Work in sSets

Start w/ Omar's remark

L23 – 4/5/13

$R_\infty X$ as $L_R X$

Localization

- Let R be an abelian group.
- Consider all maps $f: X \to Y$ s.t. $H_*(f;R): H_*(X;R) \to H_*(Y;R)$ is an equivalence. Call such an f an R-homology equivalence.
- Def: $X \to L_R X$ is a localization if $\forall$ $X \xrightarrow{f} Y$ s.t. $H_*(f,R)$ is an iso, there is a map $Y \xrightarrow{g} L_R X$ unique up to homotopy s.t.

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[ur] \arrow[r] & L_R X \end{tikzcd}$$

commutes.

$\Rightarrow$ $L_R X$ unique up to htpy.

Large picture: it is useful to work in a model category (= category equipped w/ structure to allow one to do htpy thy) with R-homology equivalences inverted. $L_R X$ is "fibrant replacement".

- $R_\infty X$ is a first approximation to $L_R X$. They are not always equal! ex: $\mathbb{RP}^2$, $R=\mathbb{Z}$

Thm: If $X \to R_\infty X$ is $H_*(\,;R)$-equivalence, then $X \to R_\infty X$ is the localization

Lemma: If $X \to Y$ is $H_*(-;R)$-equivalence, then the induced map

$$R_\infty X \to R_\infty Y \text{ is a weak equivalence.}$$

Pf: $X \to Y$ induces
$\underset{\sim}{R}X \to \underset{\sim}{R}Y$

We showed $\pi_*(RX) \cong H_*(X, R)$.

Thus hyp $\Rightarrow RX \to RY$ is a weak equiv.

$$\Rightarrow \quad R^{n+1}X \to R^{n+1}Y \quad \text{''}$$
$$\quad \| \qquad\qquad \|$$
$$\underset{\sim}{R}X \qquad \underset{\sim}{R}Y$$

Let $E^r_{-s,t}(X)$ denote spectral sequence correspond. to $R_\infty X = \varprojlim \mathrm{Tot}_m \underset{\sim}{R}X$

By last time, $E^2_{-s,t}(X) = \pi^s \pi_t \underset{\sim}{R}X$

$$\Rightarrow \quad E^2_{-s,t}(X) \to E^2_{-s,t}(Y) \text{ is iso}$$

By Bousfield mapping lemma, $R_\infty X \to R_\infty Y$ w.e. $\square$

(we're sweeping a connectedness issue under rug, but it can be dealt with)

Thus

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \arrow[dl] \\ R_\infty X \arrow[r, "R_\infty f", "\cong"'] & R_\infty Y \end{tikzcd}$$

<u>Lemma</u>: Suppose $Z \in sSet$ s.t. $\tilde{H}_*(Z;R)=0$.

Let $W \in sSet$, R = field ← This hypothesis is unnecessary

$$\mathrm{Map}_*(Z, R_\infty W] = *$$ ← ptd maps

Pf: Define $\mathrm{Map}_*(Z, R\underset{\sim}{X}) \in \mathrm{Fun}(\Delta, sSet)$

by $\mathrm{Map}_*(Z, R\underset{\sim}{X})[n] = \mathrm{Map}_*(Z, R\underset{\sim}{X}[n]) \in sSet$

(give q-simplices)

$$\mathrm{Map}_*(Z, R_\infty W) = \mathrm{Map}_*(Z, \mathrm{Map}(\underset{\sim}{\Delta}, R\underset{\sim}{X}))$$

$$= \mathrm{Map}(\underset{\sim}{\Delta}, \mathrm{Map}_*(Z, R\underset{\sim}{X})) = \mathrm{Tot}\, \mathrm{Map}_*(Z, R\underset{\sim}{X})$$

$E^2_{-s,t} = \pi^s \pi_t \mathrm{Map}_*(Z, R\underset{\sim}{X})$. By Bousfield connectivity lemma, it suffices to see $E^2_{-s,t} = *$

$$\pi_t \operatorname{Map}_*(Z, \underset{\sim}{R}X(n)) = \operatorname{Map}_*(S^t \wedge Z, \underset{\sim}{R}X(n))$$

We showed that $\overset{\underset{\sim}{R}X(n)}{\overset{\shortparallel}{R(R \cdots R X)}}$ is a product of Eilenberg MacLane spaces $K(n, R)$

Since $\tilde{H}(Z, R) = 0$, $\tilde{H}(S^t \wedge Z, R) = 0$.

$$\Rightarrow \pi_t \operatorname{Map}_*(Z, \underset{\sim}{R}X(n)) = 0. \qquad \square$$

<u>Cor</u>: If $f: X \to Y$ is an R-homology equivalence, then

$$[Y, R_\infty W] \longrightarrow [X, R_\infty W]$$

is a bijection.

Pf: It is useful to replace $X \to Y$ by an inclusion "cofibrant replacement." "mapping cylinder"

$x \longmapsto x \times 0$

$$X \longrightarrow X \times [0,1] \coprod_{X \times \{1\}} Y = \mathrm{Cyl}(f) \simeq Y$$

$$\frac{\mathrm{Cyl}(f)}{X}$$

$$X \longrightarrow X \times \{0,1\} \coprod_{X \times \{1\}} Y \longrightarrow C(X \xrightarrow{f} Y)$$

$$C(X \xrightarrow{f} Y) = \frac{X \times [0,1]}{X \times 0} \coprod_{X \times \{1\}} Y = \frac{\mathrm{Cyl}(f)}{X}$$

LES $(X, \mathrm{Cyl}(f))$ in H_*
shows $H_*(C(f), R) = 0$

Gives Fibration

$$\mathrm{Map}_*(C(f), R_\infty W) \longrightarrow \mathrm{Map}(\mathrm{Cyl}(f), R_\infty W) \simeq \mathrm{Map}(Y, R_\infty W)$$

$$\downarrow \star$$

$$\mathrm{Map}(X, R_\infty W)$$

By previous lemma $\pi_* \mathrm{Map}_*(C(f), R_\infty W) = 0$

$\square$

$\Rightarrow$ $\star$ is π_*-iso

<u>Pf thm</u>: Take $W = X$ in the corollary.

__Def__: fiber product.

Let $X_R = L_R X$

__Arithmetic Square__ Let X be nilpotent.

Then up to htpy there is a fiber square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod\limits_{p \text{ prime}} X_{\mathbb{Z}/p} \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod\limits_{p} X_{\mathbb{Z}/p}\right)_{\mathbb{Q}} \end{tikzcd}$$

reference: Dror, Dwyer, Kan

"An Arithmetic square for virtually nilpotent spaces"

Furthermore, the localizations can be replaced by completions

__Thm__ (Miller w/ Bousfield) Let Z be connected s.t. $\tilde{H}_*(Z; \mathbb{Z}[1/p]) = 0$ and

let X be nilpotent (and "Kan")

Then $X \to (\mathbb{Z}/p)_\infty X$ induces

a weak equivalence

$$\mathrm{Map}_*(Z, X) \longrightarrow \mathrm{Map}_*(Z, (\mathbb{Z}/p)_\infty X)$$

Cor: The above holds for $W = BG$ for G a p-group

- Sullivan's conjecture $X^G \to X^{hG}$ in case of trivial action is statement

$$\mathrm{Map}_*(BG, X) \simeq *$$

This corollary reduces problem to $(\mathbb{Z}/p)_\infty X$ (assuming X nilpotent)

Pf: Map(Z, arithmetic square) gives a fiber square

Let $R = \mathbb{Q}$ or $\mathbb{Z}/\ell$. Hyp $\Rightarrow \tilde{H}^*(Z,R) = 0$

By lemma, $\mathrm{Map}_*(Z, R_\infty W) \simeq *$

Only nonvanishing elts of square

$$\mathrm{map}_*(Z, X) \longrightarrow \mathrm{map}_*(Z, \mathbb{Z}/p^\wedge_\infty X) \qquad \square$$