1/28/13
Lecture 1
Introduction

• want: understand $\wedge$ category of topological spaces & maps b/w them.

• X top space, d-dim'l holes $\rightsquigarrow$ $H_d X$, $H^d X$, π_d abelian groups (except π_1)

H_1 obstructs certain maps

ex: $\nexists$
$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}"'] & D^2 \arrow[r] & S^1 \end{tikzcd}$$

$\pi_*, H_*, \& H^*$ do not control maps b/w spaces

ex: Hopf map
$$\begin{tikzcd} S^3 \arrow[r] \arrow[d, phantom, "\cap"] & S^2 \arrow[d, phantom, "\shortparallel"] \\ \mathbb{C}^2 & \mathbb{C}P^1 \\ (x,y) \arrow[r, mapsto] & x/y \end{tikzcd}$$
trivial on π_*, not null-homotopic

• Hard to convert topology to algebra (this is in some sense because $\pi_n(S^m)$ hard)

• $\pi_n(S^m)$ hard. $\mathrm{Map}(S^n, S^m)$ hard. interesting, but perhaps impossible.

• For G a $\wedge$ discrete (abstract) group, let EG be a contractible space w/ free G-action

$$BG = EG/G \Rightarrow_{i>1} \pi_i(BG) = \pi_i(\text{universal cover}) = \pi_i(EG) = 0$$

$$\pi_1 BG = G$$

Sullivan conjecture $\Rightarrow$ Map$(BG, -)$ is computable!

for G a finite p-group

How?

(1) $H^*(X)$ is an abelian group

(2) X space $\Rightarrow$ X has diagonal map

$$\begin{tikzcd} X \arrow[r, "\Delta"] & X \times X \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ & X & X \end{tikzcd}$$

$$\Rightarrow H^*(X \times X) \longrightarrow H^*(X)$$

$$\begin{tikzcd} H^*(X \times X) \\ H^*(X) \otimes H^*(X) \arrow[u, "\text{external cup product}"'] \end{tikzcd}$$

$\Rightarrow$ H^* is a ring.

(3) (1) $\Rightarrow$ H^* is a Functor to abelian groups

For each i, we will compute

$\mathrm{Nat}(H^*, H^{*+i}) \leftarrow$ "cohomology operations"

$\mathcal{A} = $ Steenrod alg $= \bigoplus_i \mathrm{Nat}(H^*, H^{*+i})$

for $H^* = H^*(-, \mathbb{Z}/p)$

Then: $H^*(X)$ is a module over the Steenrod algebra.

$\boxed{\text{(2) \& (3) can compute } \mathrm{Map}(BV, -)}$ (mostly)

Why? $H^*(BG)$ has good properties as module over $\mathcal{A}$

e.g. $- \otimes H^*(BG)$ preserves injectives in modules over $\mathcal{A}$

Thm (Lannes)

Thm (Miller $p>2$, Carlsson $p=2$) $H^*(BG)$ is a direct summand of a limit of Brown–Gitler mod $\mathbb{Z}/e$

For this & for its own sake, we will develope:

- Steenrod alg & its dual
- completion of a topological space $X^\wedge_p$
- Spectral sequences; unstable Adams SS

Application: Calculation of X^G

Fixed points do not behave well w.r.t. homotopy

ex: $\mathbb{R} \longrightarrow pt$ htpy equivalence

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & pt \\ \circlearrowleft & & \circlearrowleft \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

$\mathbb{Z}$ acting by addition

no fixed pts $\qquad$ $\exists$ fixed pt

G acts on $\mathrm{Map}(EG, X)$ by $(gf)(y) = g f(g^{-1}y)$

Def: $X^{hG} = \mathrm{Map}(EG, X)^G$

Prop: $X \to X'$ G equivariant and htpy equivalence $\Rightarrow$

$$X^{hG} \xrightarrow{\sim} (X')^{hG}$$ htpy equivalence.

Ex: $X^{h\mathbb{Z}} = \{ (x, \gamma) : x \in X$, γ path from x to $1 \cdot x \}$

In particular, $\mathbb{R}^{h\mathbb{Z}} \neq \emptyset$.

htpy fixed points can be easier to construct

Thm (Miller, Carlsson, Lannes)

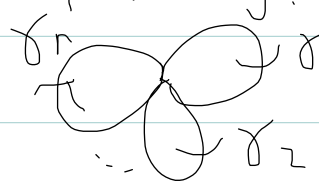
$$(X^G)^\wedge_p \xrightarrow{\sim} (X^\wedge_p)^{hG}$$

for X a finite dim'l G-cplx

• $\exists$ a spectral sequence converging to $\pi_* X^{hG}$ "good for computing fixed pts which can be important mathematically"

• why would you turn top to alg? Insert ✪

Ex: (top $\to$ alg)



$= X$ $\quad G = \mathbb{Z}/2$ "flips each circle"

(free grp n gen action $\gamma_i \mapsto \gamma_i^{-1}$)

$$H^1(\mathbb{Z}/2, \langle \gamma_1, \dots, \gamma_n \rangle) = \{0, 1, \dots, n\}$$

• That's Sullivan's Conjecture & some reasons it's useful. We're going to prove it while developing tools that are important more generally.

• Are there questions about the goals of the course?

Some logistics: problem set, (collaborate!) final paper

final grading won't be stringent, assigned work is for your own benefit.

★ Insert above

Form w/ trivial action $\underline{\text{Thm}}$ (Miller) G any finite group

X finite dim'l CW-cplx

$$\pi_* \operatorname{map}_*(BG, X) = 0$$

$$\Rightarrow \varinjlim_n [BG, \Omega^n S^n] = 0$$

Relationship to Segal's conjecture:

$$[BG, \varinjlim \Omega^n S^n] = \hat{A}(G) \neq 0$$

completed Burnside ring

- Motivation for Adams's definition of maps in the category of spectra "cells now maps later"