L10 2/21/13

Spectral Sequences

$F \in ob(\mathscr{C})$

F ab group, graded abelian group, module etc.

$d: F \to F \quad d^2 = 0$

e.g. (F, d) chain complex

S'pose F filtered meaning

$F_{-1} = 0 = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_n = F$ s.t. $d(F_p) \subset F_p$

To a filtered object, have associated graded $Gr F = \bigoplus_p F_p / F_{p-1}$

$$H(F) := \frac{\ker d}{\operatorname{Im} d}$$

Associated to (F, d) is a spectral sequence whose purpose is to (help) compute $H(F)$.

$H(F)$ has a filtration $0 = H(F)^0 \subset H(F)^1 \subset \dots \subset H(F)^n = H(F)$

$H(F)^p = \operatorname{Im}((\ker d \cap F^p) \to H(F))$

Spectral sequence computes $Gr\, H(F)$

A <u>spectral sequence</u> is a sequence (E_r, d_r)

$E_r = \bigoplus_{p=0}^{n} E_r^p \qquad d_r : E_r^p \longrightarrow E_r^{p-r} \qquad d_r^2 = 0$

s.t. $H(E_r^p) = E_{r+1}^p$

Often $E_{r_0}^p = E_{r_0+1}^p = \dots$. When this holds,

$$E_\infty^p := E_{r_0}^p$$

Important example:

(F, d) defines a spectral sequence s.t.

$E_0^p = Gr^p F = F^p / F^{p-1}$

$E_1^p = H(Gr^p F)$ $\qquad$ $E_\infty^p = H(F)^p$

easier to compute $\qquad$ want to compute

Pf: $Z_r^p = \{ x \in F^p \text{ s.t. } dx \in F^{p-r} \}$

$$E_r^p = Z_r^p \big/ \left(dZ_{r-1}^{p+r-1} + Z_{r-1}^{p-1} \right)$$

$$d^r : Z^r_p \longrightarrow Z^r_{p-r} \Big/ \left(d Z^r_{p-r+(r-1)} + Z^{r-1}_{p-r-1} \right)$$

image of Z^r_{p-1} is trivial b/c

$\quad \parallel \qquad d Z^r_{p+r-1}$ b/c $d^2 = 0$

$\Rightarrow d^r : E^r_p \longrightarrow E^r_{p-r}$ well defined

NB: $d^2 : H\left(F_p / F_{p-1} \right) \longrightarrow H\left(F_{p-1} / F_{p-2} \right)$

is boundary map from SES

$$\frac{F_{p-1}}{F_{p-2}} \to F_p / F_{p-2} \longrightarrow \frac{F_p}{F_{p-1}}$$

b/c both induced $d : F \to F$.

Usually, F is graded $F = \bigoplus_i F_i$

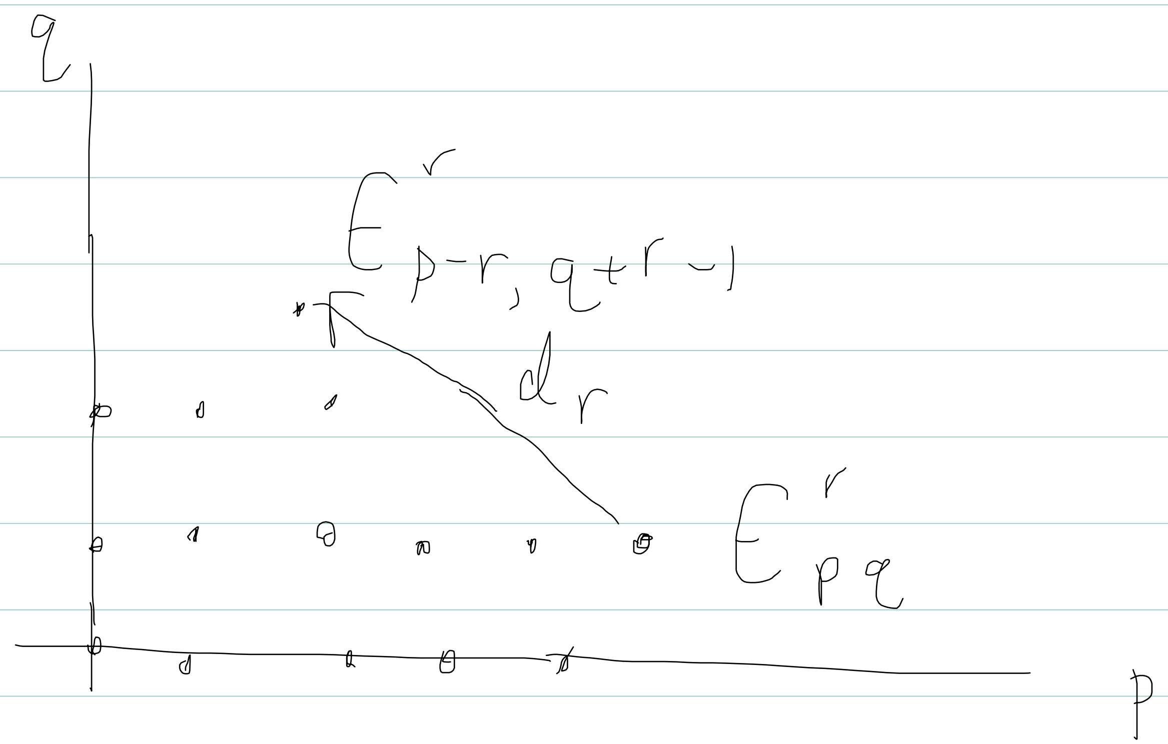
and $d: F_i \to F_{i-1}$

Insert example

Then E^r_p is also graded. One common custom is to then define

$$E^r_{pq} = \text{deg } p+q \text{ piece } (E^r_p)$$

and draw E^r



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<u>Example</u>: X CW complex, so filtered

$$0 \subset X^0 \subset X^1 \subset \ldots \subset X^n = X$$

Apply C_* = functor of singular chains

$$(C_*(X), d)$$

with filtration

$$0 \subset C_*(X^0) \subset C_*(X^1) \subset \ldots \subset C_*(X)$$

$$E^0 = Gr\, C_*(X) = \bigoplus_p C_*(X^p, X^{p-1})$$

$$E^1 = \bigoplus_p H_*(X^p, X^{p-1})$$

$$d_1 : H_*(X^p, X^{p-1}) \longrightarrow H_{*-1}(X^{p-1}, X^{p-2})$$

induced from $\dfrac{C_*(X^{p-1})}{C_*(X^{p-2})} \to \dfrac{C_*(X^p)}{C_*(X^{p-2})} \to \dfrac{C_*(X^p)}{C_*(X^{p-1})}$

So... This is CW homology complex

$$E^2 = \bigoplus_p H^p(X)$$

$E^2 = E^\infty$ b/c $d^r : E^r_{p,q} \to E^r_{p-r,q-1}$

degree $*$ in C_*

and E_{p-r} is concentrated in deg $q = p-r$.

This recovers CW-homology.

<u>Ex</u>: $0 \subset F^1 \subset F^2 = F$ say chain cplxs

$$E^0 = F_1 \oplus F/F_1$$

$$E^1 = H(F_1) \oplus H(F/F_1)$$

$$E^1 \quad \begin{array}{|ll} H_1(F_1) \xleftarrow{d} & H_1(F_2/F_1) \\ H_0(F_1) \xleftarrow{d} & H_0(F_2/F_1) \\ \hline p=0 & p=1 \end{array}$$

d is boundary from SES

$$F_1 \to F_2 \to F_2/F_1$$

$$E^2 = E^\infty \quad \begin{array}{|ll} \operatorname{coker} d & \ker d \\ \operatorname{coker} d & H_0(F_2/F_1) \\ \hline \end{array}$$

Since $E^\infty_{p,q} = Gr^p H_q(F_2)$, $E^\infty_1 = Gr^1 H_q(F_2)$

$E^\infty_2 = Gr^2 H_q(F_2)$

$$Gr^1 H(F_2) \to H(F_2) \to Gr^2 H(F_2)$$

$q \geq 1$

$$0 \to \operatorname{coker} d \to H_q(F_2) \to \ker d \to 0$$

$$\operatorname{coker} d = \operatorname{coker}(d : H_{q+1} F_2/F_1 \to H_q F_1)$$

This is precisely the information contained in LES associated to SES

$$F_1 \to F_2 \to F_2/F_1$$

"LES in H from inclusion of cplxs"

"spectral sequences are the replacement for LES when you have more than one subcomplex"

go back & give information on bigraded degrees

Reason for this:

Spectral sequence of a double complex.

$$K = \bigoplus_{p,q} K_{pq} \qquad \begin{array}{l} d_1 : K_{pq} \to K_{p-1\,q} \\ d_2 : K_{pq} \to K_{p\,q-1} \end{array}$$

$$d_1^2 = 0, \quad d_2^2 = 0, \quad d_1 d_2 + d_2 d_1 = 0$$

Associated total complex

$$\mathrm{Tot}(K)_n = \bigoplus_{p+q=n} K_{pq}$$

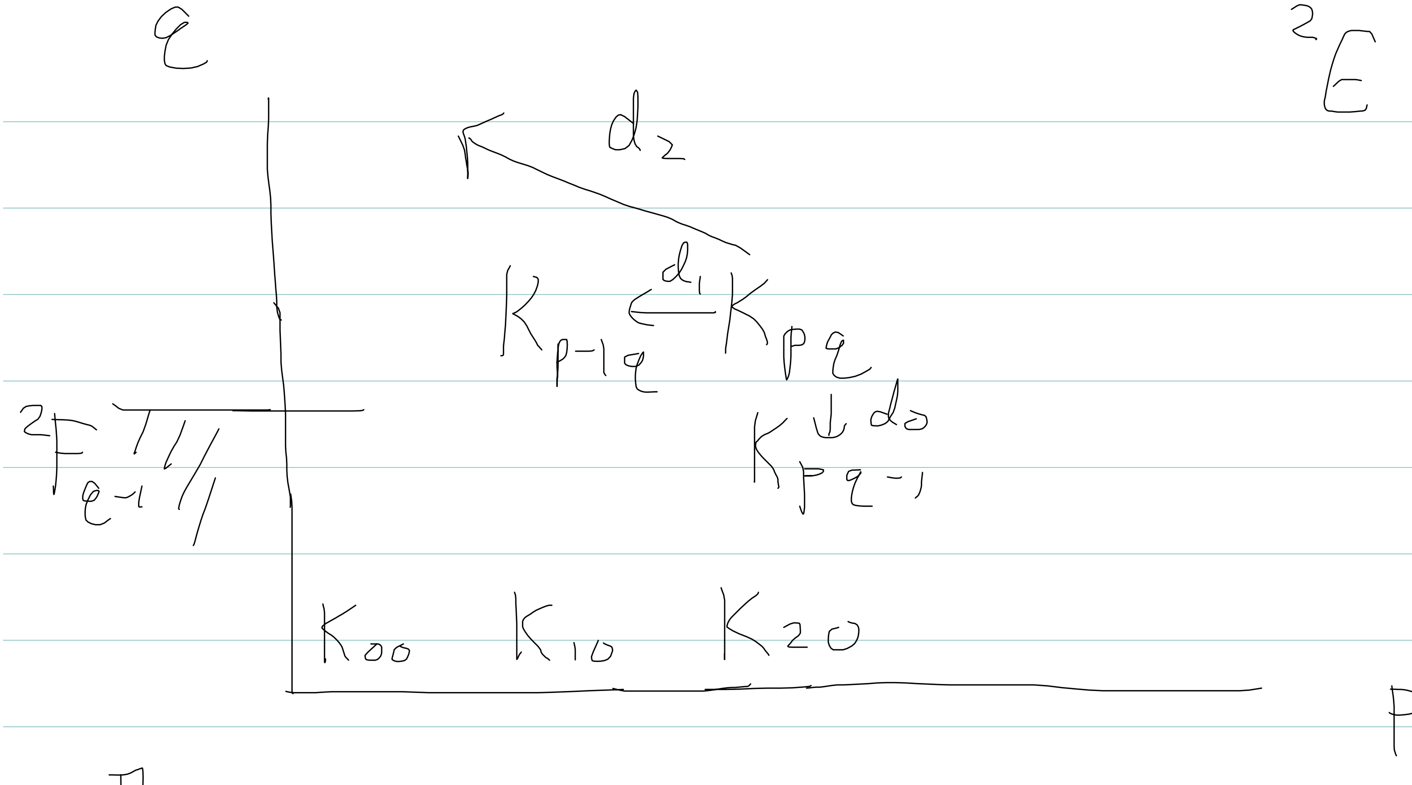
$$d = d_1 + d_2$$

Tot (K) has two natural filtrations

$$^1F_i = \bigoplus_{\substack{p,q \\ p \le i}} K_{p,q}$$

$$^2F_i = \bigoplus_{\substack{p,q \\ q \le i}} K_{p,q}$$

Giving two spectral sequences ^1E, $^2E \Rightarrow H_*(\mathrm{Tot}\, K)$



<u>Thm</u>

(Künneth Formula)

(1) Let R be a ring and X_*, Y_* be two chain complexes of f.g. free R-modules s.t. $H(Y)$ is f.g. free

Then $H(X_* \otimes Y_*) \cong H(X_*) \otimes H_*(Y)$

(2) Let X and Y be (CW complexes) s.t. $H^*(\ ; R)$ is (f.g.) free for all $*$.

Then $\underset{\smile}{\mathrm{ext}}: H^*(X;R) \otimes H^*(Y;R) \to H^*(X \times Y; R)$ is an isomorphism.

$\underline{\text{Pf}}$ (1): $\left(X_* \otimes Y_*\right) = \operatorname{Tot}\left(\begin{array}{l}\text{double complex} \\ X_p \otimes Y_q \\ d_1 = d_{X_p} \otimes \mathrm{id} \\ d_2 = (-1)^p \, \mathrm{id} \otimes d_Y\end{array}\right)$

E^0

$$\begin{tikzcd} X_{p-1} \otimes Y_q & X_p \otimes Y_q \arrow[d] \\ & X_p \otimes Y_{q-1} \end{tikzcd}$$

E^1

$$\begin{tikzcd} X_{p-1} \otimes H_q(Y) & X_p \otimes H_q(Y) \arrow[l] \\ & X_p \otimes H_{q-1}(Y) \end{tikzcd}$$

E^2

$$H_p(X) \otimes H_q(Y)$$

Given an element of $H_p(X) \otimes H_q(Y)$

Can expreess as sum of terms

$[x] \otimes [y]$ $\quad x \in \ker(d_X : X_p \to X_{p-1})$

$\quad y \in \ker(d_Y : Y_q \to Y_{q-1})$

$x \otimes y \in (X \otimes Y)_{p+q}$

has boundary 0 in $\mathrm{Tot}(X \otimes Y)$

Moreover, $[x \otimes y] \in \mathrm{Tot}(X \otimes Y)_{p+q}$

is independent of choice of x.

To see this: $(x + dx') \otimes y =$

$$x \otimes y + d(x' \otimes y).$$

Also independent of choice of y.

Thus have map $H_*(X) \otimes H_*(Y) \to H_*(X \times Y)$

This map is compatible w/ SS in sense that $[x]\otimes[y] \in E^2_{pq}$ determines $[x\otimes y] \in F^q H(\mathrm{Tot}(X\otimes Y))$

$\Rightarrow d([x]\otimes[y])=0 \Rightarrow [x]\otimes[y]$ survives to E^∞

$$\Rightarrow \quad d_2([x]\otimes[y]) = 0$$
$$d_3([x]\otimes[y]) = 0$$
$$\vdots$$

$\Rightarrow$ all differentials are 0.

$\Rightarrow E^\infty = E^2$

Thus $H(X)\otimes H(Y) \to H(X\otimes Y)$ is isomorphism on Gr. Thus iso. $\square$