Leray-Serre spectral sequence

Fiber bundle \( F \to E \xrightarrow{\pi} B \)

\( B \) covered by opens \( U \) s.t. \( p^{-1}(U) \approx F \times U \)

\( B \) CW complex. By subdividing s'pose closure of any \( \Lambda_n \)-cell contained in some \( U \).

\( \emptyset \subset B^0 \subset B^1 \subset \ldots \subset B^n \)

Apply \( \pi^{-1} \)

\( \emptyset \subset \pi^{-1}(B^0) \subset \pi^{-1}(B^1) \subset \ldots \subset \pi^{-1}(B^n) \)

Apply singular chains \( C_\ast \left(-; R\right) \), obtain filtered chain complex, thus obtain spectral sequence

\[
E^0_{pq} = \frac{C_p \left( \pi^{-1}(B^p) ; R \right)}{C_p \left( \pi^{-1}(B^{p-1}) ; R \right)} \approx C_q \left( \frac{\pi^{-1}(B^p)}{\pi^{-1}(B^{p-1})} ; R \right)
\]

grading from spectral sequence

This isomorphism depends on choices: To identify fiber over one pt of \( B \) w/ another need path
\[ d_0 : \bigoplus_{p \text{ cells}} \mathcal{C}_q (S^p \wedge F_t) \rightarrow \bigoplus_{p \text{ cells}} \mathcal{C}_{q+1} (S^p \wedge F_t) \]

\[ E^1_{p,q} = H_0 (E^0_{p,q}) = \bigoplus_{p \text{ cells}} H^{q-p} (F) \]

\[ d_1 : \bigoplus_{p \text{ cells}} \rightarrow \bigoplus_{p-1 \text{ cells}} \]

\[ E^2_{p,q} = H_1 (E^1_{p,q}) = H_p (B, H^{q-p} (F)) \]

Choice of paths gives \( \tilde{T}_1 \) \( B \) action.

This is homology w/ local coefs.

Reindex: \( i = p \) \( j = q - p \)
Thm (Serre): There is a spectral sequence

\[(E^r_{p,q}, d_r: E^r_{p,q} \to E^r_{p-r, q+r-1})\]

such that

\[E^2_{p,q} = H_p(B, \mathcal{H}_q(F; R))\]

with local coefs (irrelevant when \(\prod_i B_i\) acts trivially on \(H_q(F)\)).

\[E^\infty_{p,q} \text{ exists}\]

\[\forall n \text{ there is a filtration of } H_n(E; R)\]

\[F^i_n = 0 \subset F^0_n \subset F^1_n \subset \ldots \subset F^n_n = H_n\]

such that

\[E^\infty_{p+q} = F^p_{p+q} / F^{p-1}_{p+q}\]

\[\pi^* : H_n(E) \to F^n_n / F^{n-1}_n = E^\infty_{n,0} \to H_n(B)\]

\[i \text{ with } i : F \to E\]
There is a dual version for cohomology: there exists a spectral sequence \((E^{pq}_r, d^p_r)\)

\[ E_2^{pq} = H^p(B, H^qF) \text{ etc} \]

Furthermore, \(E_r\) is a bigraded ring, i.e. \(E\)

\[ E_r^{p_1,q_1} \otimes E_r^{p_2,q_2} \to E_r^{p_1+p_2, q_1+q_2} \]

S.t. mult on \(E_{r+1}\), induced from multiplication in \(E_r\)

mult on \(E_\infty\) induced from cup product \(H^*(E)\)

\[ d^r(\alpha \beta) = (d^r \alpha) \beta + (-1)^{p+q} \alpha \ d^r \beta \]

Example: \(S^2 X \to P X \to X\)

\[ K(\mathbb{Z}, 1) \to \ast \to K(\mathbb{Z}/2) \]

fiber bundle
Serre spectral sequence $M^* \quad \text{for } r \geq r_1$

\[ E^2_{pq} \Rightarrow H^q(K(L_2)) \]

Only non-zero rows b/c $H^q(K(L_2)) = 0$ for $q \neq 0$.

By Hurewicz: $H^i(K(L_2)) = 0$.

$\text{d}_2 \neq 0$ b/c $H^1(*) = 0$ so $E_{01}^\infty = 0$.

$\ker(d_2: E_{01}^\infty \rightarrow E_{20}^\infty)$

$\Rightarrow M^*(K(L_2), 2) = \mathbb{Z} [L_2]_1$.