GROTHENDIECK'S ANABELIAN CONJECTURES

KIRSTEN WICKELGREN

I get told, rather more frequently than I would like, that mathematicians don't really work on Grothendieck's anabelian program [Gro97b] anymore, that it is too hard to produce real progress, or even that it didn't work. None of these statements are true in my opinion. The Newton institute currently is running a program on the Grothendieck-Teichmüller group, which is an important character in Grothendieck's anabelian program, and I think it's fair to say that Mochizuki's recent and stunning work on the abc-conjecture is related. With a view to reassure the hypochondriac of intellectual stagnation, the subject produced many papers (I count at least 19 on the arXiv), and at least one book [Sti13] last year, which contained a lot of valuable results. If anything, it is getting closer to mainstream mathematics rather than farther from it. More to the point, the conjectures themselves are lovely, and provide a beautiful point of view on solving polynomial equations.

The classification of the covering spaces of a topological space in terms of its fundamental group can be used to define a notion of fundamental group in another category given a notion of covering space. Here's how. Let X be a non-pathological,[1] connected topological space with a chosen point x_0. Given a connected covering space $f : Y \to X$, choose y_0 mapping to x_0 under f, which is always possible. Changing the choice of y_0 changes the subgroup $f_*\pi_1(Y, y_0) \subseteq \pi_1(X, x_0)$ to $\gamma f_*\pi_1(Y, y_0)\gamma^{-1}$ for some γ in $\pi_1(X, x_0)$. The resulting association of the conjugacy class of the subgroup $\pi_1(Y, y_0)$ to the isomorphism class of the connected covering space $Y \to X$ is a bijection [Mun75, 82.1, 79.2]. This can be rephrased more functorially as follows. Note that $\pi_1(X, x_0)$ acts on $f^{-1}(x_0)$ by lifting a path representing an element of $\pi_1(X, x_0)$ starting at a point of $f^{-1}(x_0)$ and recording the point of $f^{-1}(x_0)$ where the path ends. This action defines a natural $\pi_1(X, x_0)$-equivariant bijection between the cosets of $f_*\pi_1(Y, y_0)$ in $\pi_1(X, x_0)$ and $f^{-1}(x_0)$. Sets with a $\pi_1(X, x_0)$-action form a category, and the association of the fiber $f^{-1}(x_0)$ to the covering space $f : Y \to X$ is a functor, which we'll call $\mathcal{F}$. One rephrases the previous bijection by the equation

$$(1) \qquad \pi_1(X, x_0) = \mathrm{Aut}(\mathcal{F}),$$

where $\mathrm{Aut}(\mathcal{F})$ denotes the group of natural transformations from $\mathcal{F}$ to itself. Equation (1) then allows us to define a fundamental group given an appropriate $\mathcal{F}$, so equipped with a notion of covering space and fiber, we obtain a definition of the fundamental group.

Grothendieck discovered and used this process [SGAI], producing the étale fundamental group for schemes. A scheme is an object which records the solutions to some set of polynomial equations. For example, $\{(x, y) \in \mathbb{C}^2 : y = x^2\}$ fits naturally into a scheme denoted $\mathrm{Spec}\,\mathbb{C}[x, y]/\langle y - x^2 \rangle$. For any ring R, there is an associated scheme $\mathrm{Spec}\,R$, and

---

[1]locally path connected and semi-locally simply connected

a scheme X is said to be *over* R if there is a map $X \to \operatorname{Spec} R$. This corresponds to the requirement that the associated polynomial equations have coefficients in R.

The étale fundamental group combines both Galois groups and topological fundamental groups. Here is one example. Let k be a subfield of $\mathbb{C}$. View a finite type scheme X over k as the total space of a fibration over the classifying space[2] $B \operatorname{Gal}(\overline{k}/k)$ of the absolute Galois group whose fiber is the profinite completion of the complex points $X(\mathbb{C})$. The associated long exact sequence of homotopy groups (assuming $X(\mathbb{C})$ connected) gives

(2) $$1 \to \pi_1(X(\mathbb{C}))^{\wedge} \to \pi_1 X \to \operatorname{Gal}(\overline{k}/k) \to 1,$$

where $\pi_1(X(\mathbb{C}))^{\wedge}$ denotes the profinite completion of $\pi_1(X(\mathbb{C}))$. See [SGAI, IX Thm 6.1, XII Thm 5.1] and [AM69] [Fri82]. The map $\pi_1(X) \to \operatorname{Gal}(\overline{k}/k)$ is the functor π_1 applied to the structure map $X \to \operatorname{Spec} k$.

For any scheme, the étale fundamental group has a natural topology compatible with the group structure, and all the morphisms between étale fundamental groups will be understood to be continuous. For simplicity, we will also assume from now on that k is a number field, i.e. k is a finite dimensional field extension of $\mathbb{Q}$.

Grothendieck's anabelian conjectures identify schemes, called *anabelian*, which are predicted to be determined by their étale fundamental groups. To be more precise about the phrase "determined by," consider again the category of topological spaces. A map of topological spaces $f : Y \to X$ and a chosen point $y_0 \in Y$ induces a map

$$\pi_1(Y, y_0) \to \pi_1(X, f(y_0)).$$

If we have some chosen point x_0 of X, we can choose a path from $f(y_0)$ to x_0 and obtain a map $\pi_1(X, f(y_0)) \to \pi_1(X, x_0)$. The composite map $\pi_1(Y, y_0) \to \pi_1(X, x_0)$ is determined up to post-composition by an inner automorphism of $\pi_1(X, x_0)$. Let

$$\operatorname{Map}^{\operatorname{out}}(\pi_1(Y, y_0), \pi_1(X, x_0))$$

denote the set of group homomorphisms up to post-composition by these inner automorphisms, i.e. $\operatorname{Map}^{\operatorname{out}}(\pi_1(Y, y_0), \pi_1(X, x_0))$ denotes the set of equivalence classes of group homomorphisms where two homomorphisms f and f' are equivalent when there exists γ in $\pi_1(X, x_0)$ such that for all η in $\pi_1(Y, y_0)$,

$$f'(\eta) = \gamma f(\eta) \gamma^{-1}.$$

The previous comments show π_1 induces a map

(3) $$\operatorname{Map}_{\operatorname{top}}(Y, X) \to \operatorname{Map}^{\operatorname{out}}(\pi_1(Y, y_0), \pi_1(X, x_0)).$$

Furthermore, up to post-composition by an inner automorphism of $\pi_1(X, x_0)$, there is a canonical isomorphism $\pi_1(X, x_1) \to \pi_1(X, x_0)$ for any other point x_1 of X, so one often omits the base points, writing $\pi_1(X)$ for $\pi_1(X, x_0)$. A nice exercise in understanding CW-complexes[3] is to prove that for X such that $\pi_* X$ is trivial for $* \neq 1$, (3) induces a bijection from homotopy classes of continuous maps $Y \to X$, assuming Y a connected CW complex, to $\operatorname{Map}^{\operatorname{out}}(\pi_1(Y, y_0), \pi_1(X, x_0))$. By Yoneda's lemma, the fact that (3) induces this bijection determines X up to homotopy. Such X are called $K(\pi, 1)$'s and they are the topological

---

[2]The classifying space BG of G is the quotient by G of a contractible space EG with a free G-action, so $BG = EG/G$.
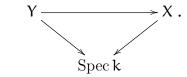
[3]CW-complexes are spaces built up from disks of various dimensions.

analogues of anabelian schemes. The notion of an anabelian scheme was introduced by Grothendieck in a letter to Faltings [Gro97a], which became important mathematically, and famous in a quiet way. It also contains the anabelian conjectures, which we'll state after defining an anabelian scheme.

2. Definition [Gro97a, $\frac{3}{4}$]. *A finite type scheme X over k is said to be* anabelian *if it can be constructed by successive smooth fibrations of curves with negative Euler characteristic.*

The complex points of an anabelian scheme form a $K(\pi, 1)$, because this is true of curves with negative Euler characteristic, and the property holds for the total space of a fibration when it holds for the base and fiber.
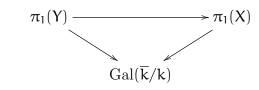
To state the anabelian conjectures, we need the analogue of (3) in the category of schemes over k. The morphisms in this category are commutative triangles

$$Y \longrightarrow X \, .$$
$$\searrow \quad \swarrow$$
$$\mathrm{Spec}\, k$$

Applying π_1 with chosen base-points as above produces a map

(4) $$\mathrm{Map}_{k-\mathrm{scheme}}(Y, X) \to \mathrm{Map}^{\mathrm{out}}_{\mathrm{Gal}(\overline{k}/k)}(\pi_1(Y), \pi_1(X)),$$

where $\mathrm{Map}^{\mathrm{out}}_{\mathrm{Gal}(\overline{k}/k)}(\pi_1(Y), \pi_1(X))$ denotes the set of equivalence classes of triangles of group homomorphisms

$$\pi_1(Y) \longrightarrow \pi_1(X)$$
$$\searrow \quad \swarrow$$
$$\mathrm{Gal}(\overline{k}/k)$$

which commute up to an inner automorphism of $\mathrm{Gal}(\overline{k}/k)$, and where two triangles are equivalent if their morphisms $\pi_1(Y) \to \pi_1(X)$ differ by an inner automorphism of $\pi_1(X)$. The map (4) is a good analogue of (3), but because of difficulties when X is not proper (which is the scheme-theoretic analogue of the condition that a topological space be compact), we need a refinement.

A map between schemes whose image is dense is called *dominant*. One can refine (4) to a map

(5) $$\mathrm{Map}^{\mathrm{dom}}_{k-\mathrm{scheme}}(Y, X) \to \mathrm{Map}^{\mathrm{out,open}}_{\mathrm{Gal}(\overline{k}/k)}(\pi_1(Y), \pi_1(X)),$$

from the dominant morphisms of k-schemes $Y \to X$ to the subset

$$\mathrm{Map}^{\mathrm{out,open}}_{\mathrm{Gal}(\overline{k}/k)}(\pi_1(Y), \pi_1(X)) \subset \mathrm{Map}^{\mathrm{out}}_{\mathrm{Gal}(\overline{k}/k)}(\pi_1(Y), \pi_1(X))$$

consisting of triangles such that $\pi_1(Y) \to \pi_1(X)$ has open image (which is the same as being an open map in this case).

3. Conjecture (Grothendieck [Gro97a, $\frac{5}{6}$, P.S.]**).** *For X and Y anabelian, the map* (5) *is bijective.*

Note that the domains of (4) and (5) are not sets of homotopy classes of maps, but rather the sets of maps themselves, with no equivalence relation imposed. The reason for this is that continuous deformations of algebraic maps are rarely algebraic. For instance, two algebraic maps between hyperbolic Riemann surfaces which are homotopic, i.e. for which there is a continuous map $[0,1] \times Y \to X$ whose restriction to $\{0,1\} \times Y$ is as expected, must be the same,[4] whence passing to homotopy classes of algebraic maps has no effect.

4. Thereom (Mochizuki [Moc99]). *For* Y *any smooth scheme and* X *a smooth curve with negative Euler characteristic,* (5) *is bijective.*

An algebraic map to a curve X which is not dominant has image consisting of a single point (when Y is assumed to be connected), which reduces the problem of understanding such maps to studying the maps $\operatorname{Spec} L \to X$ for X a field. By replacing X with $X \otimes L$, one can further reduce to $L = k$.

5. The section conjecture (Grothendieck [Gro97a, $\frac{5}{6}$]). *For* $Y = \operatorname{Spec} k$ *and* X *a smooth, proper curve with genus* > 1, (4) *is a bijection.*

The section conjecture says that the solutions to polynomial equations underlying a curve are determined by the curve's loops.

There is a version of the section conjecture without the assumption that X is proper. It involves *tangential basepoints*, which construct sections associated with k-tangent vectors at k-points of the smooth compactification of X, as well as sections associated with limits of such tangent vectors.

The section conjecture (with or without the assumption that X is proper) is unknown. It is analogous to an equivalence between fixed points and homotopy fixed points of an action of $\operatorname{Gal}(\overline{k}/k)$ on $B\pi_1(X(\mathbb{C}))^{\wedge}$ as follows. For a group G acting on a space X, the definition of the homotopy fixed points X^{hG} is $X^{hG} = \operatorname{holim}_G X = \operatorname{Map}(EG, X)^G$, where EG denotes a contractible space with a free G-action, and $\operatorname{Map}(EG, X)^G$ denotes the topological space of G-equivariant maps from EG to X. Consider the target $\operatorname{Map}^{\mathrm{out}}_{\operatorname{Gal}(\overline{k}/k)}(\operatorname{Gal}(\overline{k}/k), \pi_1(X))$ of (4) in the setting of the section conjecture. Since the classifying space $B\pi_1(X)$ of $\pi_1(X)$ is a $K(\pi, 1)$, it follows that the homotopy classes of maps $B\operatorname{Gal}(\overline{k}/k) \to B\pi_1(X)$ are in natural bijection with $\operatorname{Map}^{\mathrm{out}}(\operatorname{Gal}(\overline{k}/k), \pi_1(X))$ as discussed above. Similarly, the homotopy classes of sections of

(6) $$B\pi_1(X) \to B\operatorname{Gal}(\overline{k}/k)$$

are in natural bijection with $\operatorname{Map}^{\mathrm{out}}_{\operatorname{Gal}(\overline{k}/k)}(\operatorname{Gal}(\overline{k}/k), \pi_1(X))$. By the homotopy exact sequence (2),

$$B\pi_1(X(\mathbb{C}))^{\wedge} \to B\pi_1 X \to B\operatorname{Gal}(\overline{k}/k)$$

is a fiber sequence. (One useful way of seeing this fiber sequence is to use $E\pi_1 X \times E\operatorname{Gal}(\overline{k}/k)$ as a contractible space with both a $\pi_1(X(\mathbb{C}))^{\wedge}$-action and a $\pi_1 X$-action. First quotient $E\pi_1 X \times E\operatorname{Gal}(\overline{k}/k)$ by $\pi_1(X(\mathbb{C}))^{\wedge}$ producing $B\pi_1(X(\mathbb{C}))^{\wedge}$. This $B\pi_1(X(\mathbb{C}))^{\wedge}$ maps

---

[4]See [McM00, §III p.127] for the analogous result when X is the moduli space of genus g curves. The same argument works for X a hyperbolic curve.

to the quotient $(E\pi_1 X \times E\operatorname{Gal}(\overline{k}/k))/\pi_1 X = B\pi_1 X$, which in turn maps to $B\operatorname{Gal}(\overline{k}/k)$.)
Pulling back a section of (6) along $E\operatorname{Gal}(\overline{k}/k) \to B\operatorname{Gal}(\overline{k}/k)$ gives a section of

$$B\pi_1(X(\mathbb{C}))^{\wedge} \times E\operatorname{Gal}(\overline{k}/k) \to E\operatorname{Gal}(\overline{k}/k)$$

respecting the actions of $\operatorname{Gal}(\overline{k}/k)$, whence an element of

$$(B\pi_1(X(\mathbb{C}))^{\wedge})^{h\operatorname{Gal}(\overline{k}/k)}.$$

In fact, this correspondence identifies the homotopy type of the space of sections of (6) with

$$(B\pi_1(X(\mathbb{C}))^{\wedge})^{h\operatorname{Gal}(\overline{k}/k)},$$

giving the identification $\operatorname{Map}^{\mathrm{out}}_{\operatorname{Gal}(\overline{k}/k)}(\operatorname{Gal}(\overline{k}/k), \pi_1(X)) = \pi_0((B\pi_1(X(\mathbb{C}))^{\wedge})^{h\operatorname{Gal}(\overline{k}/k)})$. It is a fact about the étale fundamental group that $\pi_1(X \otimes \overline{k}) \cong \pi_1(X(\mathbb{C}))^{\wedge}$, so the previous can be rewritten

$$\operatorname{Map}^{\mathrm{out}}_{\operatorname{Gal}(\overline{k}/k)}(\operatorname{Gal}(\overline{k}/k), \pi_1(X)) = \pi_0((B\pi_1(X \otimes \overline{k}))^{h\operatorname{Gal}(\overline{k}/k)}).$$

Since X is a hyperbolic curve, the $\mathbb{C}$-points of X, denoted $X(\mathbb{C})$, are $B\pi_1(X(\mathbb{C}))$. View the $\overline{k}$-points of X as $B\pi_1(X \otimes \overline{k})$ by analogy. The k-points of X, written $X(k) = \operatorname{Map}_{k-\mathrm{scheme}}(\operatorname{Spec} k, X)$, are the $\operatorname{Gal}(\overline{k}/k)$-fixed points of $X(\overline{k})$. Thus we have an analogy between $\operatorname{Map}_{k-\mathrm{scheme}}(\operatorname{Spec} k, X)$ and $B\pi_1(X \otimes \overline{k})^{\operatorname{Gal}(\overline{k}/k)}$.

The G-equivariant map from EG to a point induces a natural map $X^G \to X^{hG}$ from the fixed points to the homotopy fixed points, so we have

$$B\pi_1(X \otimes \overline{k})^{\operatorname{Gal}(\overline{k}/k)} \to B\pi_1(X \otimes \overline{k})^{h\operatorname{Gal}(\overline{k}/k)}.$$

Applying π_0 to this map and precomposing with the map $X(k) \to \pi_0(X(k))$ yields a map analogous to (4) in the setting of the section conjecture.

There is a beautiful result comparing fixed points to homotopy fixed points for G a finite p-group.

6. Sullivan's conjecture. *Let G be a finite p-group. For X a finite G-complex, the natural map $(X^G)^{\wedge}_p \to \operatorname{Map}(EG, X^{\wedge}_p)^G$ induces an isomorphism after applying π_* for all $*$. Here $(-)^{\wedge}_p$ denotes Bousfield-Kan mod p completion.*

Sullivan's conjecture was proven independently by Haynes Miller [Mil84], Gunnar Carlsson [Car91], and Jean Lannes [Lan92]. Also see [DMN89].

The real section conjecture saying that

$$\pi_0(X(\mathbb{R})) \to \operatorname{Map}^{\mathrm{out}}_{\operatorname{Gal}(\mathbb{C}/\mathbb{R})}(\operatorname{Gal}(\mathbb{C}/\mathbb{R}), \pi_1^{\mathrm{orb}}(X(\mathbb{C})/\operatorname{Gal}(\mathbb{C}/\mathbb{R})))$$

is a bijection, where $\pi_1^{\mathrm{orb}}(X(\mathbb{C})/\operatorname{Gal}(\mathbb{C}/\mathbb{R}))$ can denote either the étale fundamental group of $X \to \operatorname{Spec}\mathbb{R}$ or the orbifold fundamental group of $X(\mathbb{C})/\operatorname{Gal}(\mathbb{C}/\mathbb{R})$,[5] is true by similar

---

[5]The orbifold fundamental group $\pi_1^{\mathrm{orb}}(X(\mathbb{C})/\operatorname{Gal}(\mathbb{C}/\mathbb{R}))$ can be defined as the group of continuous automorphisms of the universal covering space of $X(\mathbb{C})$ which lie over either the identity map $X(\mathbb{C}) \to X(\mathbb{C})$ or the map induced by complex conjugation.

arguments. If $X(\mathbb{R}) \neq \emptyset$, this can be rewritten as a natural bijection

$$\pi_0(X(\mathbb{R})) \to \mathrm{Map}^{\mathrm{out}}_{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \ltimes \pi_1(X(\mathbb{C}))) \cong \mathrm{H}^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \pi_1(X(\mathbb{C}))),$$

where $\pi_1(X(\mathbb{C}))$ can denote either the topological fundamental group of $X(\mathbb{C})$ or the étale fundamental group of $X \otimes \mathbb{C}$.

REFERENCES

[AM69]   M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Mathematics, No. 100, Springer-Verlag, Berlin, 1969. MR 0245577 (39 #6883)

[Car91]   Gunnar Carlsson, *Equivariant stable homotopy and Sullivan's conjecture*, Invent. Math. **103** (1991), no. 3, 497–525. MR MR1091616 (92g:55007)

[DMN89]   William Dwyer, Haynes Miller, and Joseph Neisendorfer, *Fibrewise completion and unstable Adams spectral sequences*, Israel J. Math. **66** (1989), no. 1-3, 160–178. MR 1017160 (90i:55034)

[Fri82]   Eric M. Friedlander, *Étale homotopy of simplicial schemes*, Annals of Mathematics Studies, vol. 104, Princeton University Press, Princeton, N.J., 1982. MR 676809 (84h:55012)

[Gro97a]   Alexander Grothendieck, *Brief an G. Faltings*, Geometric Galois actions, 1, London Math. Soc. Lecture Note Ser., vol. 242, Cambridge Univ. Press, Cambridge, 1997, With an English translation on pp. 285–293, pp. 49–58. MR 1483108 (99c:14023)

[Gro97b]   Alexandre Grothendieck, *Esquisse d'un programme*, Geometric Galois actions, 1, London Math. Soc. Lecture Note Ser., vol. 242, Cambridge Univ. Press, Cambridge, 1997, With an English translation on pp. 243–283, pp. 5–48. MR 1483107 (99c:14034)

[Lan92]   Jean Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 135–244, With an appendix by Michel Zisman. MR MR1179079 (93j:55019)

[McM00]   Curtis T. McMullen, *From dynamics on surfaces to rational points on curves*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 2, 119–140. MR 1713286 (2000j:11093)

[Mil84]   Haynes Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. (2) **120** (1984), no. 1, 39–87. MR 750716 (85i:55012)

[Moc99]   Shinichi Mochizuki, *The local pro-p anabelian geometry of curves*, Invent. Math. **138** (1999), no. 2, 319–423. MR 1720187 (2000j:14037)

[Mun75]   James R. Munkres, *Topology: a first course*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1975. MR 0464128 (57 #4063)

[SGAI]   *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3, Société Mathématique de France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)]. MR MR2017446 (2004g:14017)

[Sti13]   J. Stix, *Rational points and arithmetic of fundamental groups: Evidence for the section conjecture*, Lecture Notes in Mathematics, No. 2054, Springer-Verlag, Berlin, 2013.

DEPT. OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE MA