Problem set 6 posted on problem sets page
Ex: Had Kummer map $X^G \to H^1(G, \mathbb{T}_1)$

$X = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$

$\mathbb{Z}/2 \cong X \quad \mathbb{Z}/2 = \mathbb{Z}/2$

by $\subset (x+iy) = x - iy$. This is $Gal(\mathbb{C}/\mathbb{R})$-action

\[ X^{\mathbb{Z}/2} \xrightarrow{K} H^1(\mathbb{Z}/2, \mathbb{R}/\mathbb{R}) \cong H^1(\mathbb{Z}/2, \mathbb{Z} \oplus \mathbb{Z}) \]

local coefs action is by

$\subset (1 \oplus 0) = 1 \oplus 0$

$\subset 0 \oplus 1$

write $\mathbb{Z} \oplus \mathbb{Z}(1)$

Take twist notation

We're about to make this part of a spectral sequence.

We'll get $\Pi_t(X^G, b) = H^0(\mathbb{Z}/2, \mathbb{Z} \oplus \mathbb{Z}(1))$

$= (\mathbb{Z} \oplus \mathbb{Z}(1))^{\mathbb{Z}/2} = \mathbb{Z}$

This looks good 😊
We're making a spectral sequence

\[ E^2 \]

\[ H^i(G, \pi_j X) \rightarrow \pi_{j-i}(X^{hG}) \]

In example: only non-vanishing \( \pi_\ast X \) is \( \pi_1 \)

\[ \pi_0(X^{hG}) = H^1(G, \pi_1) \quad \pi_1(X^{hG}) = H^0(G, \pi_1) \]

(hide convergence issues - which are a big deal)

Let's get spectral sequence:

\[ \rightarrow X_{n+1} \rightarrow X_n \rightarrow X_{n-1} \]

\[ X = \lim X_n \quad \text{Define} \]

\[ F_n = \text{Fib}(X_n \rightarrow X_{n-1}) \]
\[ \to \Pi_{i} F_n \to \Pi_{i} X_n \to \Pi_{i} X_{n-1} \to \Pi_{i-1} F_n \to \]

\[ \to \Pi_{i-1} X_{n-1} \to \Pi_{i} F_n \to \Pi_{i} X_n \to \Pi_{i} X_{n-1} \to \Pi_{0} F_n \to \Pi_{0} X_{n} \to \Pi_{0} X_{n-1} \]

s.t. 1) \( \Pi_{2} X_{n-1} \to \text{center} \left( \Pi_{1} F_n \right) \)

2) \( \text{Ker} = \text{image} \quad \text{everywhere} \)

3) \( \Pi_{1} X_{n-1} \) acts on \( \Pi_{0} F_{n} \) & two elts of \( \Pi_{0} F_{n} \) have same image in \( \Pi_{0} X_{n} \) ⇔ they are in the same orbit

argument from 1) illustrates general principle

To see 1)

S'pose \( \Omega \) top group id 1

\( X \) top space base pt b

with \( \Omega \times X \to X \) action

Then \( \Pi_{1} \Omega \times \Pi_{1} X \to \Pi_{1} X \) group hom
Take $Y \in \Pi_1 \Sigma$

**Claim:**

$M_\ast(\emptyset \times b) \subset \text{Center} \Pi_1 X$

**Proof:**

Take $\gamma \in \Pi_1 X$

paths in $\Sigma \times X$

\[ \gamma \times \{b\} \]

all homotopic

\[ \gamma \times \{b\} \]

\[ \gamma \times \{b\} \]

$M_\ast(\text{first}) = M_\ast(\emptyset \times b) \circ \text{\textit{fig}}$

$M_\ast(\text{last}) = \text{\textit{fig}} \circ M_\ast(\emptyset \times b)$
Take $X = S^2$, in multiplication

Cor: $\pi_1 S^2$ is abelian

(Even though $S^2$ might be non-abelian)

$F \rightarrow E \rightarrow B$ fibration

where fiber

$F = \{ (e, \gamma) : e \in I, \gamma : I \rightarrow B \ \gamma(1) = (b), \gamma(0) = p(e) \}$

$\Rightarrow$ loops of $B$ based at $B$ acts on $F$

$S^2 B \times F \rightarrow F$

$\Rightarrow \pi_1 (S^2 B) = \pi_2 B$ maps to center $\pi_1 F$
Interested in $\tilde{\Omega}^*_X$

consider $\tilde{\Omega}^*_X x_0$. Some elts come from $\tilde{\Omega}^*_F_n$.

Let's write those down

\[
\begin{array}{c}
\tilde{\Omega}_0 F_3 \quad \tilde{\Omega}_2 F_1 \\
\tilde{\Omega}_0 F_2 \quad \tilde{\Omega}_1 F_1 \quad \tilde{\Omega}_1 F_0 \\
F_2 \quad F_2 \quad F_1 \quad F_0
\end{array}
\]

Rmk: If might have been more intuitive

\[
\tilde{\Omega}_1 F_i, \tilde{\Omega}_1 F_0 \quad \text{i.e.}
\]

If $d \in \tilde{\Omega}^*_X x_0$ is in image of $\tilde{\Omega}^*_X X$

$d$ lifts to $\tilde{\Omega}^*_X X_{n+1} \Rightarrow \alpha$ in $\ker(\tilde{\Omega}^*_X X)$

The indexing makes diff same as same in $H^*_X$

So we want, not elts of $\tilde{\Omega}^*_F_n$ but

elts $\ker \tilde{\Omega}^*_F_n \xrightarrow{d_1} \tilde{\Omega}^*_{-1} F_{n+1}$

Lift to $x_{n+1}$, test against $F_{n+2} d_2$
More systematically, \( \times \)

Interested in \( \pi_\ast \left( \lim X_n \right) \)

\( \Rightarrow \)Interested in \( \pi_\ast (X) \rightarrow \pi_\ast (X_n) \)

\( \Rightarrow \)Interested in \( \pi_\ast (X_{n+r}) \rightarrow \pi_\ast (X_n) \)

Define \( \pi_i X_n^r = \text{image}(\pi_i X_{n+r} \rightarrow \pi_i X_n) \)

We will "measure" \( \pi_i X_n^r \) using \( F_n \)

Consider elts of \( \pi_i F_n \) mapping to \( \pi_i X_n^r \)

Define \( \pi_i F_n^r = \ker \left( \pi_i F_n \rightarrow \pi_i X_n / \pi_i X_n^r \right) \)

\[ \underbrace{\text{action of}}_{\ker \left( \pi_i \ast X_{n-1} \rightarrow \pi_i \ast X_{n-1-r} \right)} \]
Prop/Def "Extended Hotpy spectral sequence"

\[ \exists \tilde{E}^r_{s+t}, \delta_r : \tilde{E}^r_{s+t} \rightarrow \tilde{E}^{-s-r, t+r-1}_{s} \quad t \geq s \geq 0 \]
\[ r \geq 1 \]

\[ \tilde{E}^r_{s, t} = \prod_{-s+t} F_s^{(r-1)} \]

\[ \delta_r \text{ is composite} \]

\[ \prod_{-s+t} F_s^{(r-1)} \rightarrow \prod_{-s+t} X_s^{(r-1)} \rightarrow \prod_{-s+t-1} F_{s+r}^{(r-1)} \]

satisfies the properties

1) \[ \tilde{E}^r_{s+t} \] is a group \( -s+t \geq 1 \)
   abelian \( \geq 2 \)

2) \[ \tilde{E}^{-s+t}_{r+1} = \ker \frac{dr}{\text{im} \, dr} \text{ for } -s+t \geq 1 \]

\[ \text{dr hom } +s \geq 3 \text{ in center} \]
\[ t - s = 1 \Rightarrow \text{dr extends to an action} \]

\[ E_{r, s - r + 1} \supset E_{r, s} \]

and \[ E_{r, s} \supset E_{r, s - s + r} / E_{s - r, s - r + 1} \]

Not :=

We're missing the obstructions from the first day of class.

We'll add these in

**Convergence**

possibly \( \infty \) many differentials leave

\[ E \supset E_{-s, t} \]

no differentials hit for \( r > s \)

Define \( E_{-s, t}^\infty = \bigcap_{r > s} E_{-s, t}^r \)
$Q_s \mathcal{U}_i X = \text{Image} \left( \pi_i X \to \pi_i X_s \right)$

$\pi_i X$ is often $\lim_{\leftarrow s} Q_s \pi_i X$

$\mathcal{E}_{s,s+i} \text{ "tries to be" } \ker \left( Q_s \pi_i X \to Q_{s-1} \pi_i X \right)$