Denis Nardin: 4:30 2-151 "Fgl cplx cobordism"
MIT

L12 - $H^*(K(\mathbb{Z}/2,q);\mathbb{Z}/2)$
2/26/13

Serre spectral sequence

$$H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)$$

useful fibration: $P(B) \to * \to B$

Ex: $H^*(\mathbb{C}P^\infty)$ via $K(\mathbb{Z},1) \to * \to K(\mathbb{Z},2)$

Thm: $H^*(K(\mathbb{Z}/2,q);\mathbb{Z}/2)$ is a polynomial algebra over $\mathbb{Z}/2$ with generators $\{ Sq^I(\iota_q) : I=(a_1,a_2,\dots,a_k)$ admissible with $e(I)<q \}$

Cor: Sq^I generate $\mathcal{A}^*$

Notation: I is admissible when $a_i \geq 2a_{i+1} \geq 0$

$$e(I) = (a_1-2a_2)+(a_2-2a_3)+\dots+a_k$$
$$= 2a_1 - d(I)$$
$$d(I) = \Sigma a_i$$

$\iota_q \in H^q(K(\mathbb{Z}/2,q),\mathbb{Z}/2)$ fundamental class

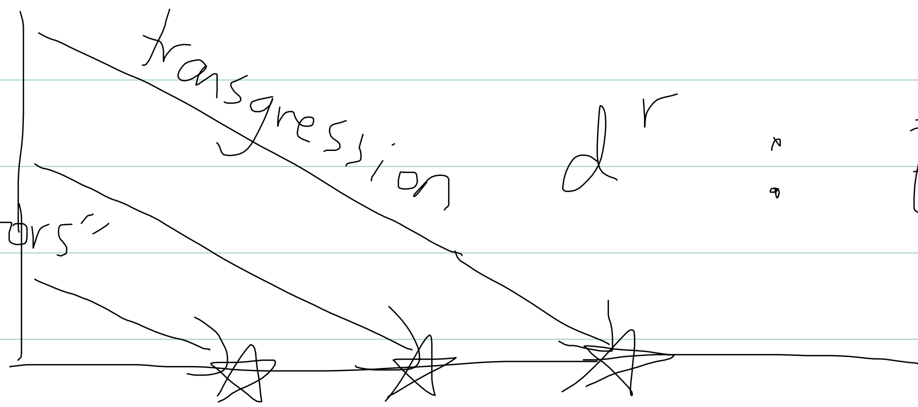
By induction on q. $q=1$: $Sq^0\iota_1$ is poly gen $H^*(K(\mathbb{Z}/2,1))$

$H^* = H^*(-;\mathbb{Z}/2)$

$$K(\mathbb{Z}/2,q) \to * \to K(\mathbb{Z}/2,q+1)$$

What happens is

"simple system generators"



$$d^r : E^r_{0\,r-1} \longrightarrow E^r_{r\,0}$$

to be defined

polynomial generators.

<u>Transgression</u>:

$$(E, F) \xrightarrow{p} (B, *)$$

$$p^* : H^*(B, *) \longrightarrow H^*(E, F)$$

$$\rightarrow H^*(F) \xrightarrow{\delta} H^{*+1}(E, F) \rightarrow H^{*+1}(E) \rightarrow$$

$x \in H^*(F)$, $y \in H^{*+1}(B)$

s.t. $p^* y = \delta x$

$\Rightarrow d^r x = 0$ for $r < * + 1$

$$d^{*+1} x = y$$

write $\tau x = y$

Say "x transgressive"

$$Sq^I \tau x = \tau Sq^I x$$

- By induction, $H^*(K(\mathbb{Z}/2, q)) = \mathbb{Z}/2[Sq^I : I \in \mathcal{I}_{<q}]$

- Since every integer has a base-2 expansion, every elt $H^*(K(\mathbb{Z}/2, q))$

is a finite product of $\vee$ $(Sq^I \iota_q)^{2^n}$

of terms of the form

for $I \in \mathcal{I}_{<q}$ and $n \geq 0$, where each term is taken once.

Hurewicz thm $\Rightarrow$ $\tau \iota_q = \iota_{q+1}$

$Sq^I \iota_q$ has dim $d(I) + q$

$$\Rightarrow Sq^{2^{n-1}(d(I)+q)} Sq^{2^{n-2}(d(I)+q)} \cdots Sq^{d(I)+q} (Sq^I \iota_q)$$

$$= (Sq^I \iota_q)^{2^n}$$

$\Rightarrow$ $(Sq^I \iota_q)^{2^n}$ is transgressive and

$$\tau\left(\left(Sq^I \iota_q\right)^{2^n}\right) = Sq^{L(d(I)+q,\, n)}\, Sq^I \iota_{q+1}$$

<u>Lemma</u>: $\{ Sq^{L(d(I)+q,\, n)} Sq^I :$

$I \in \mathcal{I}_{<q} \quad n = 0, 1, \ldots \}$

$= \{ Sq^I : I \in \mathcal{I}_{q+1} \}$

<u>Pf</u>: $\subset$: calculate

$$\tilde{e} = e\left(L(d(I)+q, n)\, Sq^I \right).$$

$n = 0 \qquad \tilde{e} = e(I) < q+1$

$n > 0 \qquad \tilde{e} = \left(d(I)+q\right) - 2a_1 + e(I)$

$= q \qquad (\text{b/c } e(I) = 2a_1 - d(I))$

" We now show every $J \in I_{<q+1}$ can be expressed uniquely in the form $L(d(I)+q, n)\, I$:

$e(J) < q$, unique expression is $J = I$
$n=0$

$e(J) = q$: $J = (J_1, \dots, J_K)$

Let j_0 be smallest such that $J_{j-1} = 2J_j$

$I = (J_{j_0+1}, \dots J_K)$ $\rightarrow$ $\forall j \leq j_0$ & $j-1 \geq 1$

$e(J) = J_{j_0} - \sum_{j > j_0} J_j = J_{j_0} - d(I)$

Since $e(J) = q$, $J_j = d(I)+q$

Thm now follows from thm of Borel:

S'pose $F \to E \to B$ w/ $E \simeq *$ and $H^*(F)$ has a simple system $\{x_\alpha\}$ of transgressive generators. Then

$$H^*(B) = \mathbb{Z}/2\,[\tau\{x_\alpha\}]$$

<u>Def</u>: A graded ring R over $\mathbb{Z}/2$ has $\{x_\alpha\}_{\alpha \in A \text{ ordered}}$ as a simple system of generators if $\{x_{i_1} x_{i_2} \cdots x_{i_r} : i_1 < i_2 < \dots i_r\}$ is $\mathbb{Z}/2$-basis for r & for each n, only finitely many x_i have gradation n.