L17 3/15/13

PS 4 Q8 : changed n-k connected to n-k-1 connected

Thanks to I. Dai

PS 6 Q6 $H^1(G, \mathbb{F}_p) \to H^1(F, \mathbb{F}_p)$ surjective

will fix soon

Thanks to G. Lee

convergence of spectral sequence associated to a tower of fibrations

$$\to X_{n+1} \to X_n \to X_{n-1} \to$$
$$F_n \to X_n$$

Tower fib

$$X = \varprojlim X_n$$

$$\pi_i X_n^r = \mathrm{Image}(\pi_i X_{n+r} \to \pi_i X_n)$$

$$\pi_i F_n^r = \ker\left(\pi_i F_n \to \frac{\pi_i X_n}{\pi_i X_n^r}\right)$$?

action of ker $\pi_{i+1} X_{n-1} \to \pi_{i+1} X_{n-r-1}$

want: $\pi_{i+1} F^r_{n-1-r} \dashrightarrow \pi_{i+1} X_{n-1} \to \pi_i F_n$

choose lift

lifts differ by elts ker $\pi_{i+1} X_{n-1} \to \pi_{i+1} X_{n-1-r}$? =

r^{th} derived sequence
(consistent with exact couple notation except where non-abelian)

$$\cdots \to \pi_1 X_{n-r}^{(r)} \to \pi_1 X_{n-1-r}^{(r)} \to \pi_0 F_n^{(r)} \to \pi_0 X_n^{(r)} \to \pi_0 X_{n-1}^{(r)} \qquad (\star)$$

<u>Prop/def</u>: The extended htpy spectral sequence of tower of fibrations

$$E_r^{-s,s+i} = \pi_{s+i} F_s^{(r-1)} \qquad \text{for } s, i \geq 0$$

$$d_r^{-s,t} : E_r^{-s,t} \longrightarrow E_r^{-(s+r),\, t+r-1}$$

is $\pi_t F_s^{(r-1)} \to \pi_t X_s^{(r-1)} \dashrightarrow \pi_t X_{s+r-1} \to \pi_{t-1} F_{s+r}$

$d_r^{-s,t}$ is a homomorphism of ab groups $-s+t-1 \geq 2$

hom of groups with image in center $-s+t-1 \geq 1$

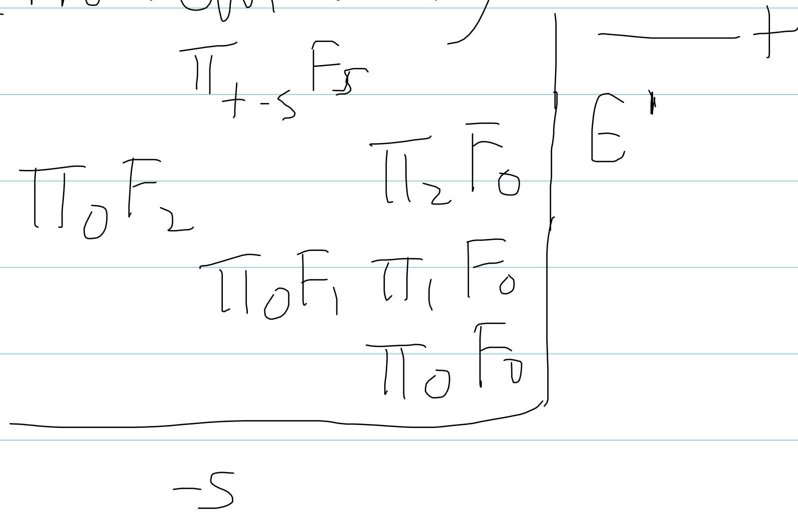
$d_r : E^r_{-s+r,\, s-r+1} \longrightarrow E^r_{-s,s}$ extends to an action

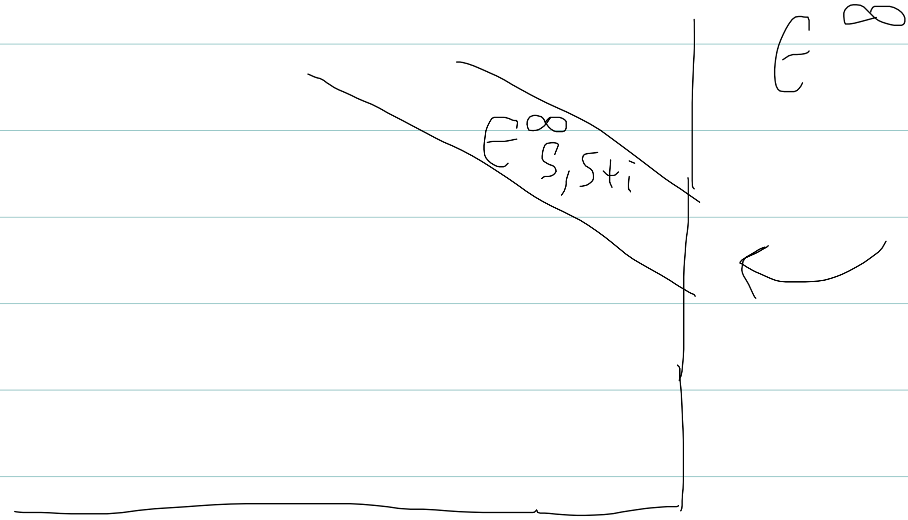
$E^{r+1}_{-s,t} = \ker d^r_{-s,t} \Big/ \text{action } d^r_{-s+r,\, t-r+1}$

define spectral sequence to mean this property

The fact that this is spectral sequence follows from exactness (★)

(Problem set)





gives information about $\pi_i X$

$\underline{Ex}$: $X_n = Map(EG^{(n)}, X)^G \leftarrow Map(EG^{(n)}/EG^{(n-1)}, X)^G$

$$\downarrow$$

$X_{n-1} = Map(EG^{(n-1)}, X)^G \ni$ constant map

$$\downarrow$$

$\prod_{t-n} F_n = \prod_{t-n} Map(EG^{(n)}/EG^{(n-1)}, X)^G$

$= \prod_{t-n} Map(\bigvee_{\substack{n \text{ cells} \\ \text{of nerve} = \\ G^{n+1}}} S^n, X)^G = Map(\bigvee_{\substack{n \text{ cells} \\ \text{of nerve} = \\ G^{n+1}}} S^t, X)^G$

$= \left(\bigoplus_{G^{n+1}} \pi_t(X)\right)^G$

$= Map\left(\mathbb{Z}[G^{n+1}], \pi_t(X)\right)^G$

$= C^n_{\substack{\text{local} \\ \text{coef}}}\left(BG, \pi_t(X)\right)$

$\Rightarrow E^1_{-n,t} = C^n_{\substack{\text{local} \\ \text{coef}}}(BG, \pi_t X)$

$$\begin{array}{ccccc} \pi_{t-n}\,\mathrm{Map}(EG^{(n)}/EG^{(n-1)}, X)^G & \to & \pi_{t-n}\,\mathrm{Map}(EG^{(n)}, X)^G & \to & \pi_{t-n-1}\,\mathrm{Map}(EG^{(n+1)}/EG^{(n)}, X)^G \\ \| & & & & \| \\ C^n(BG, \pi_t(X)) & & \xrightarrow{\quad d_1 \quad} & & C^{n+1}(BG, \pi_t(X)) \end{array}$$

boundary map
from group coh

$$E^2_{-n,t} = H^n(G, \pi_t X)$$

Convergence

$$E^r_{-n,t} \xleftarrow{d_r^{-n+r,\,t-r+1}} E_{-n+r,\,t-r+1}$$

$d_r^{-n+r,\,t-r+1} = 0$

for $r > n$

Define $E^\infty_{-n,t} = \bigcap_{r>n} E^r_{-n,t}$

view as
subset of
$E^{n+1}_{-n,t}$

Let $Q_n \pi_i X = \text{image}(\pi_i X \to \pi_i X_n)$

$$E^r_{-n,n+i} = \frac{\ker\left(\pi_i F_n \to \pi_i X_n / \pi_i X_n^r\right)}{\text{action}\left(\ker \pi_{i+1} X_{n-1} \to \underset{\overset{\|}{0}}{\pi_{i+1} X_{n-1-r}}\right)}$$

$$= \frac{\sim}{\text{action } \pi_{i+1} X_{n-1}} = \text{Image } \pi_i F_n \cap \pi_i X_n^r$$

$$= \left(\ker \pi_i X_n \to \pi_i X_{n-1}\right) \cap \left(\pi_i X_n^r\right)$$

Define $e^\infty_{-n,n+i} = \ker\left(Q_n \pi_i X \to Q_{n-1} \pi_i X\right)$

$$e^\infty_{-n,n+i} \subset E^\infty_{-n,n+i} = \left(\ker \pi_i X_n \to \pi_i X_{n-1}\right) \cap \left(\bigcap_r \pi_i X_n^r\right)$$

We want $=$. i.e. want $\bigcap_r \pi_i X_n^r = Q_n \pi_i X$

Here's what can go wrong

$\sigma \in \bigcap_r \prod_i X_n^r$ i.e. $\forall\ r\ \exists\ \sigma_{n+r} \in \prod_i X_{n+r}$

s.t. $\sigma_{n+r} \mapsto \overline{\sigma}$

i.e.

$$\begin{tikzcd} & X_{n+r} \arrow[d] \\ S^i \arrow[ur, bend left, "\sigma_{n+r}"] \arrow[r, "\sigma"] & X_n \end{tikzcd}$$

We would like:

$$\begin{tikzcd} & X_{n+r+1} \arrow[d] \\ S^i \arrow[ur, bend left, "\sigma_{n+r+1}"] \arrow[r, bend left, "\sigma_{n+r}"'] & X_{n+r} \end{tikzcd}$$

But we may not be able to arrange this:

Derived functor of limit

$\varprojlim^1$: Given $\cdots \to G_{n+1} \xrightarrow{p_{n+1}} G_n \to G_{n-1} \to \cdots$

$$0 \to \varprojlim_n \to \prod G_n \to \prod G_n \to \varprojlim^1 \to 0$$

composed of maps $\prod G_n \to G_i$ this is diffrnce

More generally if the G_n are groups

$$\prod G_n \circlearrowright \prod G_n$$

$$\begin{array}{cc} \prod G_n & \prod G_n \\ \cup & \cup \\ (g_1, \dots) & (x_1, x_2, \dots) \end{array}$$

$$g \cdot x = (g_1 x_1 j_2(g_2)^{-1}, g_2 x_2 j_3(g_3)^{-1}, \dots$$

$\varprojlim$ = stabilizer $(1,1,1,\dots)$

$\varprojlim^1 = \prod G_n /$ action

<u>Prop</u>: $* \to \{G_n'\} \to \{G_n\} \to \{G_n''\} \to *$

$1 \to \varprojlim \to \varprojlim$

Ex: $p^n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^n$

$$\lim^1 p^n\mathbb{Z} = \mathbb{Z}_p/\mathbb{Z}$$

Milnor exact sequence

Prop: $\varprojlim^1 \pi_{i+1} X_n \to \pi_i X \longrightarrow \varprojlim_n \pi_i X_n$

$$\varprojlim_n \pi_i X_n = \varprojlim Q_n \pi_i X$$

Pf: surjective, htpy lifting

identify kernel

Def: $\{E^r_{-s,t}\}$ converges completely to $\pi_i X$ $i \geq 1$ if

(i) $\varprojlim^1 \pi_i X = 0$ (i)' $\pi_i X = \varprojlim Q_n \pi_i X$

(ii) $E^\infty_{-n,n+i} = e^\infty_{-n,n+i} = \ker(Q_n\pi_i X \to Q_{n-1}\pi_i X)$

<u>Prop:</u> Let $\{X_n\}$ be a tower of fibrations. Then

$$\varprojlim_r{}^{1} E_r^{s,s+i} = *$$

$\forall s$

$$\varprojlim_r{}^{1} E_r^{s,s+i+1} = *$$

$\Rightarrow \{E_r\}$ converges completely

so $\pi_i X$ is inverse limit of epimorphisms with $\ker E_\infty^{s,s+i}$