L_R X \end{tikzcd}$$

s.t. $H_*(f, R)$ iso

$$\begin{tikzcd} & L_R X \arrow[d] \\ X \arrow[ur] \arrow[r] & R_\infty X = \mathrm{Tot}\, \underset{\sim}{RX} \end{tikzcd}$$

Last time

Prop: $X \to Y$ $H^*(-,R)$-equiv

$\Rightarrow$ $\mathrm{Map}_*(Y, R_\infty W) \simeq \mathrm{Map}_*(X, R_\infty W)$

(True for any $R$ – we proved for $R = \mathbb{Z}$ or field)

Categorical aside: Fiber product:

Given

$$\begin{tikzcd} & X \arrow[d, "\pi_X"] \\ Y \arrow[r, "\pi_Y"'] & S \end{tikzcd}$$

the fiber product $X \times_S Y$ is $\mathrm{Hom}(Z, X \times_S Y)$

$= \{ (f_X, f_Y) \in \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y) \mid \pi_X f_X = \pi_Y f_Y \}$

$= \mathrm{Hom}(Z, X) \times_{\mathrm{Hom}(Z,S)} \mathrm{Hom}(Z, Y)$

e.g.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \\ * \arrow[r] & B \end{tikzcd}$$

$F$ = fiber product $= * \times_B E$

Fiber square is

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

fiber square up to htpy.

Algebraic aside: $\mathbb{Z}_{(p)} = \mathbb{Z}[\frac{1}{s} : s \neq np]$

$\mathbb{Z}_p = \mathbb{Z}_p^\wedge = \varprojlim \mathbb{Z}/p^n$ p-adic integers

$\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$ p-adic rationals

$\hat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N = \prod_p \mathbb{Z}_p^\wedge$

# Arithmetic square D. Sullivan

Bousfied, Dror, Dwyer, Kan

**obs from algebra:**

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r] \arrow[d] & \hat{\mathbb{Z}}_p \arrow[d] \\ \mathbb{Q} \arrow[r] & \mathbb{Q}_p \end{tikzcd}$$

is a fiber square

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \hat{\mathbb{Z}} = \prod \hat{\mathbb{Z}}_p \arrow[d] \\ \mathbb{Q} \arrow[r] & \hat{\mathbb{Z}} \otimes \mathbb{Q} \end{tikzcd}$$

← finite Adeles

is a fiber square

For any $M$ f.g. module

$$\begin{tikzcd} M_{(p)} \arrow[r] \arrow[d] & \hat{M}_p \arrow[d] \\ M_{\mathbb{Q}} \arrow[r] & \hat{M}_{\mathbb{Q}} \end{tikzcd}$$

is a Fiber Square

To simplify notation, let $X_R = L_R X$

<u>Thm</u> (Dror-Dwyer-Kan) Let $X$ be nilpotent. Then up to htpy there is a fiber square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_{p \text{ prime}} X_{\mathbb{Z}/p} \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left( \prod_p X_{\mathbb{Z}/p} \right)_{\mathbb{Q}} \end{tikzcd}$$

<u>ref</u>: "An Arithmetic square for virtually nilpotent spaces"

includes any space w/ finite $\pi_1$

Furthermore, the localizations can be replaced by completions

A hint of proof: Functoriality $\Rightarrow \exists$ square

Form pullback, obtain

$$\begin{tikzcd}
X \arrow[dr, "\phi"'] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& W \arrow[r] \arrow[d] & \prod X_{\mathbb{Z}/p} \arrow[d] \\
& X_{\mathbb{Q}} \arrow[r] & \left(\prod X_{\mathbb{Z}/p}\right)_{\mathbb{Z}}
\end{tikzcd}$$

$X \cong X_{\mathbb{Z}}$, so it suffices to show $H_*(\phi, \mathbb{Z})$ iso.

$\Rightarrow$ it suffices to show $H_*(\phi, \mathbb{Z}/p)$ and $H_*(\phi, \mathbb{Q})$ are iso

$\Rightarrow$ suffices to show $H_*(W \to \prod X_{\mathbb{Z}/p}; \mathbb{Z}/p)$ and $H_*(W \to X_{\mathbb{Q}}; \mathbb{Q})$ iso

work to show this.

<u>Thm</u> (Miller w/ thanks to Bousfield)

Let $Z$ be connected s.t.

$$\widetilde{H}_*(Z; \mathbb{Z}[\tfrac{1}{p}]) = 0$$

and $X$ nilpotent (and "Kan"). Then

$$X \to (\mathbb{Z}/p)_\infty X$$

induces a weak equivalence

$$\mathrm{Map}_*(Z, X) \to \mathrm{Map}_*(Z, (\mathbb{Z}/p)_\infty X)$$

insert proof

<u>Cor</u>: The above holds for $W = BG$ with $G$ a p-group.

Sullivan's conjecture concerns $X^G \to X^{hG}$. In case of trivial action is statement

$$\mathrm{Map}_*(BG, X) \simeq *$$

(Thm of H. Miller)

Cor above reduces problem to $(\mathbb{Z}/p)_\infty X$ under the assumption that $X$ nilpotent.

Miller takes $X$ f.d. CW-complex, and reduces to case of universal cover, which is nilpotent.

Upshot: We wish to understand

$$\mathrm{Map}(Z, (\mathbb{Z}/p)_\infty X)$$

Insert above:

Pf: $\mathrm{Map}_*(Z, \text{arithmetic square})$ gives fiber square

Let $R = \mathbb{Q}$ or $\mathbb{Z}/\ell$. Hyp $\Rightarrow \tilde{H}_*(Z,R)=0$

We had a lemma on Friday, saying

$$\tilde{H}_*(Z,R)=0 \Rightarrow \mathrm{Map}_*(Z, R_\infty X)\simeq *$$

$\Rightarrow$ only non-contractible elts of $\mathrm{Map}_*(Z, \text{arithmetic square})$

are $\mathrm{map}_*(Z,X) \longrightarrow \mathrm{map}_*(Z, (\mathbb{Z}/p)_\infty X)$ $\square$

$\rightsquigarrow$

## Unstable Adams spectral sequence

identify as derived functors

$$\pi^s \pi_t \mathrm{Map}_*(Z, R^n X)_\varphi \Longrightarrow \pi_* \mathrm{Map}(Z, R_\infty X)$$

## Non-abelian homological algebra (Quillen)

- Many categories have natural analogues of concepts htpy thy spaces

e.g. many categories $\mathcal{C}$ have a natural notion of weak equivalence.

- leads to notion of homotopy category

$$\mathrm{Ho}\,\mathcal{C} = \mathcal{C}[W^{-1}]$$

invert all weak equivalences

- e.g. Abelian category $\mathcal{A} \rightsquigarrow Ch_+(\mathcal{A})$ quasi-iso
- Many functors of interest do not preserve natural notion of weak equivalence

Fiber product of spaces

e.g.

$$\begin{array}{ccc} & & * \\ & & \downarrow \\ * & \to & X \end{array}$$

e.g. $Ch_+(\mathbb{Z})$

$$\begin{array}{c} \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \\ \text{quasi-iso} \\ \to 0 \to \mathbb{Z}/p \end{array} \qquad \overset{\deg 0}{\otimes \mathbb{Z}/p}$$

$\rightsquigarrow$ Approximate such a functor by a w.e. preserving one

Def: (Derived functor) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor b/w categories w/ notions w.e.

A right derived functor $RF$ is

$$\mathcal{C} \xrightarrow{RF} \mathcal{D}$$

preserving weak equivalences together with $F \Rightarrow RF$ and universal with this property. To state universal formally:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \arrow[d, "\delta"] \\ \mathrm{Ho}\,\mathcal{C} \arrow[r] & \mathrm{Ho}\,\mathcal{D} \end{tikzcd}$$

$$\mathrm{Lan}_\gamma\, \delta \circ F$$

Riehl "Cat htpy thy" 2.1

Left Kan extn

$$F \overset{\lambda}{\Rightarrow} RF, \quad \delta \circ F \overset{\delta \circ \lambda}{\Rightarrow} \delta \circ RF = \delta \circ \mathrm{Lan}_\gamma\, \delta \circ F$$

is $\delta \circ F \Rightarrow \mathrm{Lan}_\gamma\, \delta \circ F$

In practice:

$c \in \mathrm{ob}(\mathcal{C})$ $\qquad c \to I$

I resolution (injective)
or fibrant replacement.

**Ex** $\mathrm{Ext}_{\mathcal{K}}(-,-)$
⇝ Unstable Adams.

**Ex**: $\mathrm{Alg}_a \xrightarrow{\text{indec}}$ vector spaces

augmented alg

$$A \longmapsto I/I^2$$

augmented alg

$$I \to A \to k$$

⇝ André–Quillen coh

**Ex** $\mathcal{K}_a \to \mathcal{U}$ indec functor

studied by Miller, Goerss