Recall: ι_q, Sq^I, e

office hours Friday

Fix for PS 3 #7 posted – please add $\text{coprod}(\sum B^i x^i) = \sum B^i \text{coprod}(x^i)$

L14 – 3/6/13

Finishing $H^*(K(q,\mathbb{Z}/2);\mathbb{Z}/2)$

<u>Thm</u>: $H^*(K(q,\mathbb{Z}/2);\mathbb{Z}/2) \cong \mathbb{Z}/2[Sq^I \iota_q : e(I) < q]$

<u>Pf</u>: By induction. S'pose true for q.

order the elements of the set $\{(Sq^I \iota_q)^{2^n} : n = 1,2,\ldots,\ I \text{ s.t. } e(I) < q\}$

and call them $x_1, x_2, \ldots$

Note that $\{x_{i_1} x_{i_2} \cdots x_{i_n} : i_1 < i_2 < \ldots < i_n\}$ is a basis for $H^*(K(q,\mathbb{Z}/2);\mathbb{Z}/2)$

$$\underset{F}{\underset{\shortparallel}{K(q,\mathbb{Z}/2)}} \to * \to \underset{B}{\underset{\shortparallel}{K(q+1,\mathbb{Z}/2)}}$$

$H^*(B, H^*F) \Rightarrow H^*(*) = \mathbb{Z}/2$ in degree 0

B simply connected $\Rightarrow E_2^{pq} \cong H^p(B) \otimes H^q(F)$

Last wednesday we showed: x_i is transgressive $\forall\, i$ and $\{\tau(x_i)\} = \{Sq^I \iota_{q+1} : e(I) < q+1\}$

Thus, it suffices to show that $H^*(B)$ is polynomial on $\tau(x_i)$.

Use Zeeman comparison thm Recall

Let $\overline{E}$ = Serre spectral sequence

We construct $f: E \xrightarrow{\cong} \overline{E}$
with $E_2^{*0} = \mathbb{Z}/2[y_1, y_2, \dots]$ with $y_i \mapsto \tau x_i$

and this shows the thm.

E is spectral sequence associated to (R, d) filtered differential ring.

$$R = \bigotimes_{x_i} \left(\mathbb{Z}/2[\overset{\substack{\deg x_i + 1 \\ \shortparallel}}{y_i}, z_i] \big/ z_i^2 \right) \quad \deg z_i = \deg x_i$$

$= P \otimes \Lambda$ ← exterior alg on generators z_i

P: polynomial algebra on generators y_i

we will have: $f(z_i) = x_i \quad f(y_i) = \tau(y_i)$

Thus, we want $d(z_i) = y_i$.

Define $d: P^* \longrightarrow P^{*+1}$

by $d(z_i) = y_i$
$d(y_i) = 0$

and satisfying the requirement

$$d(ab) = (da)(b) + (-1)^{|a|} a\, db$$

$$F^p(R) = \bigoplus_{i \geq p} P^i \otimes \Lambda$$

d increases grading.

$\Rightarrow d = 0$ on $Gr(R) \Rightarrow E_2^{p,q} = R^{p,q}$

In particular, E is a spectral sequence of the form considered in Zeeman's

Comparison thm.

Furthermore, $E_\infty^{pq} = 0$ for $(p,q) \neq (0,0)$. To see this:

i.e. $H^*(P,d) = 0$ for $* > 0$

$$P = \bigotimes_i \mathbb{Z}/2[y_i, z_i]/(z_i^2)$$

$\mathbb{Z}/2[y_i, z_i]/z_i^2$, d

has basis $\{y_i^n z_i, y_i^n\}$

$$d(y_i^n z_i) = y_i^{n+1}$$

$$d(y_i^n) = 0$$

Cocycles = coboundaries = $\mathrm{Span}_{\mathbb{Z}/2}\, y_i^{n+1}$

$\Rightarrow H^*(\mathbb{Z}/2[y_i, z_i]/z_i^2, d) = 0$ $\quad * > 0$

$\Rightarrow H^*(P) = 0$ for $* > 0$.

Define ring homomorphism $f_2^{*,0} : P \to H^*(B; \mathbb{Z}/2)$

$y_i \mapsto \tau(x_i)$

Define a $\mathbb{F}_2$-mod hom

$$f_2^{*q} : \Lambda \longrightarrow H^*(F)$$
$$z_{i_1} \cdots z_{i_n} \longmapsto x_{i_1} \cdots x_{i_n}$$

$$f_2^{pq} := f_2^{p*} \otimes f_2^{*q}$$

$f_2 d = d f_2$ b/c: • true for $z_{i_1} \cdots z_{i_n}$ and y_i (since x_i transgressive)

• Both behave as derivations on $z_{i_1} \cdots z_{i_n} \otimes y_i$

Define f_3 by $H(f_2)$.

Continue.

$\square$