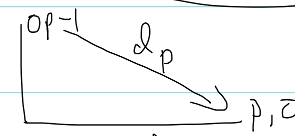
Skip

L13 starts on next page

$\delta(z_\alpha) = y_\alpha$

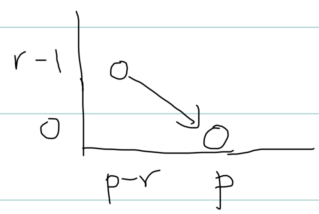
$\delta(z_\alpha^2) = y_\alpha \otimes z_\alpha + y_\alpha \otimes z_\alpha = 0$

$\delta(y) = 0$



$d_r = 0$ for $r > p$

Thus $E_r^{p,0} = E_\infty^{p,0}$ for $r > p$



assume f_{r+1}^{p0} is iso

$$\begin{tikzcd}
 & & \ker d^r \arrow[d, phantom, "\parallel"] & & \\
0 \arrow[r] & \operatorname{Im} d_r E_r^{p-r,r-1} \arrow[r] \arrow[d, "\text{claim } \cong"'] & E_r^{p,0} \arrow[r] \arrow[d, "f_r^{p0}"] & E_{r+1}^{p,0} \arrow[r] \arrow[d, "f_{r+1}^{p0}", "\cong"'] & 0 \\
0 \arrow[r] & \operatorname{Im} \bar{d}_r \bar{E}_r^{p-r,r-1} \arrow[r] & \bar{E}_r^{p,0} \arrow[r] \arrow[d, phantom, "\parallel"] & \bar{E}_{r+1}^{p,0} \arrow[r] & 0 \\
 & & \ker \bar{d}_r & &
\end{tikzcd}$$

- For A^* (even unstable), want $H^*(K(q,\mathbb{Z}/2),\mathbb{Z}/2)$
- $K(q-1,\mathbb{Z}/2) \to * \to K(q,\mathbb{Z}/2)$

induction...

office hours Wed this week

L13 – Zeeman's comparison thm

3/4/12

Consider spectral sequences of the form

$$\{E_r^{pq}, \; d_r : E_r^{pq} \to E_r^{p+r,q-r+1}\}$$

$p,q \geq 0$

s.t. $E_2^{pq} = E_2^{p,0} \otimes E_2^{0,q}$

Let $f_r^{pq} : E_r^{pq} \to \bar{E}_r^{pq}$ be a morphism of such spectral sequences

(define first)

Lemma: S'pose f_2^{0q} is iso $\forall q$

and f_2^{p0} " $p \leq k$.

Then f_r^{pq} is injective for $p \leq k$ and

iso for $p \leq k-r+1$.

Pf: True for $r=2$ (b/c f is $\otimes$-prod).

S'pose true for $r=m$.

$$E_{m+1}^{pq} = \frac{\operatorname{Ker} d_m^{p,q}}{\operatorname{Im} d_m}$$

$$\begin{tikzcd} \ker d_m^{pq} \arrow[r, phantom, "\subset"] \arrow[d, dashed] & E_m^{pq} \arrow[d, "f_m^{pq}"] \\ \ker \bar{d}_m^{p,q} \arrow[r, phantom, "\subset"] & \bar{E}_m^{pq} \end{tikzcd} \qquad p \le k$$

Since $\bar{d} f = f d$,

Injective b/c f_m^{pq} is.

$$\begin{tikzcd} E_m^{p-m,q+m-1} \arrow[r, "d_m"] \arrow[d] & \operatorname{Im} d_m^{p-m,q+m-1} \arrow[r] \arrow[d, "f|_{\operatorname{Im}}^{p-m}"] & 0 \\ \bar{E}_m^{p-m,q+m-1} \arrow[r, "d_m"] & \operatorname{Im} \bar{d}_m^{p-m,q+m-1} \arrow[r] & 0 \end{tikzcd}$$

$p \le k \Rightarrow$ left vertical arrow iso

$\Rightarrow f|_{\operatorname{Im}}^{p-m}$ surjective

$\Rightarrow f : E_{m+1}^{p,q} \longrightarrow \bar{E}_{m+1}^{p,q}$ injective.

$\Rightarrow f_{m+1}^{pq}$ injective for $p \le k$.

Consider $r = m+1$

and $p \leq k - (m+1) + 1 = k - m$.

$$E^{pq}_{m+1} = \frac{\ker d_m^{p,q}}{\operatorname{Im} d_m^{p-m,q+m-1}}$$ (and similarly for $\bar{E}$)

$$\begin{tikzcd}
E^{pq}_m \arrow[r] \arrow[d, "d_m"] & \bar{E}^{pq}_m \arrow[d, "\bar{d}_m"] \\
E^{p+m,q-m+1}_m \arrow[r, "F"'] & \bar{E}^{p+m,q-m+1}_m
\end{tikzcd}$$

injective by induction b/c $p+m \leq k$

$\Rightarrow$ $\ker d_m^{p,q} \xrightarrow{F} \ker \bar{d}_m^{p,q}$ iso ☆

$$\begin{tikzcd}
\ker d_m \arrow[r] \arrow[d, "\text{iso by ☆}"'] & E \arrow[r, "d_m^{p-m,q+m-1}"] \arrow[d, "\text{iso by hyp}"] & \operatorname{Im} \arrow[d] \\
{} & {} \arrow[r] & \overline{\operatorname{Im}}
\end{tikzcd}$$

$\square$

Thm: Suppose $f : E_r^{pq} \to \overline{E}_r^{pq}$ morphism as above s.t. f_2^{0q} is an isomorphism

and assume $E_\infty^{pq} = 0 = \overline{E}_\infty^{pq}$ for $(p,q) \neq (0,0)$

Then f is an iso.

Pf: Sufficient to show f_2^{p0} iso.

Assume by induction f_2^{p0} iso for $p \leq K$.
We wish to show $f_2^{K+1\,0}$ iso
For r sufficiently large, $f_r^{K+1\,0}$ iso
S'pose $f_{r+1}^{K+1\,0}$ iso

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Im} d_r^{K+1-r,\,r-1} \arrow[r] \arrow[d] & \ker d_r = E_r^{K+1\,0} \arrow[r] \arrow[d, "f"] & E_{r+1}^{K+1\,0} \arrow[r] \arrow[d, "\text{iso by induction}"] & 0 \\
0 \arrow[r] & \operatorname{Im} \overline{d}_r^{K+1-r,\,r-1} \arrow[r] & E_2^{K+1\,0} \arrow[r] & \overline{E}_{r+1}^{K+1\,0} \arrow[r] & 0
\end{tikzcd}
$$

try 1:

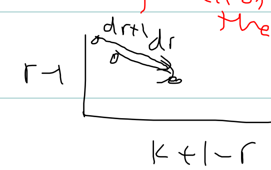
By lemma, $E_r^{k+l-r,\,r-1} \longrightarrow \bar{E}_r^{k+l-r,\,r-1}$ is an iso.

$\Rightarrow \operatorname{Im} d_r^{k+l-r,\,r-1} \longrightarrow \operatorname{Im} \bar{d}_r^{k+l-r}$ surjection.

Furthermore,

- $E_j^{k+l-r,\,r-1} \xrightarrow{f} \bar{E}_j^{k+l-r,\,r-1}$ injective (lemma)

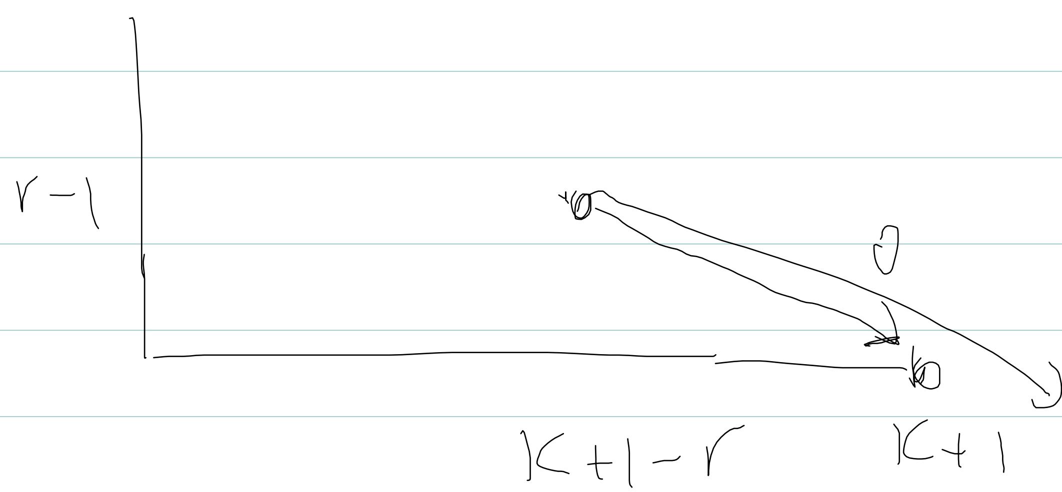
f iso on all of thes



$\Rightarrow$ • $\operatorname{Im} d_j^{(k+l-r)-j,\, r-1+j-1} \xrightarrow[\text{iso}]{f}$

$j \geq r$

- $$\begin{tikzcd}
\operatorname{Ker} d_r^{k+l-r,\,r-1} \arrow[r] \arrow[d] & \ker \bar{d}_r^{k+l-r,\,r-1} \arrow[r] \arrow[d] & \operatorname{coker} \arrow[d, equal] \\
\operatorname{Ker}/\operatorname{Im}\ E_{r+1}^{k+l-r,\,r-1} \arrow[r] & \operatorname{Ker}/\operatorname{Im} = \bar{E}_{r+1}^{k+l-r,\,r-1} \arrow[r] & \operatorname{coker}_{r+1}^{k+l-r,\,r-1}
\end{tikzcd}$$



All $d_j^{k+l-r,\,r-1}$, $\bar{d}_j^{k+l-r,\,r-1}$ are 0 for $j > r$.

$\Rightarrow \operatorname{coker}_j^{k+l-r,\,r-1} = \operatorname{coker}_{r+1}^{k+l-r,\,r-1} \quad \forall j \geq r+1.$

$\Rightarrow$ coker $= 0$. Thus $\ker d_r^{k+l-r,\,r-1} \longrightarrow \ker \bar{d}_r^{k+l-r,\,r-1}$ surjective

Injective b/c $f^{k+l-r,r-1}$ is.

$\Rightarrow \operatorname{Im} d_r^{k+l-r,r-1} \cong \operatorname{Im} \bar{d}_r^{k+l-r,r-1}$ $\square$

<u>try 2:</u> By 5 lemma, sufficient to see

$$\begin{array}{ccc} \operatorname{Im} d_r^{k+l-r,r-1} & \xrightarrow{\cong} & \operatorname{Im} \bar{d}_r^{k+l-r,r-1} \\ \text{\rotatebox{90}{$\cong$}} & & \\ E_r^{k+l-r,r-1}/\ker d_r^{k+l-r,r-1} & & \end{array}$$

By lemma, $E_r^{k+l-r,r-1} \xrightarrow{f} \bar{E}_r^{k+l-r,r-1}$ is an iso.

Thus it is sufficient to show

$$\ker d_r^{k+l-r,r-1} \longrightarrow \ker \bar{d}_r^{k+l-r,r-1}$$

is surjective.

Let Q denote cokernel

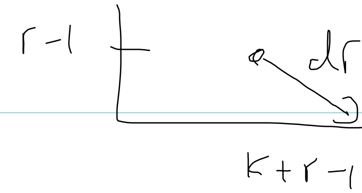
We claim that

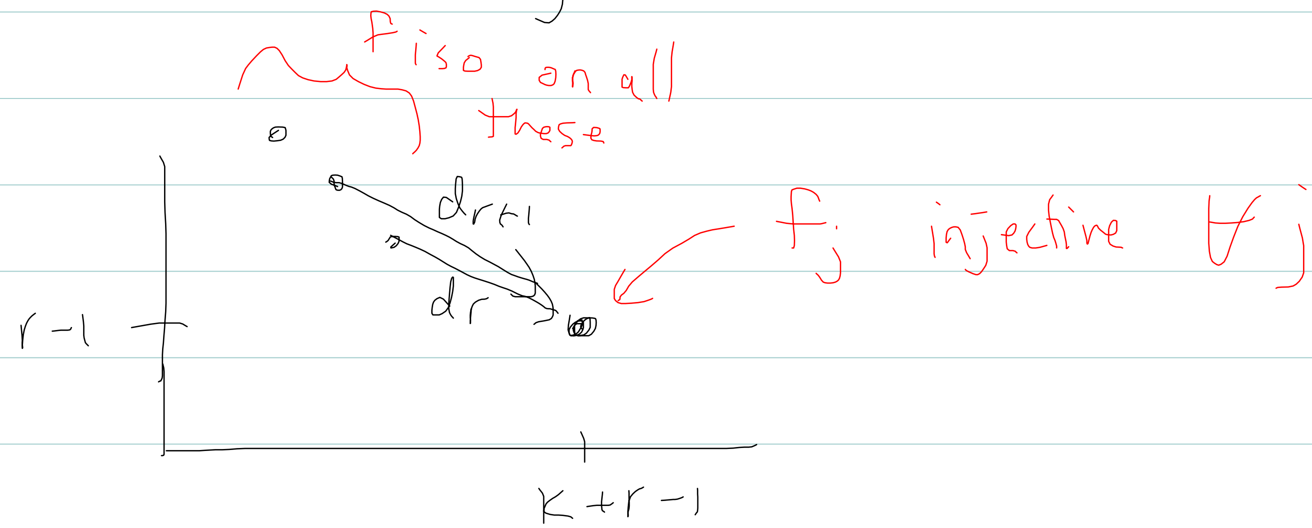
$$Q \cong \operatorname{coker}\left(E_\infty^{k+l-r,r-1} \longrightarrow \bar{E}_\infty^{k+l-r,r-1}\right)$$

Since r.h.s. $=0$, it suffices to see claim.

For $j > r$ $\quad d_j^{k+1-r,r-1} = 0$

$$\bar{d}_j^{k+1-r,r-1} = 0$$





$$\Rightarrow \quad \operatorname{Im} d_j^{(k+r-1)-j,(r-1)+(j-1)} \cong \operatorname{Im} \bar{d}$$

$j \geq r$

$$E_{j+1}^{k+1-r,r-1} = \frac{\ker d_r}{\operatorname{Im} d_j^{(k+1-r)-j,(r-1)+(j-1)} + \operatorname{Im} d_{j-1} + \dots \operatorname{Im} d_r}$$

$\square$

Zeeman's comparison thm

$$f_2^{p,q} = f_2^{p,0} \otimes f_2^{0,q}$$

or

$$\begin{tikzcd} 0 \arrow[r] & E^{p,0} \otimes E^{0,q} \arrow[r] \arrow[d, "f_2 \otimes f_2"] & E^{p,q} \arrow[r] \arrow[d, "f_2"] & \mathrm{Tor}_1(E_2^{p+1,0}, E_2^{q,q}) \arrow[d, "\mathrm{Tor}(f_2, f_2)"] \arrow[r] & 0 \end{tikzcd}$$

commutes with exact rows.

Then any two of the following imply the third

I. $f_2^{p,0}$ iso $\forall p$

II $f_2^{0,q}$ iso $\forall q$

III f_∞ iso $\forall p,q$