$F \in \text{obj}(\mathcal{C})$

$\text{F} \text{ ab group, graded abelian group, module etc.}$

d: $F \rightarrow F \quad d^2 = 0$

e.g. $(F, d)$ chain complex

S'pose $F$ filtered meaning

$F_1 = 0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = F$ s.t. $d(F_p) \subset F_p$

To a filtered object, have associated graded

$$\text{Gr } F = \bigoplus_{p} F_p / F_{p-1}$$

$H(F) := \frac{\ker d}{\text{Im } d}$

Associated to $(F, d)$ is a spectral sequence whose purpose is to (help) compute $H(F)$.

$H(F)$ has a filtration

$0 = H(F)^0 \subset H(F)^1 \subset \ldots \subset H(F)^n = H(F)$

$H(F)^p = \text{Im}(\ker d \cap F^p) \rightarrow H(F)$

Spectral sequence computes $\text{Gr } H(F)$

A spectral sequence is a sequence $(E_r, d_r)$

$E_r = \bigoplus_{p=0} \oplus E_{r,p}$

$d_r: E_{r,p} \rightarrow E_{r-1, p-r}$

$d_r^2 = 0$

s.t. $H(E_r^p) = E_{r+1}^p$
Often \( E_r^p = E_{r+1}^p = \ldots \). When this holds,

\[
E_\infty^p := E_r^p
\]

Important example:

\((F, d)\) defines a spectral sequence s.t.

\[
E_0^p = \text{Gr}_F^p F = F^p / F^{p-1}
\]

\[
E_1^p = H(\text{Gr}_F^p F) \quad E_\infty^p = H(F)^p
\]

\[\uparrow\]

easier to \quad want to compute\quad\quad\quad\quad compute

\textbf{pf.}\quad \mathbb{Z}_r^p = \{ x \in F^p \text{ s.t. } dx \in F^{p-r} \}

\[
E_r^p = \frac{\mathbb{Z}_r^p}{d\mathbb{Z}_{r-1}^p + \mathbb{Z}_{r-1}^p}
\]
\[ d^r : \mathbb{Z}_p \rightarrow \mathbb{Z}_{p-r} \]

\[ \left( d^r \mathbb{Z}_{p-r+(-1)} + \mathbb{Z}_{p-r-1} \right) \]

image of \( \mathbb{Z}_{p-1} \) is trivial \( \text{b/c} \)

\[ d \mathbb{Z}_{p-r+1} \quad \text{b/c} \quad d^2 = 0 \]

\[ \Rightarrow d^r : E^r_p \rightarrow E^r_{p-r} \quad \text{well defined} \]

\[ \text{NB: } d^2 : H(\frac{F_p}{F_{p-1}}) \rightarrow H(\frac{F_{p-1}}{F_{p-2}}) \]

is boundary map from SES

\[ \frac{F_{p-1}}{F_{p-2}} \rightarrow \frac{F_p}{F_{p-2}} \rightarrow \frac{F_p}{F_{p-1}} \]

\( \text{b/c both induced } d : F \rightarrow F. \)
Usually, $F$ is graded $F = \bigoplus F_i$ and $d : F_i \to F_{i-1}$.

**Insert example**

Then $E^r_p$ is also graded. One common custom is to then define

$$E^r_{pq} = \deg p+q \text{ piece } (E^r_p)$$

and draw $E^r$.
Example: $X$ a CW complex, so filtered $0 < X^0 < X^1 < \ldots < X^n = X$

Apply $C_\ast = \text{functor of singular chains}$

$\left( C_\ast(X), d \right)$

with filtration

$0 < C_\ast(X^0) \subset C_\ast(X^1) \subset \ldots \subset C_\ast(X)$

$E^0 = \text{Gr } C_\ast(X) = \bigoplus_p C_\ast(X^p, X^{p-1})$

$E^1 = \bigoplus_p H_\ast \left( X^p, X^{p-1} \right)$

$d_1 : H_\ast \left( X^p, X^{p-1} \right) \to H_\ast \left( X^{p-1}, X^{p-2} \right)$

Induced from

$\frac{C_\ast(X^p)}{C_\ast(X^{p-2})} \rightarrow \frac{C_\ast(X^p)}{C_\ast(X^{p-3})}$
So... This is CW homology complex

\[ E^2 = \oplus \mathcal{H}^p(X) \]

\[ E^2 = E^\infty \quad \text{bl/c} \quad d^r : E^r_{p,q} \to E^r_{p-r,q+r} \]

degree * \quad \text{in } \mathbb{C}^* \quad \text{in } \mathbb{C}^*

And \( E^r_{p-r} \) is concentrated in deg \( q = p - r \).

This recovers CW-homology.

Ex: \( 0 < P' \subseteq F \subseteq F^2 = F \) say chain pls

\[ E^0 = F_1 \oplus F/F_1 \]

\[ E^1 = \mathcal{H}(F_1) \oplus \mathcal{H}(F/F_1) \]
\[
\begin{align*}
E_1^1 & \xrightarrow{d} E_1(F_1) \\
& \xrightarrow{d} E_0(F_1)^{\oplus} \\
& \xrightarrow{d} E_0(F_1 + H_0(F/F_1)) \\
p = 0 & \quad p = 1 \\
\text{SES} & \quad F_1 \to F_2 \to F_2/F_1
\end{align*}
\]

\[
\begin{align*}
E_2 & \\
& \xrightarrow{d} \text{ker } d \\
& \xrightarrow{d} \text{ker } d \\
& \xrightarrow{d} E_0(F/F_1)
\end{align*}
\]

\[
\begin{align*}
E_\infty & = \text{Gr}^p H_\infty(F_\infty) \\
E_{\infty}^p & = \text{Gr}^p H_\infty(F_\infty) \\
E_{\infty}^q & = \text{Gr}^q H_\infty(F_\infty)
\end{align*}
\]

\[
\begin{align*}
\text{Gr}^p H(F_1) & \to H(F_2) \to \text{Gr}^q H(F_2) \\
q \geq 1
\end{align*}
\]

\[
\begin{align*}
0 & \to \text{ker } d \xrightarrow{d} H_\infty(F_2) \to \text{ker } d \to 0 \\
\text{ker } (d: H_{q+1} F_2 / F_1 \to H_q F_1)
\end{align*}
\]
This is precisely the information contained in LES associated to SES

\[ F_1 \to F_2 \to F_2/F_1 \]

"LES in $M$ from inclusion of cplxs"

"Spectral sequences are the replacement for LES when you have more than one sub complex"

--- go back & give information on bigraded degrees ---

Reason for this:

Spectral sequence of a double complex.

\[ K = \bigoplus_{p,q} K_{p,q} \]

\[ d_1 : K_{p,q} \to K_{p+1,q-1} \]

\[ d_2 : (K_{p,q} \to K_{p,q+1}) \]
\[ d_1^2 = 0, \quad d_2^2 = 0, \quad d_1d_2 + d_2d_1 = 0 \]

Associated total complex

\[ \text{Tot} (K) \big|_n = \bigoplus_{p + q = n} K_{p,q} \]

\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \]

\[ \text{Tot} (K) \text{ has two natural filtrations} \]

\[ F_i = \bigoplus_{p \leq i} K_{p,q} \]

\[ \mathcal{F}_i = \bigoplus_{q \leq i} K_{p,q} \]

Giving two spectral sequences \( \mathcal{E} \)

\[ H_\ast (\text{Tot} K) \]
\[ e \begin{array}{c} d_2 \cr \downarrow \cr K_{p-1} \end{array} \xrightarrow{d_1} K_{pq} \xrightarrow{d_0} K_{pq-1} \]

\[ \text{Koo Kio K20} \]

\[ \text{P} \]

**Thm. (Künneth Formula)**

(1) Let \( R \) be a ring and \( X_\ast, Y_\ast \) be two chain complexes of f.g. free \( R \)-modules s.t. \( H(Y) \) is f.g. free.

Then \( H(X \ast \otimes Y_\ast) \cong H(X) \otimes H(Y) \).

(2) Let \( X \) and \( Y \) be \((\text{CW complexes})\) s.t. \( H^\ast(\,; R) \) is (f.g.) free for all \( \ast \).

Then \( \text{ext} : H^\ast(X; R) \otimes H^\ast(Y; R) \to H^\ast(X \times Y; R) \) is an isomorphism.
\[ P^k (1) : (X_\ast \otimes Y_\ast) = \text{Tot} \left( \begin{array}{c}
\text{double complex} \\
X_p \otimes Y_q
\end{array} \right) \]

\[ d_1 = d \times \text{id} \]

\[ d_2 = (-1)^p \text{id} \otimes d \]

\[ E^0 \]

\[ E^1 \]

\[ E^2 \]
Given an element of $H_p(X) \otimes H_q(Y)$ can express as sum of terms

$$[x] \otimes [y], \quad x \in \ker (d_x: X_{p-1} \to X_p)$$

$$y \in \ker (d_y: Y_{q-1} \to Y_q)$$

$$x \otimes y \in (X \otimes Y)_{p+q}$$

has boundary 0 in Tot $(X \otimes Y)$

Moreover, $[x \otimes y] \in \text{Tot} (X \otimes Y)_{p+q}$ is independent of choice of $x$.

To see this: $(x + dx^1) \otimes y = x \otimes y + d(x \otimes y)$.

Also independent of choice of $y$.

Thus have map $H_p^*(X) \otimes H_q^*(Y) \to H^*_p(X)$.
This map is compatible w/ SS in sense that \([x] \otimes [y] \in E_{pq}^2\) determines \([x \otimes y] \in F^q H(T \otimes (X \otimes Y))\).

\[ \Rightarrow d([x] \otimes [y]) = 0 \Rightarrow [x] \otimes [y] \text{ survives to } E^\infty \]

\[ \Rightarrow d_2 ([x] \otimes [y]) = 0 \]

\[ d_3 ([x] \otimes [y]) = 0 \]

\[ \Rightarrow \text{ all differentials are } 0 \]

\[ \Rightarrow E^\infty = E^2 \]

Thus \(M(X) \otimes M(Y) \rightarrow H(X \otimes Y)\) is isomorphism on Gr. Thus iso. \(\square\)