Leray-Serre spectral sequence

L11 - Serre Spectral Sequence 2/25/13

Fiber bundle
$$\begin{array}{c} F \to E \\ \downarrow \pi \\ B \end{array}$$

B covered by opens U s.t. $p^{-1}(U) \approx F \times U$

B CW complex. By subdividing s'pose closure of any n-cell contained in some U.

$$\emptyset \subset B^0 \subset B^1 \subset \dots \subset B^n$$

Apply π^{-1}

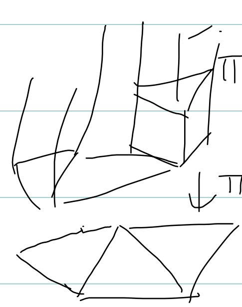
$$\emptyset \subset \pi^{-1}(B^0) \subset \pi^{-1}(B^1) \subset \dots \subset \pi^{-1}(B^n)$$

Apply singular chains $C_*(-;R)$, obtain filtered chain complex, thus obtain spectral sequence

$$E^0_{pq} = \frac{C_q(\pi^{-1}(B^p);R)}{C_q(\pi^{-1}(B^{p-1});R)} \cong C_q\left(\pi^{-1}(B^p)/\pi^{-1}(B^{p-1});R\right)$$

p ← grading from spectral sequence

q ← grading



$\pi^{-1}(B^1) \subset \pi^{-1}(B^2)$

$\downarrow \pi$

$B^1 \subset B^2$

$$\cong \bigoplus_{\substack{p \text{ cells of } B \\ e_\alpha}} C_q\left(\frac{\pi^{-1}(e_\alpha)}{\pi^{-1}(\partial e_\alpha)}\right)$$

$$\cong \bigoplus_{\substack{p \text{ cells of} \\ e_\alpha}} C_q(S^p \wedge F_+)$$

This isomorphism depends on choices: To identify fiber over one pt of B w/ another need path

$$d_0 : \bigoplus_{\substack{p \text{ cells} \\ B}} C_q(S^p \wedge F_+) \longrightarrow \bigoplus_{p \text{ cells}} C_{q-1}(S^p \wedge F_+)$$

$$E^1_{pq} = H(E^0_{pq}) = \bigoplus_{\substack{p \text{ cells} \\ B}} H^{q-p}(F)$$

$$d_1 : \bigoplus_{p \text{ cells}} \longrightarrow \bigoplus_{p-1 \text{ cells}}$$

$$\Rightarrow E^2_{p,q} = H(E^1_{pq}) = H_p(B, H_{q-p}(F))$$

choice of paths gives $\pi_1 B$ action.

This is homology w/ local coefs

reindex: $i = p \quad j = q - p$

Thm (Serre): There is a spectral sequence

$$\left(E^r_{p,q},\ d_r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}\right)$$

s.t. • $E^2_{p,q} = H_p(B, H_q(F;R))$

local coefs (irrelevant when $\pi_1 B$ acts trivially on $H_q(F)$)

• $E^\infty_{p,q}$ exists

• $\forall n$ there is a filtration of $H_n(E,R)$

$$F_n^{-1} =: 0 \subset F_n^0 \subset F_n^1 \subset \dots \subset F_n^n = H_n$$

s.t. $E^\infty_{p,q} = F^p_{p+q} / F^{p-1}_{p+q}$

• π_* is $H_n(E) \to F^n_n / F^{n-1}_n = E^\infty_{n,0} \to H_n(B)$

• i_* w/ $i : F \to E$

- There is a dual version for cohomology:
There exists a spectral sequence (E_r^{pq}, d_r^{pq})
$E_2^{p,q} = H^p(B, H^q F)$ etc

Furthermore, E_r is a bigraded ring, i.e. $\exists$

$$E_r^{p_1,q_1} \otimes E_r^{p_2,q_2} \longrightarrow E_r^{p_1+p_2,\, q_1+q_2}$$

s.t. mult on E_{r+1} induced from multiplication in E_r
mult on E_∞ induced from cup product $H^*(E)$

$$d^r(\alpha\beta) = (d^r\alpha)\beta + (-1)^{p+q}\alpha\, d^r\beta$$

<u>Example:</u> $\Omega X \to PX \to X$

$K(\mathbb{Z},1) \longrightarrow * \longrightarrow K(\mathbb{Z}/2)$

fiber bundle

Serre spectral sequence H^*

$(r, -r+1)$

$$E^2 \qquad \begin{tikzcd}[row sep=small] q & & & & & \\ 1 & \mathbb{Z}z_1 \arrow[drr, "d_2"] & 0 & z_1 \otimes z_2 \arrow[drr] & & \\ 0 & \mathbb{Z} & 0 & \mathbb{Z}z_2 & & z_2^2 \\ & 0 & 1 & 2 & 3 & 4 \quad p \end{tikzcd}$$

only non-zero rows b/c $H^q(K(\mathbb{Z},1)) = 0$ for $q \neq 0, 1$

$\Rightarrow d^r = 0$ for $r \geq 3$

By Hurewicz $H^1(K(\mathbb{Z},2)) = 0$

$d_2 z_1 \neq 0$ b/c $H^1(*) = 0$ so $E^\infty_{01} = 0$

$$E^\infty_{01} = \ker(d_2 : E_{01} \to E^\infty_{20})$$

$$\Rightarrow H^*(\underbrace{K(\mathbb{Z},2)}_{= \mathbb{C}P^\infty}, \mathbb{Z}) = \mathbb{Z}[z_2]$$