integrally closed

$\dim 1$, Noetherian rings are DVR's

$x^2 + y^2 = 2^3 \subset \mathbb{P}^2 = \text{Proj } \mathbb{C}[x,y,z]$ $p = (0,1,1)$

$\mathbb{A}^2 \subset \mathbb{P}^2$

$\text{ord}_p \frac{x}{y} = 1$

**PF:**

$k[x,y] / (x,y-1)$

$\langle x^2 + y^2 - 1 \rangle$

$(y-1)(y+1) = x^2$

$y+1$ is a unit $\implies \langle x, y-1 \rangle = \langle x \rangle$

$\implies$ maximal ideal generated by $x$

$\implies x$ uniformizer and $\text{val}_p(x) = 1$ $\text{val}_p(y-1) = 2$
Getting to know $\text{CH}^*$: X variety

$f \in K(X) = \text{rational functions}$

$\text{Div}(f) = \sum Y \in X \text{ irreducible} \: \text{ord}_Y(f) Y$

$\mathcal{Z}(X) = \text{free group generated by subvarieties}$

$\text{Rat}(X) \subset \mathcal{Z}(X)$

Subgroup generated by $\text{Div}(f)$

$\text{CH}^*(X) = \mathcal{Z}(X) / \text{Rat}(X)$

Remark: $\text{Div}(f)$ also makes sense for a section of a line bundle. Then $c_1(X) = \text{Div}(f)$

Remark: Naturally graded by dimension

Eisenbud–Harris $1,14$

Proposition (Mayer–Vietoris): If $X_1, X_2 \hookrightarrow X$

$\text{CH}^*(X_1 \cap X_2) \to \text{CH}^*(X_1) \otimes \text{CH}^*(X_2) \to \text{CH}^*(X_1 \cup X_2) \to 0$
(b) (excision) \( Y \xrightarrow{\text{cl}_Y} X \quad U = X \setminus Y \)

\[ \text{CH}(Y) \rightarrow \text{CH}(X) \rightarrow \text{CH}(U) \rightarrow 0 \]

If \( X \) is smooth, \( \text{CH}(X) \rightarrow \text{CH}(U) \) is a ring homomorphism

**day 1**: \( \text{CH}^*(\mathbb{A}^n) = \mathbb{Z} \text{CA}^n \)

**Cor**: \( U \subset \mathbb{A}^n \) non-empty open set, then \( \text{CH}^*(U) = U \)

Last time: \( \text{CH}^{(n-1)}(X) \cong \text{Pic}(X) \times \text{sm, proj} \)

**Def**: Subvarieties \( Y \) and \( Z \) of \( X \) intersect transversely at \( p \) if

- \( Y, Z, X \) are smooth at \( p \)
- \( T_p X = T_p Z + T_p Y \)
Subset of such $p$ is open

**Def:** $Y$ and $Z$ intersect generically transversely if $Y$ and $Z$ meet transversely at a general pt of each component $Y \cap Z$

**Thm/Def:** $X$ smooth, quasi-proj variety. $\prod$ product structure on $CH(X)$ s.t.

- If $Y$ and $Z$ are generically transverse, then

$$[Y] [Z] = [Y \cap Z]$$

This product makes $CH(X)$ into an associative, commutative ring, graded by codimension

**Ex:** $CH^*(\mathbb{P}^n) = \mathbb{Z} \mathbb{C} \mathbb{P}^n/ \mathbb{C} \mathbb{P}^{n-1}$
PF: Use affine stratification

EH 1.3.5

\[ \mathbb{P}^{n-1} \to \mathbb{P}^n \to \mathbb{A}^n \to 0 \]

and induction, and intersection of hyperplanes

Historically, proof based on

Moving lemma \( X \) smooth, quasi-proj

(Ca) for every \( \alpha, \beta \in CH^*(X) \) there are generically transverse cycles \( A, B \in Z(X) \) with

\[ [A] = 2 \]
\[ [B] = B \]

(b) The class \( [A \cap B] \) is independent of the choice of \( A, B \)
Prop: If $X$ is smooth and $Y \subseteq X$ is any subvariety, then
\[ c_1(X).[Y] = c_1(X/Y) \]

pf: Lemma: There is a section cycle in the class of $c_1(X)$ transverse to every comp of $Y$.

pf: Bertini's Thm can take transverse sections for ample enough

\[ \forall \mathcal{H}^0(G^1) \, \exists \xi \mathcal{H}^0(G^1) \text{ ample} \]

\[ c_1(X) = \text{Div}(\sigma) - \text{Div}(\tau) \]

\[ \text{Div}(f_u).[Y] \in c_1(X).[Y] - \text{Div}(\tau)[Y] \]

\[ \text{Transverse } \text{Div} \sigma + Y - \text{Div} \tau \neq X \]
**Example:** Self-intersection of hypersurface
\[
X = \text{deg } d \text{ hypersurface in } \mathbb{P}^n \quad X \cdot X = d^2 \mathcal{O}_{\mathbb{P}^n} = d^2 \mathcal{O}_{\mathbb{P}^n}.
\]

**Principle:** If you can move a curve, you have positive self-int.

**Example:**
\[
N_{\mathbb{P}^1} \mathbb{P}^2 = \mathcal{O}_C(1)
\]
\[
N_{\mathcal{B} \circ \mathbb{P}^2} = \mathcal{O}_C(-1)
\]

**Functionality:** proper pushforward

**Definition:** \( f : Y \to X \) is proper if it is
- Separated, of finite type, and universally closed
- Con (C-pts, inverse image of compact is compact)
- \( A \subset Y \) subvariety

\( \Rightarrow FC(A) \) is a subvariety
Def: $f: A \to B$ map between varieties is \underline{generically finite} if $K(B) \subset K(A)$ is a finite extension of fields.

In this case \[\deg f = [K(A): K(B)]\]

Def: Let $f: Y \to X$ be a proper map of schemes and let $A \subset Y$ be a subvariety

(a) If $\dim f(A) < \dim A$ \[f_* A = 0\]

(b) If $\dim f(A) = \dim A$ \[f_* A = \deg f|_A f(A)\]

(c) $f_*: \mathcal{Z}(Y) \to \mathcal{Z}(X)$ is linear

Thm: If $Y \to X$ is a proper map of schemes, $f_*: \mathcal{Z}(Y) \to \mathcal{Z}(X)$ induces a map
f_* : CHC(Y) → CHC(X)

homology / cohomology theory

Bloch’s formula: $CH^*(X) ≅ H^*_\text{Nis}(X, K^M_*)$