L20 - 3/29/13
$\underset{\sim}{R}X$

sSet = category of $\mathrm{Fun}(\Delta^{op}, \mathrm{Set})$

Substitute for Top

$$\mathrm{sSet} \underset{C_*}{\overset{|\cdot|}{\rightleftarrows}} \mathrm{Top}$$

compactly generated, Hausdorff

Quillen equivalence

$$R_\infty : \mathrm{sSet} \longrightarrow \mathrm{sSet}$$

$$R_\infty X = \mathrm{Tot}\, \underset{\sim}{R}X$$

where $\underset{\sim}{R}X$ is a cosimplicial simplicial set, i.e.

$$\underset{\sim}{R}X \in \mathrm{Fun}(\Delta, \mathrm{sSet})$$

The def of $\underset{\sim}{R}X$ is as follows:

$$\underset{\sim}{R}X[n] = \underbrace{RR\cdots R}_{n+1 \text{ times}} X$$

$$\underset{\sim}{R}X\left([n-1] \xrightarrow{d^i} [n]\right) = R^i \phi R^{n-i} \qquad \phi : \mathrm{Id} \to R$$

$$\underset{\sim}{R}X\left([n+1] \xrightarrow{s^i} [n]\right) = R^i \psi R^{n-i} \qquad \psi : RR \to R$$

Ex (not clear) $\mathbb{Z}_\infty X \simeq X$

if X is simply connected.

or more generally nilpotent meaning

- connected
- π_1 nilpotent
- $\pi_n X$ has finite $\pi_1 X$ filtration with $\pi_1 X$ acting trivially on Gr

Structure $R_\infty X$

Consider RX, $X \in sSet$

$$X[0] \underset{d_1}{\overset{d_0}{\leftleftarrows}} X[1] \overset{d_0, d_1, d_2}{\leftleftarrows\!\!\leftarrow} X[2] \leftarrow \cdots$$

$RX \in sSet$

$$RX[0] \underset{Rd_1}{\overset{Rd_0}{\leftleftarrows}} RX[1] \Lleftarrow$$

- For a simplicial abelian group B, define <u>associated chain complex</u>

$A(B_\cdot) \in Ch_+$ cat of chain complexes

$$A_0 \xleftarrow{d} A_1 \xleftarrow{d} A_2 \leftarrow \cdots$$

$$A(B) = B[0] \xleftarrow{d_0 - d_1} B[1] \leftarrow B[2] \leftarrow$$

$$\cdots B[n-1] \xleftarrow{\sum (-1)^k d_k} B[n] \leftarrow$$

Note: $H_*(|X|, R) = H_*(A(RX_\cdot))$
$$\begin{array}{c} H_*(|X|, R) \\ \| \\ H_*(X, R) \end{array}$$

- $\pi_n |X|$ can be computed in sSet

If X is "Kan"
$$\pi_n X = \frac{\{x \in X_n \mid d_i x = *\}}{x \sim x' \text{ if } \exists \; y \in X_{n+1} \text{ s.t. } d_n y = x, \; d_{n+1} y = x', \; d_i y = * \; i < n}$$

- any simplicial group is Kan
- Consider the simplicial abelian group B as a sSet

$$\pi_n B = \frac{\bigcap_i \ker d_i}{\mathrm{Image}\left(d_{n+1}\big|_{\bigcap_{i \le n} \ker d_i}\right)}$$

- This formula shows that $\pi_n B$ is also the homology of an associated chain complex

Explicitely: Define $N(B) \in Ch_+$

$$N(B)_n = B[n] \cap \ker d_0 \cap \dots \cap \ker d_{n-1}$$

$$\cdots \leftarrow N(B)_{n-1} \xleftarrow{(-1)^n d_n} N(B)_n \xleftarrow{(-1)^{n+1} d_{n+1}} \cdots$$

Then $H_n(N(B)) = \pi_n B$

Note that $N(B) \subset A(B)$

<u>Thm</u>: $N(B) \subset A(B)$ induces H_* iso.

<u>Pf</u> Filter $A(B)$

$$F^p A_n = \bigcap_{0 \le i < \min(n,p)} \ker d_i$$

$$\begin{array}{l} F^p A_n = N_n \text{ for } p \ge n \\ \cap \\ F^{p+1} A_n \\ \cap \\ \vdots \\ \cap \\ F^{-1} A_n = A_n \end{array}$$

It suffices to show $F^p \subset F^{p+1}$ is H_* iso.

Define inverse $F^p \xrightarrow{f} F^{p+1}$ up to chain htpy

$$f : F^p A(B) \to F^{p+1} A(B)$$

$$f(x) = \begin{cases} x & \text{if } x \in F^{p+1} \quad \begin{pmatrix} \text{e.g.} \\ n < p+1 \end{pmatrix} \\ x - s_p d_p x & \text{otherwise.} \end{cases}$$

useful simplicial identities:

$$d_m s_p = id \quad m = p, p+1$$

$$d_m s_p = s_p d_{m-1} \quad m > p+1$$

$$d_m s_p = s_{p-1} d_m \quad m < p$$

In particular, $d_p (x - s_p d_p x) =$

$$d_p x - d_p s_p d_p x =$$

$$d_p x - d_p x = 0.$$

$\Rightarrow$ Image$(f) \subset F^{p+1}$ as claimed.

$F^{p+1} \subset F^{p} \xrightarrow{f} F^{p+1}$ is identity by construction.

We claim that $F^{p} \xrightarrow{f} F^{p+1} \subset F^{p}$ is chain homotopic to id.

i.e. $\exists\ t : F^{p}_{*} \to F^{p}_{*+1}$

s.t. $dt - td = f - \mathrm{id}$

Define $t(x) = \begin{cases} 0 & x \in F^{p}A_n \quad n < p \\ (-1)^{p} S_p x & n \geq p \end{cases}$

$d = \sum_m (-1)^m d_m$

$$dtx = \begin{cases} 0 & n < p+1 \\ (-1)^{p} \sum (-1)^m d_m S_p x & n \geq p+1 \end{cases}$$

$$= (-1)^{p} \Big(\sum_{m<p} (-1)^m S_{p-1} d_m x + \sum_{m \geq p+1} (-1)^m S_p d_{m-1} x \Big)$$

$$= (-1)^p \sum_{m>p+1} (-1)^m S_p d_{m-1} x \qquad n \geq p+1$$

if $x \in F^p A_n$ $\quad n < p+1$

$\Leftrightarrow$

if $x \in F^p A_{n+1}$ $\quad n < p$

$\Updownarrow$

$$+\, dx = \begin{cases} 0 & \text{if } dx \in F^p A_n \quad n<p \\ (-1)^p S_p\, dx & \text{if } n \geq p+1 \end{cases}$$

$$= \begin{cases} 0 & n<p \\ (-1)^p S_p \sum_{m \geq p} (-1)^m d_m x & n \geq p+1 \end{cases}$$

$$\Rightarrow dt + td = \begin{cases} 0 & n < p+1 \\ (-1)^p (-1)^p S_p d_p x & n \geq p+1 \end{cases}$$

$$(-1)^p (-1)^p S_p d_p x = S_p d_p x = x - f$$

$\square$

<u>Cor:</u> $\pi_n B \cong H_n(A(B))$

$B = RX \Rightarrow \pi_n RX = H_n(X, R)$

Dold-Kan correspondence

N admits an inverse functor

$$sAb \underset{\Gamma}{\overset{N}{\rightleftarrows}} Ch_+ \qquad N(X \times Y) = N(X) \times N(Y)$$

$\Rightarrow$ same for Γ

Prop: Let $R = \mathbb{Z}$ or a field.

$$RX = \prod_{n \geq 0} K(H_n(X,R),n)$$

Pf: $N(RX) \in Ch_+$

Lemma: Let $R = \mathbb{Z}$ or a field, and Ch_+ denote chain cplxs over R. For any $A \in Ch_+$ there is a diagram of H_*-isos

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[dr] & \\
\prod_n \Sigma^n H_n A & & A
\end{tikzcd}$$

Pf: choose $0 \to R^{m_0} \to R^{n_0} \to M_0 \to 0$

$$\begin{tikzcd}
R^{m_0} \arrow[r] \arrow[d] & A_1 \arrow[d] \\
R^{n_0} \arrow[r] & A_0
\end{tikzcd}$$

Let $Z_0 = R^{m_0} \longrightarrow R^{n_0}$

continue

Let $Z = \prod Z_n$

Important hyp: R mod have fre res of length 2

Apply Lemma to $N(RX)$

Then apply Γ $\square$