L21 – 4/1/13
$\mathrm{Map}(-, RX)$
&
Tot_m

$sSet$ = cat of simpl. sets $= \mathrm{Fun}(\Delta^{op}, Set)$

$X, Y \in sSet$, form $X \times Y \in sSet$ $(X \times Y)[n] = X[n] \times Y[n]$

cat product. Note $C_*(\mathbb{X} \times \mathbb{Y}) = C_*(\mathbb{X}) \times C_*(\mathbb{Y})$ $\quad \mathbb{X}, \mathbb{Y} \in Top$

b/c C_* is a right adjoint it preserves Π & lim

$s Ab$ = " simp abelian groups $= \mathrm{Fun}(\Delta^{op}, Ab)$ also have product

Ch_+ = cat of chain cplxs of form $A_0 \xleftarrow{d_1} A_1 \leftarrow \dots$ "

Last time:

$$s Ab \xrightarrow{N} Ch_+$$

$N(B) \in Ch_+$ defined

$$N(B)_m = \bigcap_{i=0}^{m-1} \ker d_i$$

$$N(B)_{m-1} \xleftarrow[(-1)^m d_m]{} N(B)_m$$

Defined to have the property $\pi_n B = H_n N(B)$

$\Rightarrow N$ sends a weak equivalence to H_* iso

Def: equivalence relation on Ch_+ generated by H_* iso is called "quasi-isomorph.

Note: $N(B_1 \times B_2) = N(B_1) \times N(B_2)$

Dold-Kan theorem N has an inverse functor Γ

$$\begin{tikzcd} sAb \arrow[r, shift left, "N"] & Ch_+ \arrow[l, shift left, "\Gamma"] \end{tikzcd}$$

Reference: Goerss, Jardine "Simplicial homotopy theory"

$X \in sSet$, R ring

$RX \in sAb$ defined $(RX)[n] = R\,X[n]$ meaning free R-mod w/ basis $X[n]$

Prop: Let $R = \mathbb{Z}$ or a field.

Then $RX \cong \prod_n K(H_n(X,R), n)$

start ✪

Pf: Since N preserves products, so does Γ.

It suffices to show
$$N(RX) \underset{\nwarrow\ \text{quasi-iso}}{\simeq} \prod_{n=0}^{\infty} A(n)$$

where $A(n) \in Ch_+$ is s.t.

$$H_* A(n) = \begin{cases} H_n(N(RX)) & * = n \\ 0 & \text{otherwise} \end{cases}$$

✪ Last time,
$$\underset{\pi_n RX}{\underset{\cong}{H_n\big(N(RX)\big)}} \cong H_n(X,R)$$

In fact, We can take $A(n) = \Sigma^n H_n(N(RX))$ $\leftarrow 0 \leftarrow H_n(N(R(X)))\ \overset{0}{\swarrow}$

It is true more generally that for any $A \in Ch_+$
$$A \simeq \prod_n \Sigma^n H_n(A).$$

To see this:

Case 1: R a field

Choose cycles in A_n representing a basis of $H_n(A)$.

This defines a map

$$\Sigma^n H_n(A) \longrightarrow A.$$

These maps combine to form

$$\coprod \Sigma^n H_n(A) \longrightarrow A$$

a chain map b/c $d=0$ on l.h.s. and maps to elts w/ 0-boundary on r.h.s.

This is H_* iso as desired.

case 2: $R = \mathbb{Z}$.

Can Choose a presentation

$$0 \leftarrow H_n A \leftarrow R^{m_n} \leftarrow R^{o_n} \leftarrow 0$$

Let $z_n \in Ch_+$ be defined

$$\begin{tikzcd}
\cdots & 0 & R^{m_n} \arrow[l] & R^{o_n} \arrow[l] & \arrow[l] \cdots \\
& & \text{deg } n \arrow[u] & \text{deg } n+1 \arrow[u] &
\end{tikzcd}$$

Choose cocycles in A_n representing image of basis of R^{m_n}.

This defines

$$\begin{tikzcd}
R^{m_n} \arrow[r] \arrow[d, "\partial"] & A_n \arrow[d, "\partial"] \\
{} \arrow[r, "z"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
R^{\oplus m} \arrow[r, dashed] \arrow[d] \arrow[dr, bend left, "\text{image is}", "\text{coboundaries}"'] & A_{n+1} \arrow[d] \\
R^{m_n} \arrow[r] \arrow[d] & A_n \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
{} & {}
\end{tikzcd}$$

gives map $Z_n \to A$

Thus obtain map

$$\begin{tikzcd}
H^* \cong & \prod Z_n \arrow[dl] \arrow[dr] & \cong H^* \\
\prod \Sigma^n H_n A & & A
\end{tikzcd}$$

$\Rightarrow A \cong \prod \Sigma^n H_n A$

quasiiso $\square$

"objects of derived cat of modules /$\mathbb{Z}$ or field are equivalent to product of homology"

$$\Rightarrow \pi_n \mathrm{Map}(Y, RX) =$$

$$\pi_0 \mathrm{Map}\left(\Sigma^n Y, \prod_m K(H_m(X,R), m)\right)$$

$$= \prod_m H^m\left(\Sigma^n Y, H_m(X,R)\right)$$

$\Rightarrow$ We understand $\mathrm{Map}(-, RX)$.

The formulation of this understanding we will use for Sullivan's conjecture is: $R = \mathbb{Z}/2$

Let $\mathcal{K}$ be category of unstable algebras over $\mathcal{A}^*$ = steenrod alg

$$H^* : \underset{\text{sSet}}{\overset{\text{Top}}{\text{or}}} \longrightarrow \mathcal{K}$$

S'pose $H_*(X, R)$ is finite dimensional for all $*$

<u>Prop</u>: $\pi_n Map(Y, RX) = Hom_{\mathcal{K}}(H^*(RX), H^*(\Sigma^n Y))$

Recall $H^*(K(\mathbb{Z}/2, m))$

<u>Ex</u>: $R = S^m$ $\quad RX = K(\mathbb{Z}/2, 0) \times K(\mathbb{Z}/2, m)$

$$\begin{tikzcd}
\pi_0 Map(\Sigma^n Y, RX) \arrow[d, "="', "\text{by above}"'] \arrow[r] & Map_{\mathcal{K}}(H^*(K(\mathbb{Z}/2,0)) \otimes H^*(K(\mathbb{Z}/2,m)), H^* \Sigma^n Y) \arrow[dl, "\phi"] & f = f_1 \otimes f_2 \arrow[d, mapsto] \\
H^0(\Sigma^n Y) \times H^m(\Sigma^n Y) & & f_1(2_0) \times f_2(2_m)
\end{tikzcd}$$

ϕ is injective by calculation

<u>Pf</u> Let $E_m = H_m(X, R)$, $\quad K(E) = \prod_m K(E_m, m)$

$$H^*(K(E)) \cong \bigotimes_m H^*(K(E_m, m))$$

Have map $\leftarrow$ iso at H^* b/c $E = \prod_{m < * + 1} K(E_m, m) \times (*\text{-connected})$ using Künneth formula

$$E_m \cong \mathbb{Z}/2^{e_m} \Rightarrow K(E_m, m) \cong K(\mathbb{Z}/2, m)^{e_m}$$

$$\Rightarrow H^*(K(E_m, m)) = \bigotimes^{e_m} H^*(K(\mathbb{Z}/2, m))$$

$$\begin{tikzcd}
\pi_n Map(Y, RX) \arrow[r, phantom, "\cong"] \arrow[dd, "="', "\text{by previous}"'] & \pi_0 Map(\Sigma^n Y, RX) \arrow[d] \\
& Map_{\mathcal{K}}(H^*(RX), H^*(\Sigma^n Y)) \arrow[dl, "\phi"] \\
\prod^m H^m(\Sigma^n Y)^{e_m} & \overset{\shortparallel}{\bigotimes_m \bigotimes^{e_m} H^* K(\mathbb{Z}/2, m))}
\end{tikzcd}$$

ϕ injective as above

$\square$

This will become: $\pi_* \mathrm{Map}(Y, R_\infty X) \Leftarrow \mathrm{Ext}^s_{\mathcal{K}}(H^*(X), H^*(\Sigma^{t}_{+}))$

For $Y = BV$ $\quad V$ 2-group
Ext^s for $s > 0$ will vanish.

$$R_\infty X = \mathrm{Tot}\, \underset{\sim}{R}X \qquad \underset{\sim}{R}X \in \mathrm{Fun}(\Delta, sSet)$$

$$\underset{\sim}{R}X[n] = \underbrace{R \cdots R}_{n+1 \text{ times}} X$$

More generally, $\underset{\sim}{Y} \in \mathrm{Fun}(\Delta, sSet)$

$\mathrm{Tot}\, \underset{\sim}{Y} \in sSet \qquad \mathrm{Tot}\, \underset{\sim}{Y} := \mathrm{Map}(\underset{\sim}{\Delta}, \underset{\sim}{Y})$

$$\underset{\sim}{\Delta} \in \mathrm{Fun}(\Delta, sSet)$$

$$\underset{\sim}{\Delta}[n] = \Delta^n \quad n\text{-simplex}$$

$$\mathrm{Map}(\underset{\sim}{\Delta}, \underset{\sim}{Y})[q] = \mathrm{Mor}_{\mathrm{Fun}(\Delta, sSet)}(\underset{\sim}{\Delta} \times \Delta^q, \underset{\sim}{Y})$$

This is the same as

$$\mathrm{Tot}\, \underset{\sim}{Y} \underset{\text{subset satisfying compatibility}}{\subset} \prod_n \underbrace{\mathrm{Map}(\overset{\underset{\sim}{\Delta}[n]}{\overset{\shortparallel}{\Delta^n}}, \underset{\sim}{Y}[n])}_{sSet \text{ of maps}}$$

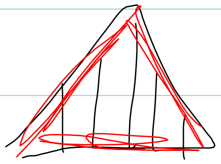
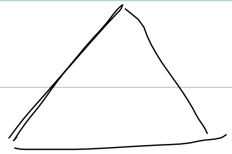
subset of a mapping space

(geometric realization is a quotient of a product)

Tot $\underset{\sim}{Y}$ is an inverse limit of a tower of fibrations:

For $X \in sSet$,

m-skeleton X = informally, subcomplex generated by ~~m~~-simplices ~~*~~ $< n$

e.g.  $sk_1 \Delta^2 =$ 

Formally, $Fun(\Delta^{op}, Set) \xrightarrow[T]{} Fun(\Delta^{op}_{\leq m}, Set)$

T admits a left adjoint, L

$sk_m = L \circ T : sSet \to sSet$

Let $\mathrm{sk}_m \underset{\sim}{\Delta} \in \mathrm{Fun}(\Delta, s\mathrm{Set})$ $\quad \mathrm{sk}_m \underset{\sim}{\Delta}[n] = \mathrm{sk}_m \Delta^n$

$$\mathrm{Tot}\, \underset{\sim}{Y} = \mathrm{Map}(\underset{\sim}{\Delta}, \underset{\sim}{Y}) = \varprojlim_m \mathrm{Map}(\mathrm{sk}_m \underset{\sim}{\Delta}, \underset{\sim}{Y})$$

Define $\mathrm{Tot}_m \underset{\sim}{Y} = \mathrm{Map}(\mathrm{sk}_m \underset{\sim}{\Delta}, \underset{\sim}{Y})$

$$\mathrm{Tot}\, \underset{\sim}{Y} = \varprojlim \mathrm{Tot}_m \underset{\sim}{Y}$$

$$\begin{tikzcd}
\mathrm{Tot}_m \underset{\sim}{Y} \arrow[r] \arrow[d, phantom, "||"] & \mathrm{Tot}_{m-1} \underset{\sim}{Y} \\
\mathrm{Map}_*(S^m, N\underset{\sim}{Y}[m]) &
\end{tikzcd}$$

$$N\underset{\sim}{Y}[m] = \underset{\sim}{Y}[m] \cap \bigcap_{i=0}^{m-1} \ker s_i$$