Localization

- Let R be an abelian group.
- Consider all maps $F: X \to Y$ s.t. $H_*(F; R): H_*(X; R) \to H_*(Y; R)$ is an equivalence. Call such an F an R-homology equivalence.

**Def:** $X \to L_R X$ is a localization if $\forall X \to Y$ s.t. $H_*(F; R)$ is an iso, there is a map $Y \to L_R X$ unique up to homotopy s.t.

$$
\begin{array}{c}
x \\
\downarrow \\
L_R X
\end{array}
\Rightarrow Y
\begin{array}{c}
x \\
\downarrow \\
L_R X
\end{array}
$$

Commutes.

$\Rightarrow L_R X$ unique up to htpy.

Large picture: it is useful to work in a 1-category with R-homology equivalences inverted. $L_R X$ is “fibrant replacement.”

- $R \to X$ is a first approximation to $L_R X$. They are not always equal!

**Thm:** If $X \to R \to X$ is $H_*(\cdot; R)$-equivalence, then $X \to R \to X$ is the localization

**Lemma:** If $X \to Y$ is $H_*(\cdot; R)$-equivalence, then the induced map $R \to X \to R \to Y$ is a weak equivalence.
If \( X \to Y \) induces \( RX \to RY \)

We showed \( \pi_* (RX) \cong H_* (X, R) \).

Thus \( RX \to RY \) is a weak equiv.

\[ \Rightarrow \ R^{n+1}X \to R^{n+1}Y \]

\[ \cong \ R^{n+1}X \cong R^{n+1}Y \]

Let \( E_{-s,t} (X) \) denote spectral sequence converging to \( R_{\infty}X = \lim_{\to m} \text{Tot}_{m} RX \)

By last time, \( E_{-s,t}^{2} (X) = \pi^{s} \pi^{t} RX \)

\[ \Rightarrow \ E_{-s,t}^{2} (X) \to E_{-s,t}^{2} (Y) \] is iso

By Bousfield mapping lemma, \( R_{\infty}X \to R_{\infty}Y \) we...
(we’re sweeping a connectedness issue under the rug, but it can be dealt with)

Thus \( f : X \to Y \)

\[ \begin{array}{c}
\downarrow \\
\cong \\
R_0X \cong X \\
R_0Y \cong Y
\end{array} \]

Lemma: Suppose \( Z \in \mathbf{sSet} \) st. \( \tilde{H}_*(Z; R) = 0 \)

Let \( W \in \mathbf{sSet}, \; R = \text{field} \)

\[ \text{pf: Define } \text{Map}(Z, R_X) \in \text{Fun}(\Delta, \mathbf{sSet}) \]

by \( \text{Map}(Z, R_X)(n) = \text{Map}(Z, R_X^n) \in \mathbf{sSet} \)

(give \( n \)-simplices)

\[ \text{Map}(Z, R_{\infty}W) = \text{Map}(Z, \text{Map}(\Delta, R_X)) \]

\[ = \text{Map}(\Delta, \text{Map}(Z, R_X)) = \text{Tot} \text{Map}(Z, R_X) \]

\[ E^2_{s,t} = \prod^s \prod^t \text{Map}(Z, R_X) \]. By Bousfield connectivity
Lemma: it suffices to see $E^{2}_{S^{+}Z, R^{X}Cn} = ^{\ast}\Pi_{+}\text{Map}^{\ast}(Z, R^{X}Cn) = \text{Map}^{\ast}(S^{+}Z, R^{X}Cn)$.

We showed that $R^{X}R\ldots R^{X}$ is a product of Eilenberg Maclane spaces $K(n, R)$.

Since $\tilde{H}(Z, R) = 0$, $\tilde{H}(S^{+}Z, R) = 0$.

$$\Rightarrow \Pi_{+}\text{Map}^{\ast}(Z, R^{X}Cn) = 0.$$ \hfill \Box

Cor: If $f: X \to Y$ is an $R$-homology equivalence, then

$$[Y, R_{\ast}W] \to [X, R_{\ast}W]$$

is a bijection.

Proof: It is useful to replace $X \to Y$ by an inclusion "cofibrant replacement." "Mapping cylinder"
\[
X \rightarrow X \times \text{cof} \quad Y = \text{cyl}(f) \sim Y
\]

\[
X \times X \overset{\text{fib}^2}{\longrightarrow} Y \quad C(f) \rightarrow Y
\]

LES \; (X, \text{cyl}(f)) \; \text{in} \; H_*

Shows \; H_*(C(f), R) = 0

Gives Fibration

\[
\text{Map} (C(f), R_\infty W) \rightarrow \text{Map} (\text{cyl}(f), R_\infty W) \cong \text{Map}(Y, R_\infty W)
\]

By previous lemma \; \text{Map}(C(f), R_\infty W) = 0

\Rightarrow \bigcirc \; \text{is } \mathbb{T}_*\text{-iso}

Pf: \text{Take } W = X \text{ in the corollary.}
Def: Fiber product. Let \( X_p = L_p X \)

Arithmetic Square Let \( X \) be nilpotent.

Then up to htpy there is a fiber square

\[
\begin{array}{ccc}
X & \rightarrow & \prod_{p \text{ prime}} X_{2/p} \\
\downarrow & & \downarrow \\
X_\mathbb{Q} & \rightarrow & (\prod_{p \text{ prime}} X_{2/p}) \mathbb{Q}
\end{array}
\]

Reference: Dror, Dwyer, Kan

"An Arithmetic square for virtually nilpotent spaces"

Furthermore, the localizations can be replaced by completions by Bousfield

Thm (Miller) Let \( Z \) be connected
\[ \tilde{\pi}_*(\mathbb{Z};\mathbb{Z}[\frac{1}{p}]) = 0 \text{ and let } X \text{ be nilpotent (and "Kan")}\]

Then \[ \mathbb{Z} \to (\mathbb{Z}/p) \otimes X \] induces a weak equivalence

\[ \text{Map}_*(\mathbb{Z}; X) \to \text{Map}_*(\mathbb{Z}/(\mathbb{Z}/p)); X) \]

Cor: The above holds for \( W = B\mathcal{G} \) for \( G \) a \( p \)-group

- Sullivan's conjecture in case of trivial action is statement

\[ \text{Map}_*(B\mathcal{G}, X) = * \]

This corollary reduces problem to \( (\mathbb{Z}/p) \otimes X \) (assuming \( X \) nilpotent)
pf: Map(\mathbb{Z}, \text{arithmetic square}) gives a fiber square

Let R = \mathbb{Q} or \mathbb{Z}/l. \quad \text{Map}^* = \bigoplus \mathbb{Z}/l \rightarrow \mathbb{Z}/l \rightarrow \mathbb{Z}/l \rightarrow \mathbb{Z}/l \rightarrow \mathbb{Z}/l

By lemma, Map^*(\mathbb{Z}, R \otimes W) = \ast

Only nonvanishing els of square

\text{Map}^*(\mathbb{Z}, X) \rightarrow \text{Map}^*(\mathbb{Z}, \mathbb{Z}/l \rightarrow X) \rightarrow \cdots