Serre spectral sequence

\[ H^p(B, H^q(F)) \Rightarrow H^{p+q}(E) \]

Useful fibration: \( p(B) \to \ast \to B \)

Example: \( H^\ast(G, \mathbb{P}^n) \) via \( K(\mathbb{Z}, 1) \to \ast \to K(\mathbb{Z}, 2) \)

Theorem: \( H^\ast(K(\mathbb{Z}/2, q); \mathbb{Z}/2) \) is a polynomial algebra over \( \mathbb{Z}/2 \) with generators \( \Sigma \mathbb{Z}/2^\ast \), \( I = (a_1, a_2, \ldots, a_k) \) admissible with \( e(I) \leq q \).

Corollary: \( \Sigma_1^\ast \) generate \( A^\ast \)

Notation: \( I \) is admissible when \( a_i \geq 2q_{i+1} \geq 0 \)

\[ e(I) = (a_1 - 2a_2) + (a_2 - 2a_3) + \ldots + a_k \]

\[ d(I) = \Sigma a_i \]

\( 2q \in H^q(K(\mathbb{Z}/2, q); \mathbb{Z}/2) \) fundamental class

By induction on \( q \), \( q = 1 \): \( \Sigma_1^\ast \mathbb{Z}/2 \) is poly gen \( H^\ast(K(\mathbb{Z}/2, 1)) \)

\( H^\ast = H^{\ast-} \mathbb{Z}/2 \)

\[ K(\mathbb{Z}/2, q) \to \ast \to K(\mathbb{Z}/2, q+1) \]
What happens is

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Transgression \text{ dr: } \mathbb{E}_r \to \mathbb{E}_{ro}
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to be defined polynomial generators.

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Transgression:

\[(E, F) \xrightarrow{p} (B, *)\]

\[p^*: H^*(B, *) \to H^*(E, F)\]

\[\Rightarrow H^*F \xrightarrow{\delta} H^*(E, F) \to H^{*+1}(E) \to \]

\[x \in H^*(F), \; y \in H^{*+1}(B)\]
s.t. \( \delta^* y = \delta x \)

\[ \Rightarrow d^r x = 0 \text{ for } r < *+1 \]

\[ d^{*+1} x = y \]

write \( \exists x = y \)

Say "\( x \) transgressive"

\[ \mathbb{Z} \sqcup \mathbb{Z} = \mathbb{Z} \sqcup \mathbb{Z} \]

\[ \mathbb{Q} \sqcup \mathbb{Z} = \mathbb{Z} \sqcup \mathbb{Q} \]

* By induction, \( H^*(K(\mathbb{Z}/2, q)) = \mathbb{Z}/2[s_t] \)

\[ \text{Ie } \tau \sqcup \mathbb{Z} \]

. Since every integer has a base-2 expansion, every elt \( H^*(K(C\mathbb{Z}/2, q)) \)
finite is a product of \((\text{Sq}^I \cdot 2q)^{2^n}\)
of terms of the form

for \(I \leq I \leq q\) and \(n \geq 0\), where each term is taken once.

Hurewicz thm \(\Rightarrow\) \(\tau \cdot 2q = 2q + 1\)

\(\text{Sq}^I \cdot 2q\) has dim \(d(I) + q\)

\(\Rightarrow\) \(\text{Sq}^I \text{Sq}^I \text{Sq}^I \cdots \text{Sq}^I \cdot 2q = 2^{n-1}d(I + q) 2^{n-2}d(I + q) \cdots d(I + q) \cdot 2q\)

\(\Rightarrow\) \((\text{Sq}^I \cdot 2q)^{2^n}\)

\(\Rightarrow\) \((\text{Sq}^I \cdot 2q)^{2^n}\) is transgressive and
\[ \psi \left( \left( \text{Sq}^I 2^n \right)^2 \right) = \text{Sq}^I \left( d(I) + q, n \right) \] 

**Lemma:** \( \exists \text{ Sq}^I \left( d(I) + q, n \right) \) 

\[ I \leq I \leq q, n = 0, 1, \ldots, 3 \] 

\[ = \exists \text{ Sq}^I : I \in I_{q+1} \] 

**Pf:** Calculate 

\[ \tilde{e} = e \left( L \left( d(I) + q, n \right) \text{ Sq}^I \right) \] 

\[ n = 0 \] 

\[ \tilde{e} = e(I) < q + 1 \] 

\[ n > 0 \] 

\[ \tilde{e} = (d(I) + q) - 2q_1 + e(I) \] 

\[ = q \quad \text{(b/c e(I)} = 2q_1 - d(I) \)} \]
We now show every $J \subseteq I \leq q + 1$
can be expressed uniquely in the form $L(d(I)+q, n) I$.

$e(J) < q$, unique expression is $J = I \ n = 0$

$e(C) = q : J = (J_1, \ldots, J_k)$

Let $j_0$ be smallest
such that $J_{j-1} = 2J_j$

$I = (J_{j_0+1}, \ldots, J_k)$ \quad $\forall \ j \leq j_0 \ & \ j-1 \geq 1$

$e(J) = J_{j_0} - \sum_{j>j_0} J_j = J_{j_0} - d(I)$

Since $e(C) = q$, $J_j = d(I) + q$
Thm now follows from thm of Browd:

S'pose \( F \to E \to B \) w/ \( E = * \) and

\( H^*(F) \) has a simple system \( \mathbb{Z} \times \mathbb{Z} \)
of transgressive generators. Then

\[
H^*(B) = \mathbb{Z}/2 \left[ \mathbb{Z} \times \mathbb{Z} \right]
\]

Def: A graded ring \( R \) over \( \mathbb{Z}/2 \) has

\( \mathbb{Z} \times \mathbb{Z} \) as a simple system of

ordered generators if \( \mathbb{Z} x_1 x_2 \cdots x_r : i_1 \leq i_2 \leq \cdots \leq i_r \)
is \( \mathbb{Z}/2 \)-basis for \( R \) & for each \( n \), only

finitely many \( X_i \) have gradation \( n \).