L19 - R-completion construction & Dold-Kan

$\mathcal{G}$: Set $\longrightarrow$ Group

set $X \longmapsto$ free group with generators elts of X

$\mathcal{O}$: Group $\longrightarrow$ Set

group $G \longmapsto$ set of elts

$\mathcal{G}, \mathcal{O}$ are <u>adjoint functors</u> (D. Kan) $\underset{\text{group}}{Map(\mathcal{G}X, G)} = \underset{\text{set}}{Map(X, G)}$

Construction: adjoint functors + $\underset{\substack{\text{set,}\\ \text{space,}\\ \text{object of } \mathcal{C}\\ \text{simplicial set}}}{X}$ $\rightsquigarrow$ cosimplicial $\substack{\text{set,}\\ \text{space,}\\ \text{object of } \mathcal{C}\\ \\ \text{simplicial set}}$

Let's just forget about $\mathcal{C}$

$$\begin{tikzcd}
 & \mathcal{G}\mathcal{G}(X) \\
(\star) \quad X \arrow[r] & \mathcal{G}(X) \arrow[u, shift left, "d_0"] \arrow[u, shift right, "d_1"']
\end{tikzcd}$$

$X = \{ x_i \ : i = 1, 2, .. \}$

$\mathcal{O}\mathcal{G}(X) = \{ [x_i x_j \dots] \quad \}$

$d_0([x_1^2 x_2 x_3]) = [x_1^2 x_2 x_3]$ $\leftarrow$ generators of $\mathcal{G}(\mathcal{G}(X))$

$d_1([x_1^2 x_2 x_3]) = [x_1][x_1][x_2][x_3]$

Not so hard to see: $(\star)$ is equalizer diagram

$X = \mathcal{O}(G)$ $\qquad f: X \to \underset{\nwarrow\,\text{group}}{H}$

$$\begin{tikzcd}
 & \mathcal{G}\mathcal{G}(X) \arrow[dr, dashed, "\tilde{f}"] & \\
X \arrow[r] & \mathcal{G}(X) \arrow[u, "d_0"] \arrow[r, "f"'] & H
\end{tikzcd}$$

f group homomorphism $\Longleftrightarrow \tilde{f} d_1 = \tilde{f} d_0$

- encode info of group in free group
- resolve X. No need to stop at $\mathcal{G}\mathcal{G}(X)$

$$\begin{array}{l} X \\ \downarrow \\ \underbrace{\mathcal{G}(X) \rightrightarrows \mathcal{G}\mathcal{G}(X) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathcal{G}\mathcal{G}\mathcal{G}(X)}_{\text{cosimplicial object}} \cdots \end{array}$$

- This creates derived functors in non-abelian contexts!

Recall: Δ def $\quad [n] \in \mathrm{ob}(\Delta)$

$\mathcal{C}$ cat. Cosimplicial ob $\mathcal{C}$ is def

Def: $X \in \mathcal{C}$

$\mathcal{G}X \in \mathrm{Fun}(\Delta, \mathcal{C})$

$$(\mathcal{G}X)([k]) = \underbrace{\mathcal{G}\mathcal{G}\cdots\mathcal{G}}_{} X = \mathcal{G}^{k+1}X$$

k+1 times,
i.e. once for every elt [k]

$\phi : \mathrm{id} \to \mathcal{G}$

$\psi : \mathcal{G}\mathcal{G} \to \mathcal{G}$

$$(\mathcal{G}X)([k-1] \xrightarrow{d^i} [k]) = \mathcal{G}^i \phi \mathcal{G}^{k-i}$$

$$(\mathcal{G}X)([k+1] \to [k]) = \mathcal{G}^i \psi \mathcal{G}^{k-i}$$

<u>R-completion</u> R ring. set $\xrightarrow{\mathcal{G}}$ R-mod

$\mathcal{C}$ = S~~ets~~ simplicial sets = $\mathrm{Fun}(\Delta^{op}, \mathrm{set})$

$X \in \mathrm{sSet}$

$$RX = \Delta^{op} \xrightarrow{X} \mathrm{set} \xrightarrow{\mathcal{G}} R\text{-mod}$$

We have same structure as above.

Obtain:

$\underset{\sim}{RX} \in \mathrm{Fun}(\Delta, sSet)$ cosimplicial simplicial set

think cosimplicial space

Def

(R-completion of X)

$$X \to R_\infty X = \mathrm{Tot}\, \underset{\sim}{RX}$$

i.e. n-simplices of $R_\infty X$ form the set of maps of cosimplicial spaces

$$\underbrace{\Delta^n}_{n\text{-simplex}} \times \underset{\sim}{\Delta} \longrightarrow \underset{\sim}{RX}$$

cosimplicial space

$$\subset \prod_n \mathrm{Map}(\Delta^n, R^{n+1}X)$$

↑

It is subset of product of those tuples of maps whose restrictions are appropriate

Ex (not obvious) $\mathbb{Z}_\infty X \cong X$

if X is simply connected.

Or more generally *nilpotent* meaning

- connected
- $\pi_1 X$ nilpotent
- $\pi_n X$ has finite $\pi_1 X$ filtration with $\pi_1 X$ acting trivially on Gr

I) Structure of $R_\infty X$

II) Purpose of $R_\infty X$ (= capture info up to $H_*(-;R)$) equivalence)
— sometimes —

I) First step: X sSet, R = field or $\mathbb{Z}$

Let's look at RX simplicial R-mod

Prop: $$RX = \prod_{n\geq 0} K(H_n(X,R),n)$$

Consequence: $\mathrm{Map}(-, RX)$ is well-controlled. (Put after pf)

— Go to next —

To prove this: Dold-Kan correspondence

There are inverse functors

$$\begin{tikzcd} sAb \arrow[r, shift left, "N"] & Ch_+ \arrow[l, shift left] \end{tikzcd}$$

- insert above -

RX : 0 simplices | simplices

$$\begin{tikzcd} RX_0 & RX_1 \arrow[l, shift left, "d_0"'] \arrow[l, shift right, "d_1"] & {} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

- For a simplicial ab group $B_\cdot$, define <u>associated chain cplx</u> $A(B_\cdot)$

$$0 \leftarrow B_0 \xleftarrow{d_0 - d_1} B_1 \xleftarrow{d_0 - d_1 + d_2} B_2 \leftarrow \cdots$$

<u>Note</u>: $H_*(X, R) = H_*(A(RX_\cdot))$

- we can also consider $B_\cdot$ as a simplicial set. Any simplicial group has property "Kan"

$$\pi_n|B_\cdot| = \frac{\{x \in B_n \mid \partial_i x = id\}}{x \sim x' \text{ if } \exists\, y \in B_{n+1}}$$

s.t. $\partial_n y = x$, $\partial_{n+1} y = x'$, $\underset{i<n}{\partial_i y} = id$

This involves the theory where top is just replaced by cat of simpsets. Let's just gloss over the difference b/w Top & sSet

Write $\pi_n B.$ for $\pi_n |B.|$. There are notions fib, weak equivalence, etc. for sSet.

$$\text{Thus} \quad \pi_n RX = \frac{\bigcap_i \ker \partial_i}{\text{Image}\left(\partial_{n+1}\Big|_{\bigcap_{i\le n} \ker \partial_i}\right)}$$

Define $N(B.)$ a chain complex s.t.

$$N(B.)_n = B_n \cap \ker \partial_0 \cap \cdots \cap \ker \partial_{n-1}$$

$$\leftarrow N(B.)_{n-1} \xleftarrow{(-1)\partial^n} N(B.)_n \leftarrow$$

- We have made definitions s.t.

$$H_n(N(B_\cdot)) = \pi_n B$$

- $N(B_\cdot) \subset A(B_\cdot)$

Thm The inclusion $N(B_\cdot) \subset A(B_\cdot)$ induces an iso on H_*.

Cor: $\pi_* B_\cdot \xrightarrow{\cong} H_*(A(B_\cdot))$
Whence for $B = RX$, $\pi_n RX = H_n(X, R)$

Pf: $F^p A(B)_n = \bigcap_{0 \le i < \min(p,n)} \ker \partial_i$

$F^{-1} A(B) = A(B_\cdot)$
$F^m A(B)_n = N(B_\cdot)$ for $m \ge n$
$F^{m+1} \supset F^m$. It suffices to show this

inclusion is H_* iso.

Define $f^p : F^pA(B_\cdot) \longrightarrow F^{p+1}A(B_\cdot)$

$$f^p(x) = \begin{cases} x \in F^pA_n(B) & n < p+1 \\ x - s_p\partial_p x \in F^pA_n(B) & n \geq p+1 \end{cases}$$

$$\partial_p(x - s_p\partial_p x) = \partial_p x - \underbrace{\partial_p s_p}_{id} \partial_p x \; x = 0$$

map of chain complexes

$F^p \xrightarrow{f^p} F^{p+1} \subset F^p$ is identity.

It suffices to find a chain htpy b/w

$F^{p+1} \subset F^p \xrightarrow{f^p} F^{p+1}$ & id

$t^p : F^{p+1}_* \longrightarrow F^{p+1}_{*+1}$

$$t^p(x) = \begin{cases} 0 & x \in F^p A_n \quad n < p \\ (-1)^p s_p x & x \in F^p A_n \quad n \geq p \end{cases}$$

we wish to see

$$\partial t^p x + t^p \partial x = x - f^p x$$

$$\partial = \sum (-1)^n \partial_n$$

$\partial_m s_p$ = map induced by

$$\begin{tikzcd}
{[0, \ldots, n]} \arrow[d, "s_p"', "\text{p \& p+1 identified}"] \arrow[dd, bend right=60, "\text{p, p+1 identified missing m}"'] \\
{[0, \ldots, n-1]} \arrow[d, "\text{missing m in image}"] \\
{[0, \ldots, n]}
\end{tikzcd}$$

$s_p \partial_m$ = map induced by

$$\begin{tikzcd}
{[0, \ldots, n]} \arrow[d, "\text{missing m}"] \arrow[dd, bend right=60, "\text{identify p and p+1 missing m-1 in image}"'] \\
{[0, \ldots, n+1]} \arrow[d, "\text{identify p, p+1}"] \\
{[0, \ldots, n]}
\end{tikzcd}$$

$\Rightarrow \partial_m s_p = s_p \partial_{m-1}$

$m > p+1$

first leave m or m+1 out , then identify m&m+1

$$\partial_m s_m = id = \partial_{m+1} s_m$$

$$\partial_m s_p = s_p \partial_{m-1} \qquad m > p+1$$

$$\partial_m s_p = s_{p-1} \partial_m \qquad m < p$$

For $m < p$, $\partial_m = 0$ for $x \in F^p$

$$\Rightarrow \partial_m s_p x = s_{p-1} \partial_m x = 0$$

For $m = p$ & $p+1$ $\partial_m s_p = id$

$$\Rightarrow \partial (-1)^p s_p x = \sum_{m > p+1}^{n} (-1)^{m+p} \partial_m s_p x = \sum_{m>p+1}^{n} (-1)^{m+p} s_p \partial_{m-1} x$$

$$= -(-1)^p s_p \underset{\overset{\|}{\sum_{m>p}}}{\partial} x - (-1)^{p+1} (-1)^p s_p \partial_p x$$

$\square$